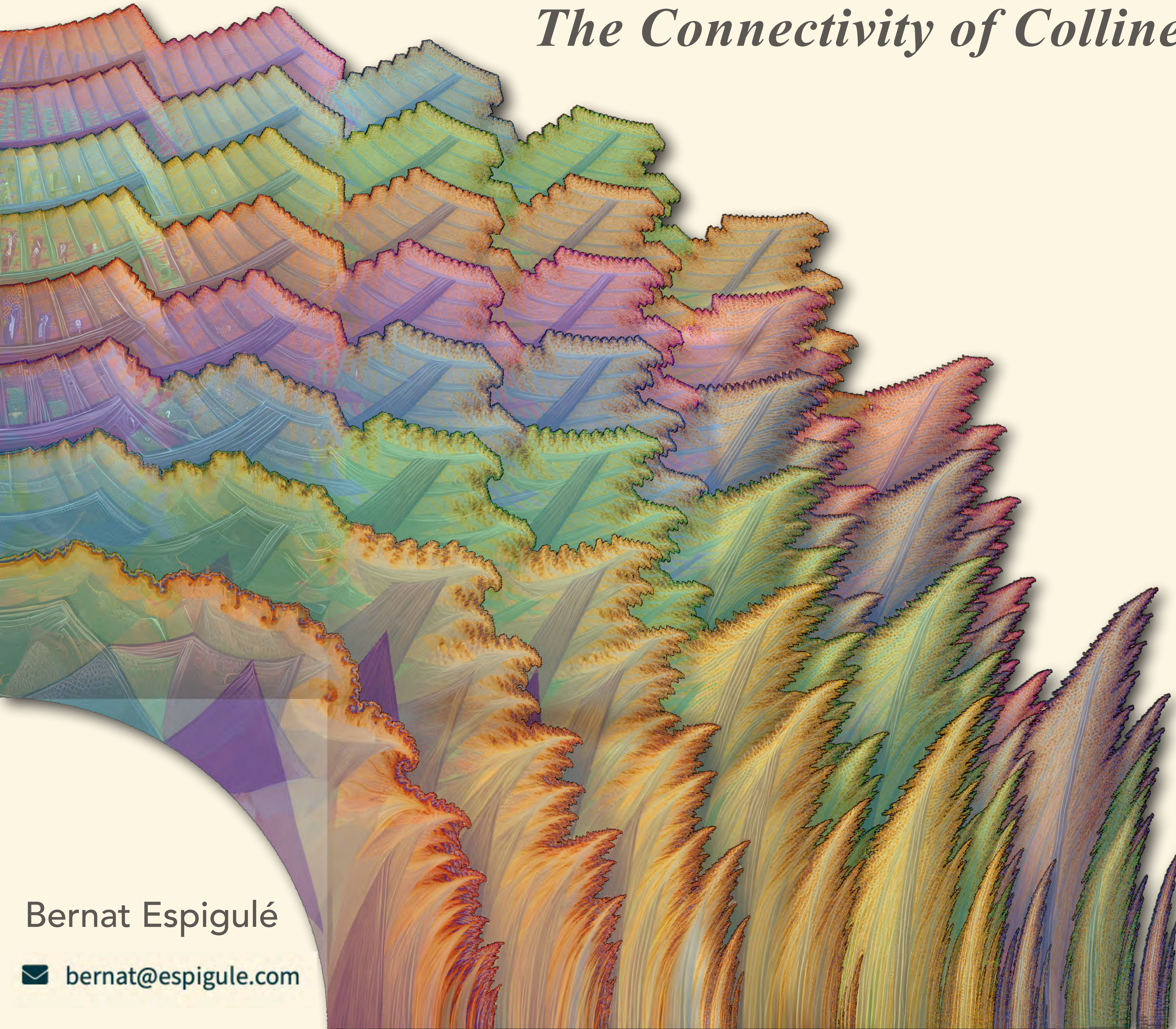
Beyond Roots of Integer Polynomials: The Connectivity of Collinear Fractals



Joint work with my PhD supervisors, Joan Saldaña, and David Juher

Bernat Espigulé

bernat@espigule.com

Universitat de Girona

IFUdG 2022–2024 Santander

Low dimensional dynamical systems: topology, geometry, combinatorics and bifurcations

PID2023-146424NB-I00

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

AGENCIA ESTATAL DE INVESTIGACIÓN

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought.

SHORT STORIES

The Beauty of Roots

John C. Baez, J. Daniel Christensen, and Sam Derbyshire

Figure 1. Roots of all polynomials of degree 23 whose coefficients are ± 1. The brightness shows the number of roots per pixel.

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought. For example, if we plot the roots of all polynomials of degree 23 whose coefficients are all 1 or -1, we get an astounding picture, shown in Figure 1.

More generally, define a **Littlewood polynomial** to be a polynomial $p(z) = \sum_{i=0}^{d} a_i z^i$ with each coefficient a_i equal to 1 or -1. Let $\mathbf{X}_n$ be the set of complex numbers that are roots of some Littlewood polynomial with n nonzero terms (and thus degree $n-1$). The 4-fold symmetry of Figure 1 comes from the fact that if $z \in \mathbf{X}_n$ so are $-z$ and $\overline{z}$. The set $\mathbf{X}_n$ is also invariant under the map $z \mapsto 1/z$, since if z is the root of some Littlewood polynomial then $1/z$ is a root of the polynomial with coefficients listed in the reverse order.

It turns out to be easier to study the set

$$\mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n = \{z \in \mathbb{C} | z \text{ is the root of some Littlewood polynomial}\}.$$

If n divides m then $\mathbf{X}_n \subseteq \mathbf{X}_m$, so $\mathbf{X}_n$ for a highly divisible number n can serve as an approximation to $\mathbf{X}$, and this is why we drew $\mathbf{X}_{24}$.

Some general properties of $\mathbf{X}$ are understood. It is easy to show that $\mathbf{X}$ is contained in the annulus $1/2 < |z| < 2$. On the other hand, Thierry Bousch showed [2] that the closure of $\mathbf{X}$ contains the annulus $2^{-1/4} \leq |z| \leq 2^{1/4}$. This means that the holes near roots of unity visible in the sets $\mathbf{X}_n$ must eventually fill in as we take the union over all n. More surprisingly, Bousch showed in 1993 that the closure $\overline{\mathbf{X}}$ is connected and locally path-connected [3]. It is worth comparing the work of Odlyzko and Poonen [7], who previously showed similar result for roots of polynomials whose coefficients are all 0 or 1.

The big challenge is to understand the diverse, complicated and beautiful patterns that appear in different regions of the set $\mathbf{X}$. There are websites that let you explore and zoom into this set online [4, 5, 8]. Different regions raise different questions.

For example, what is creating the fractal patterns in Figure 2 and elsewhere? An anonymous contributor suggested a fascinating line of attack which was further developed by Greg Egan [5]. Define two functions from the complex plane to itself, depending on a complex parameter q:

$$f_{+q}(z) = 1 + qz, \qquad f_{-q}(z) = 1 - qz.$$

When $|q| < 1$ these are both contraction mappings, so by a theorem of Hutchinson [6] there is a unique nonempty compact set $D_q \subseteq \mathbb{C}$ with

$$D_q = f_{+q}(D_q) \cup f_{-q}(D_q).$$

We call this set a **dragon**, or the **q-dragon** to be specific. And it seems that *for $|q| < 1$, the portion of the set $\mathbf{X}$ in a small neighborhood of the point q tends to look like a rotated version of D_q.*

Figure 3 shows some examples. To precisely describe what is going on, much less prove it, would take real work. We invite the reader to try. A heuristic explanation is known, which can serve as a starting point [1, 5]. Bousch [3] has also proved this related result:

Theorem. *For $q \in \mathbb{C}$ with $|q| < 1$, we have $q \in \overline{\mathbf{X}}$ if and only if $0 \in D_q$. When this holds, the set D_q is connected.*

John C. Baez is a professor of mathematics at UC Riverside. His email address is john.baez@ucr.edu.

J. Daniel Christensen is a professor of mathematics at the University of Western Ontario. His email address is jdc@uwo.ca.

Sam Derbyshire is a Haskell consultant at Well-Typed. His email address is sam@well-typed.com.

DOI: https://doi.org/10.1090/noti2789

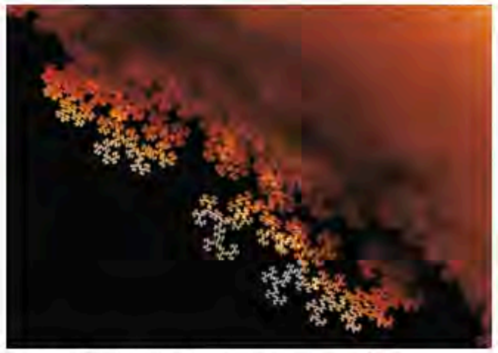


Figure 2. The region of $\mathbf{X}_{24}$ near the point $z = \frac{1}{2}e^{i/8}$.

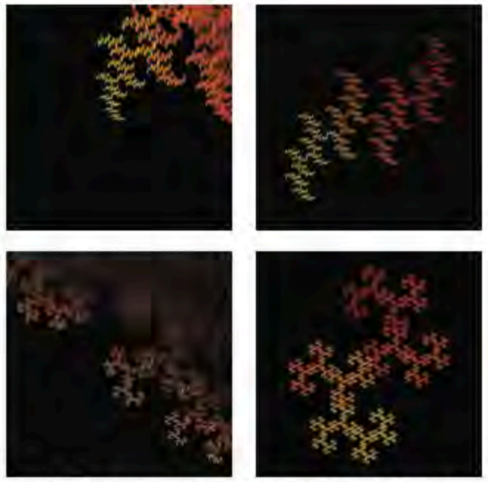


Figure 3. Top: the set $\mathbf{X}$ near $q = 0.594 + 0.254i$ at left, and the set D_q at right. Bottom: the set $\mathbf{X}$ near $q = 0.375453 + 0.544825i$ at left, and the set D_q at right.

References

[1] J. C. Baez, The beauty of roots. Available at http://math.ucr.edu/home/baez/roots.

[2] T. Bousch, Paires de similitudes $Z \to SZ+1, Z \to SZ-1$, January 1988. Available at https://www.imo.universite-paris-saclay.fr/~thierry.bousch/preprints/.

[3] T. Bousch, Connexité locale et par chemins hölderiens pour les systèmes itérés de fonctions, March 1993. Available at https://www.imo.universite-paris-saclay.fr/~thierry.bousch/preprints/.

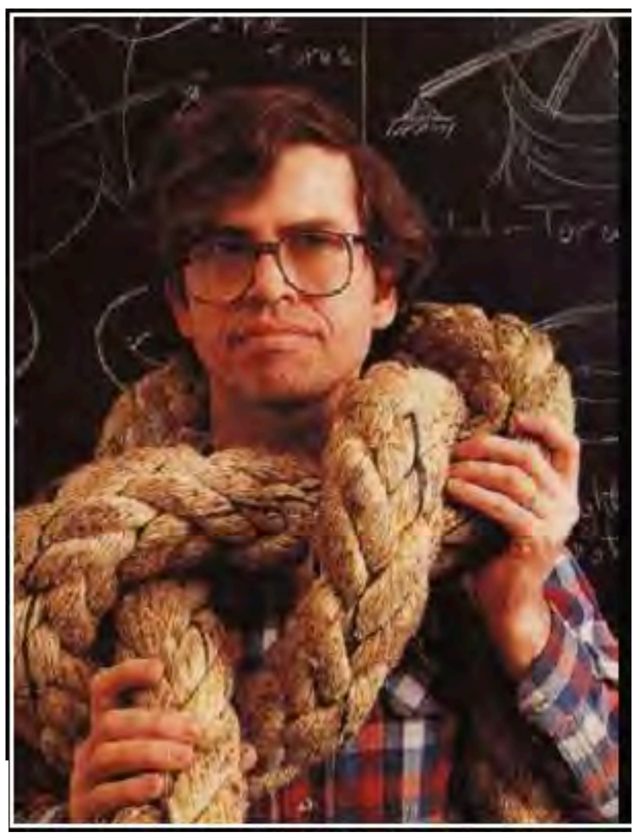
[4] J. D. Christensen, Plots of roots of polynomials with integer coefficients. Available at http://jdc.math.uwo.ca/roots/.

[5] G. Egan, Littlewood applet. Available at http://www.gregegan.net/SCIENCE/Littlewood/Littlewood.html.

[6] J. E. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747. Also available at https://maths-people.anu.edu.au/~john/Assets/Research%20Papers/fractals_self-similarity.pdf.

[7] A. M. Odlyzko and B. Poonen, Zeros of polynomials with 0,1 coefficients, *L'Enseignement Math.* 39 (1993), 317–348. Also available at http://dx.doi.org/10.5169/seals-60430.

[8] R. Vanderbei, Roots of functions $F(z) = \sum_{j=0}^{n} \alpha_j f_j(z)$ where $\alpha_j \in \{-1, 1\}$. Available at https://vanderbei.princeton.edu/WebGL/roots_PlusMinusOne.html.



Bill Thurston

(1946 - 2012)

mathoverflow

Questions Tags Users Badges Una

Perron number distribution

60

18

A *Perron number* is a real algebraic integer λ that is larger than the absolute value of any of its Galois conjugates. The Perron-Frobenius theorem says that any non-negative integer matrix M such that some power of M is strictly positive has a unique positive eigenvector whose eigenvalue is a Perron number. Doug Lind proved the converse: given a Perron number λ, there exists such a matrix, perhaps in dimension much higher than the degree of λ. Perron numbers come up frequently in many places, especially in dynamical systems.

My question:

> What is the limiting distribution of Galois conjugates of Perron numbers λ in some bounded interval, as the degree goes to infinity?

share cite improve this question

edited Jan 11 '11 at 4:04

asked Jan 11 '11 at 3:50

Bill Thurston

21.4k • 11 • 82 • 113

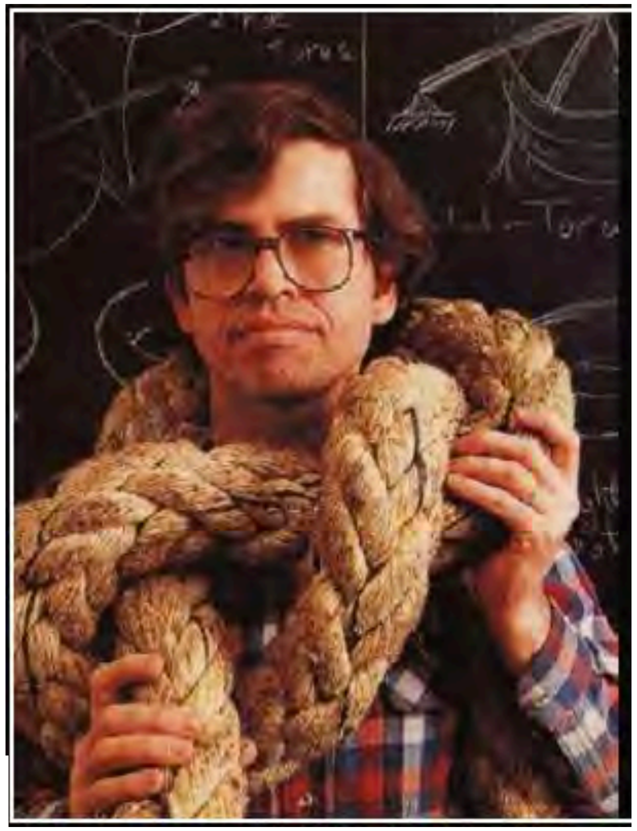
2 Should be related to the distribution at math.ucr.edu/home/baez/roots ; there are references at that link, I think. – Qiaochu Yuan Jan 11 '11 at 4:20

2 @Qiaochu Yuan: Thanks for bringing it up. I actually intended to check out and point to those references, until my question got too long. I was trying to take a slice of things in a way that eliminates the fractal distribution of roots of polynomials with bounded coefficients. My motivation for this question originated in trying understand topological entropy for postcritically finite iterated polynomials, where a Mandelbrot-like distribution comes up that is very related to those exhibited by Baez (and others). – Bill Thurston Jan 11 '11 at 4:41

Thurston's last paper

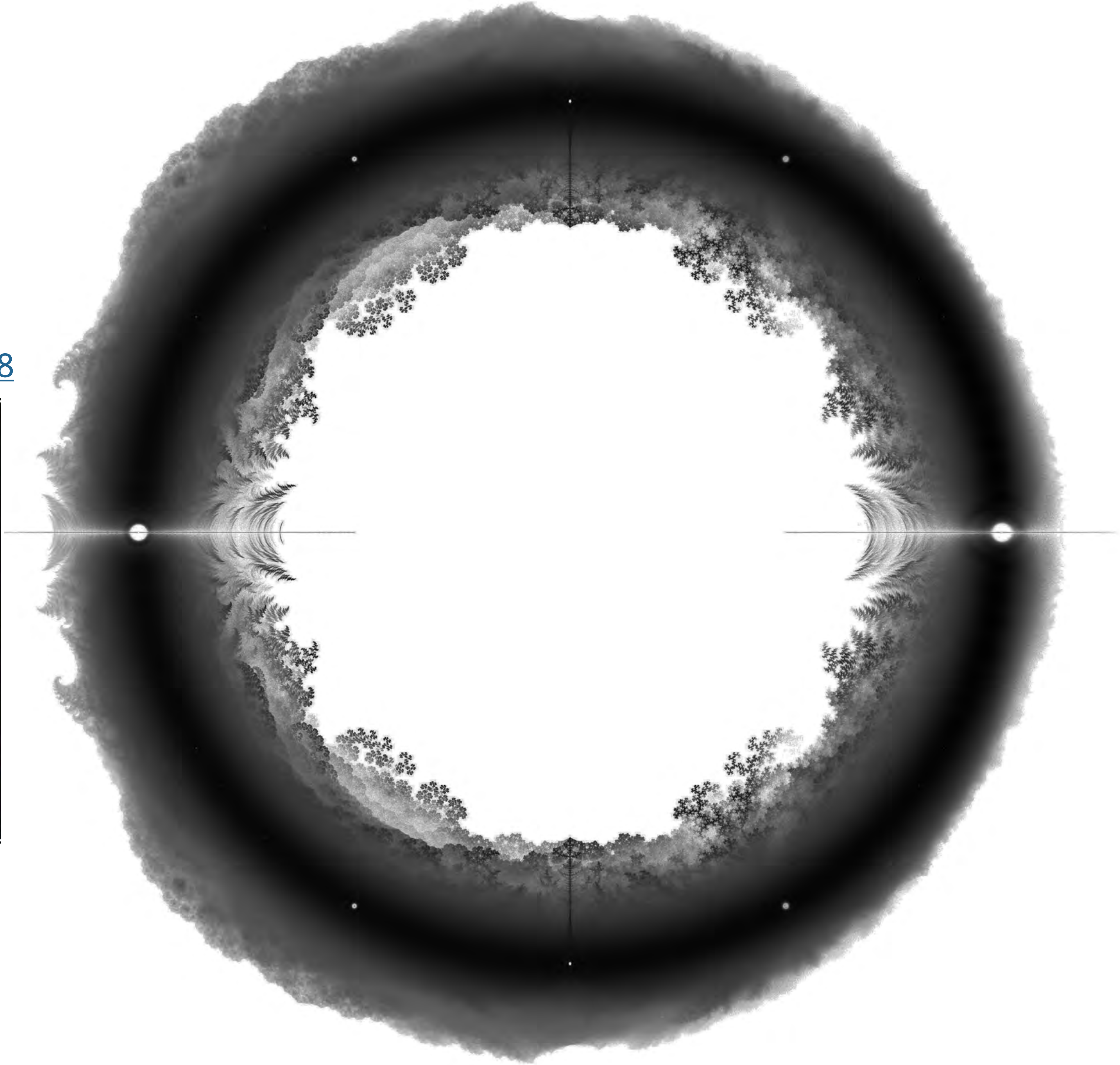
ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



Bill Thurston

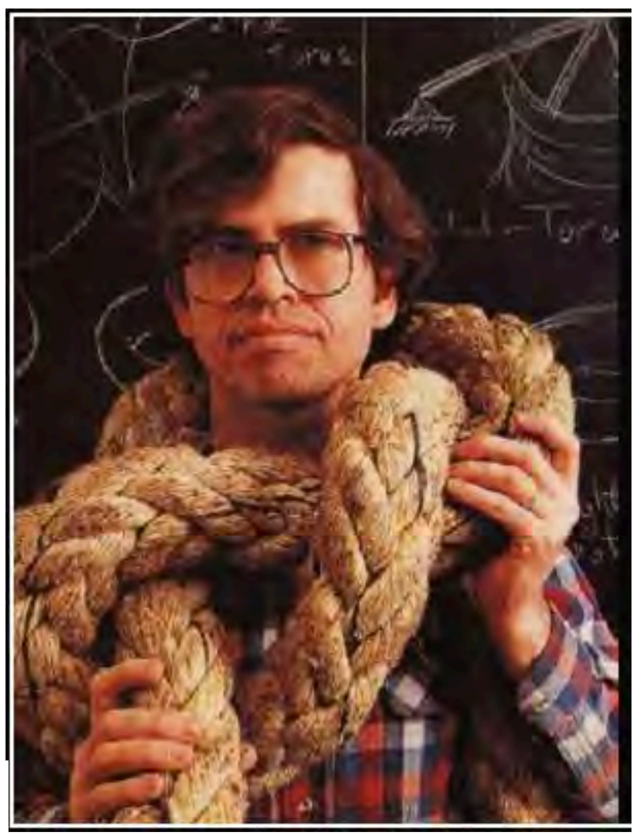
(1946 - 2012)



Thurston's last paper

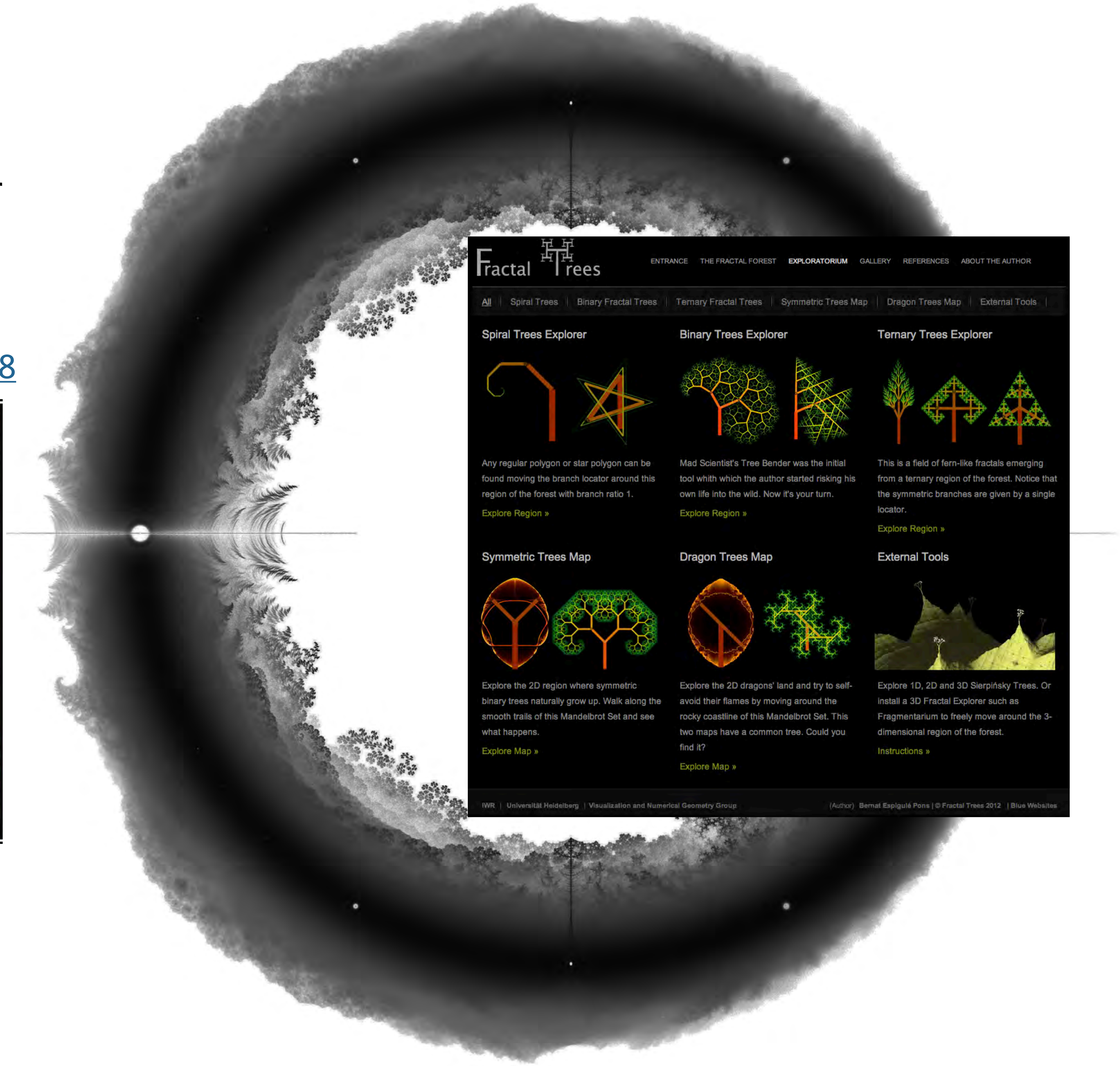
ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



Bill Thurston

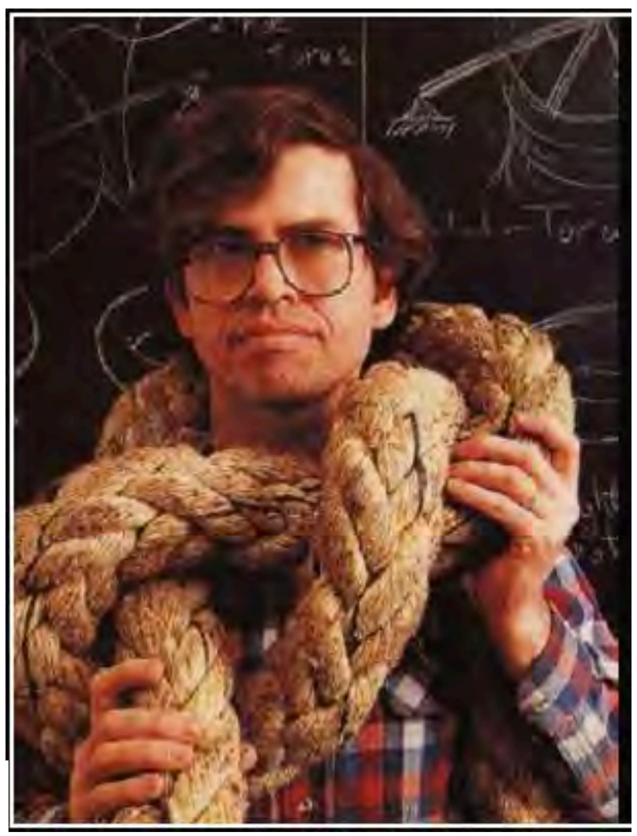
(1946 - 2012)



Thurston's last paper

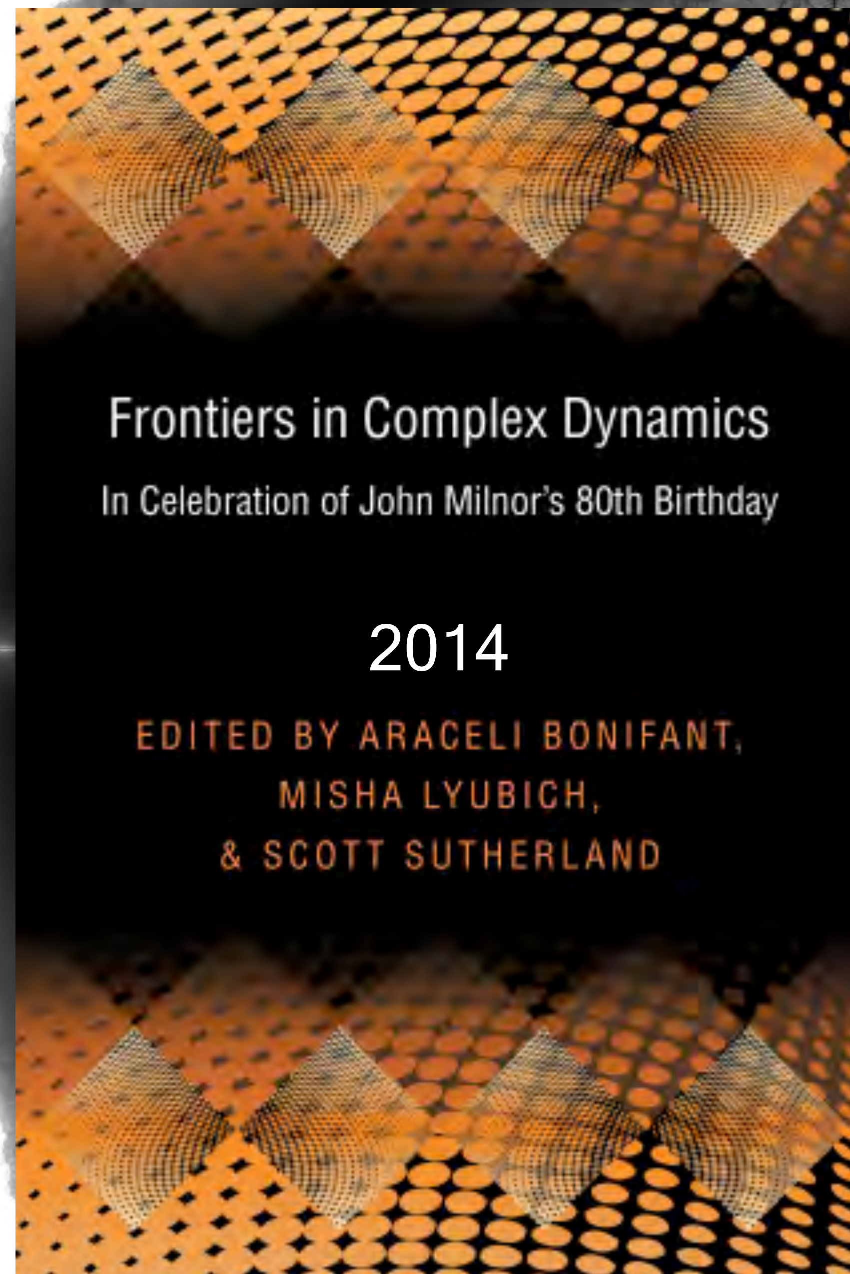
ENTROPY IN DIMENSION ONE

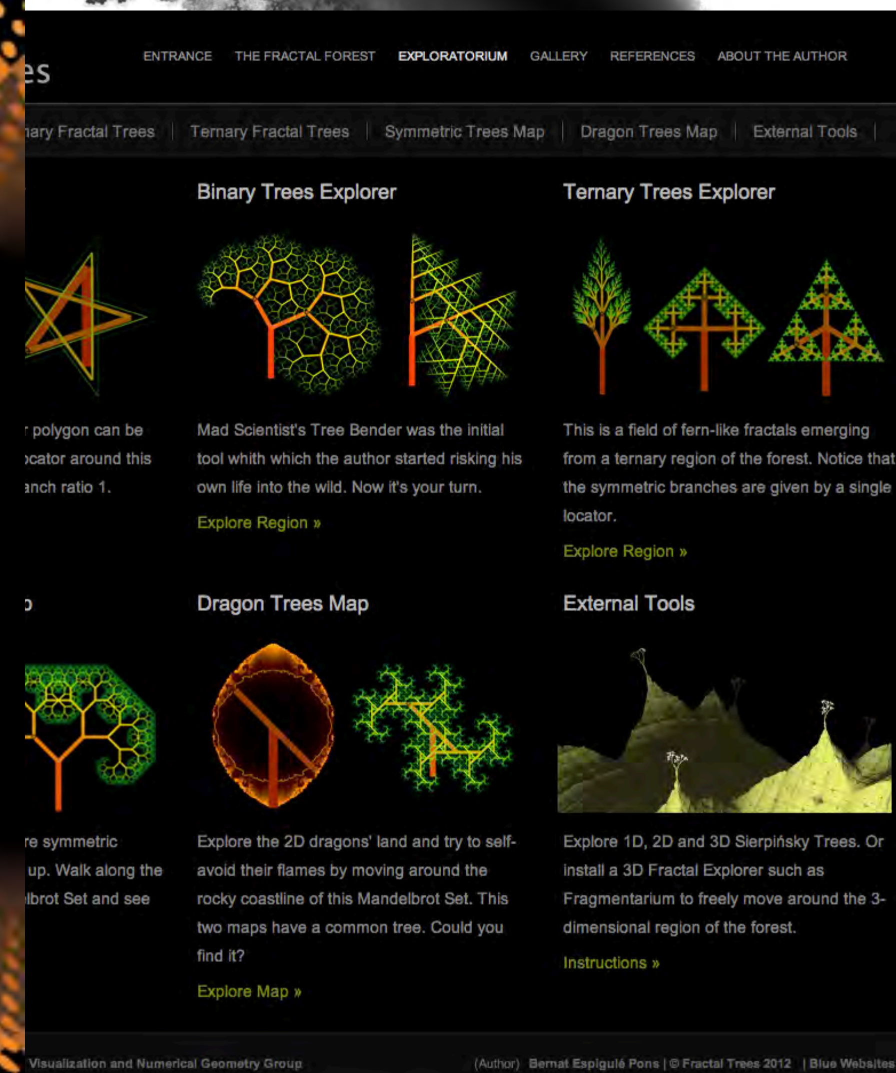
arxiv.org/abs/1402.2008



Bill Thurston

(1946 - 2012)

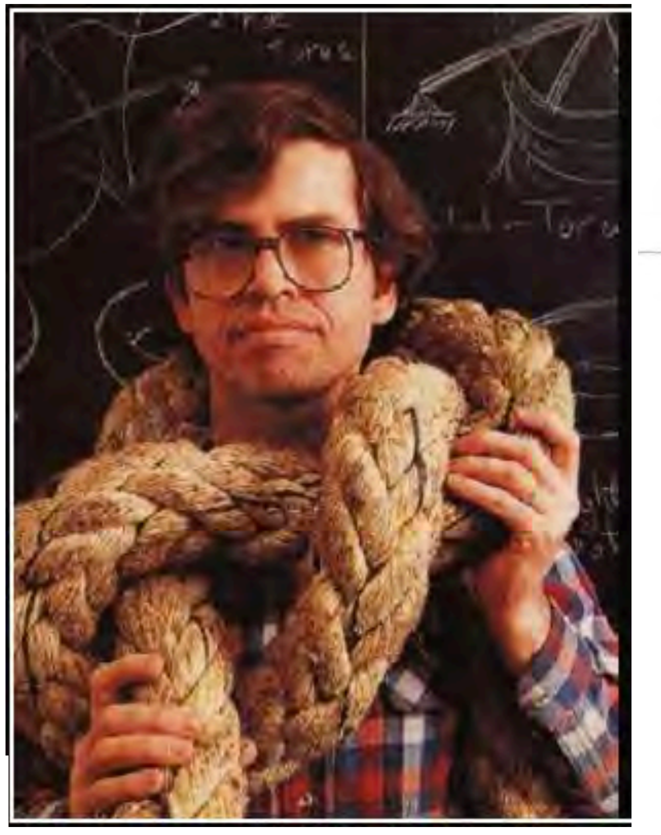




Thurston’s last paper

ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



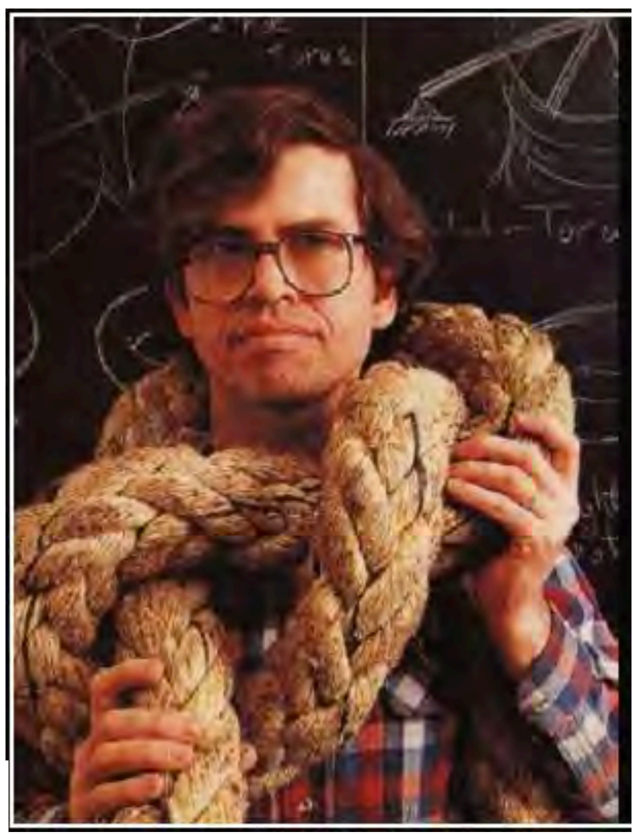
Bill Thurston

(1946 - 2012)

Thurston's last paper

ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



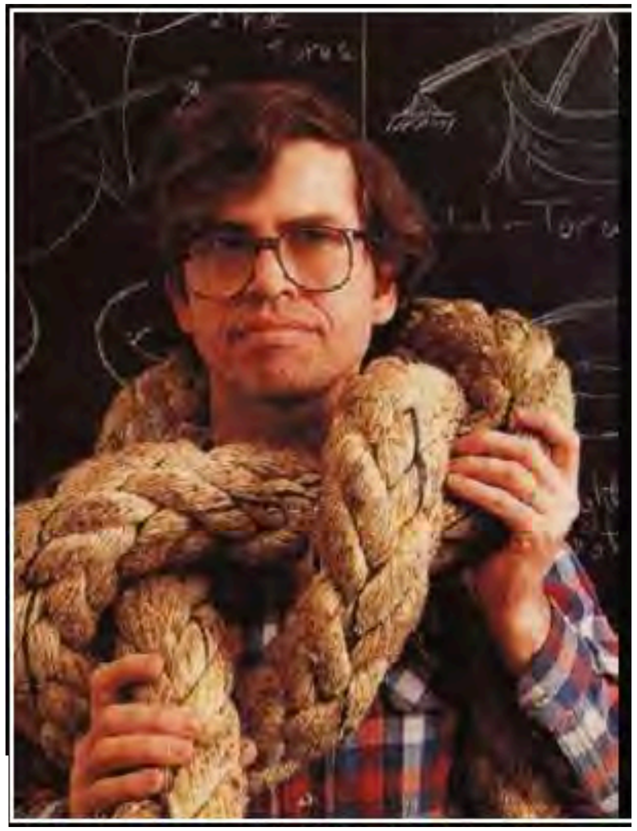
Bill Thurston

(1946 - 2012)

Thurston's last paper

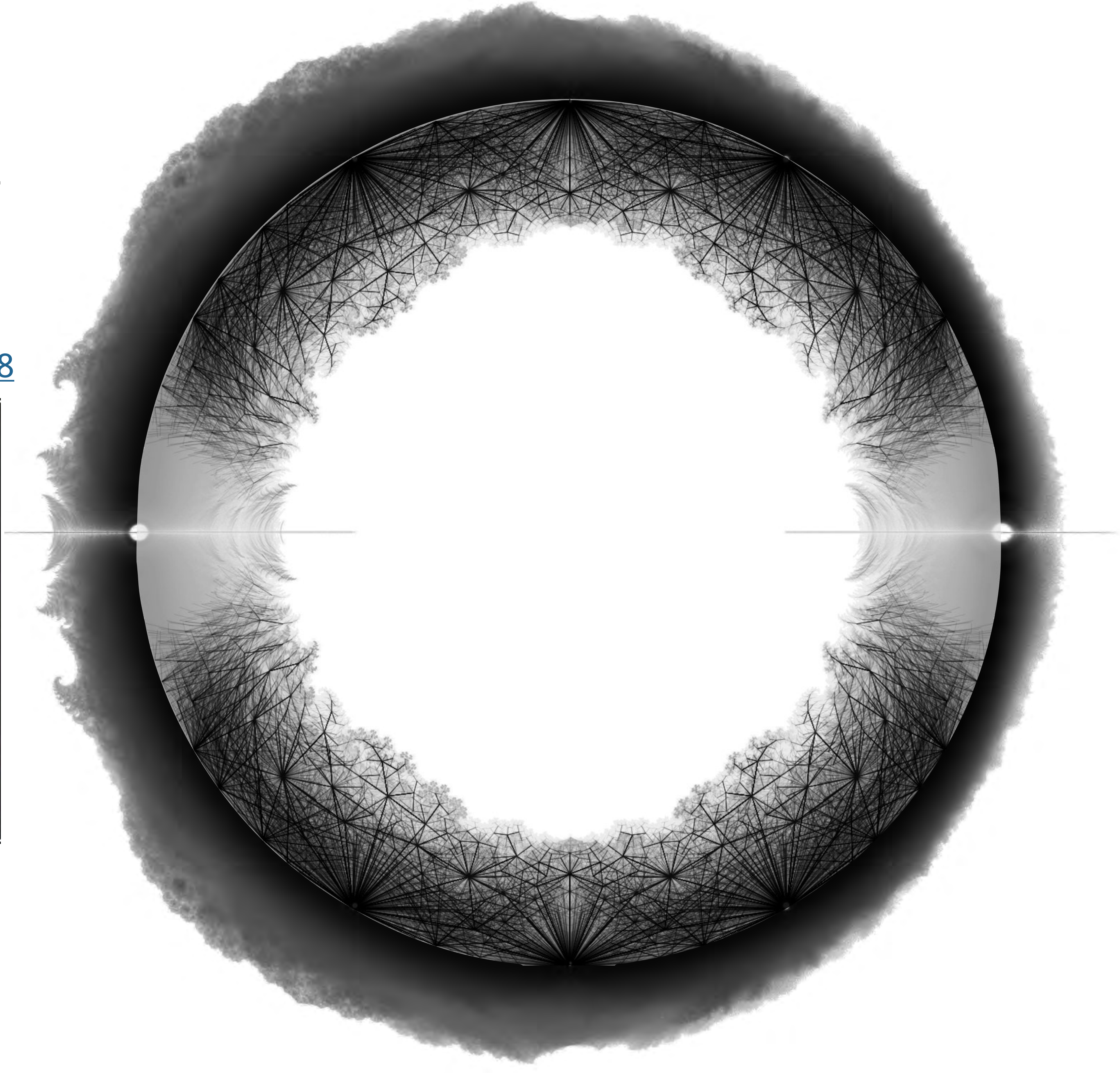
ENTROPY IN DIMENSION ONE

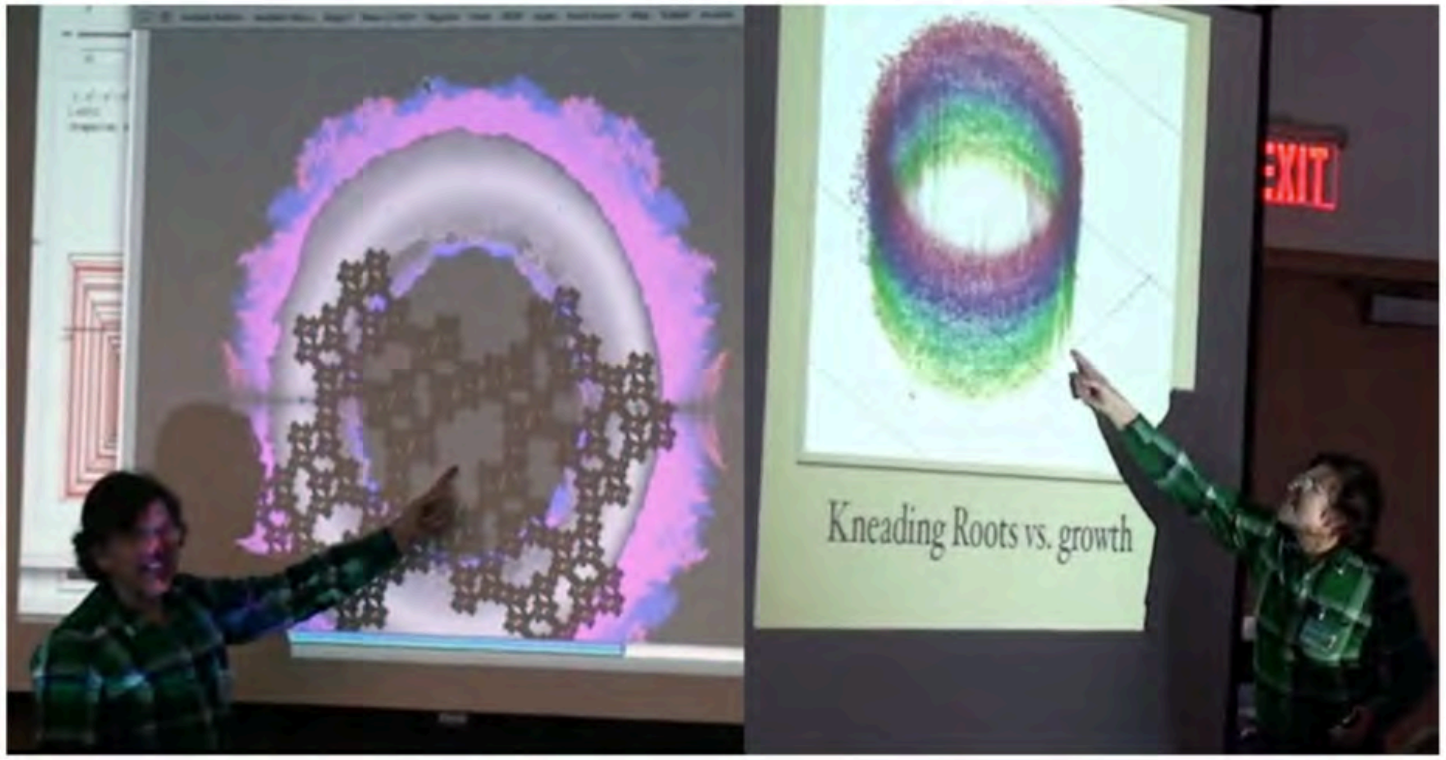
arxiv.org/abs/1402.2008



Bill Thurston

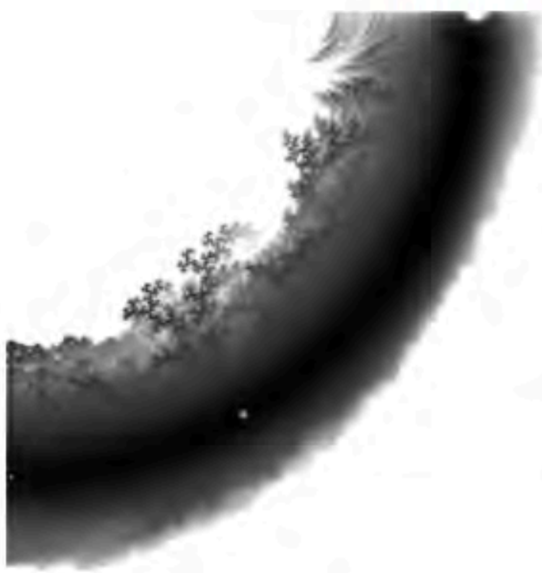
(1946 - 2012)

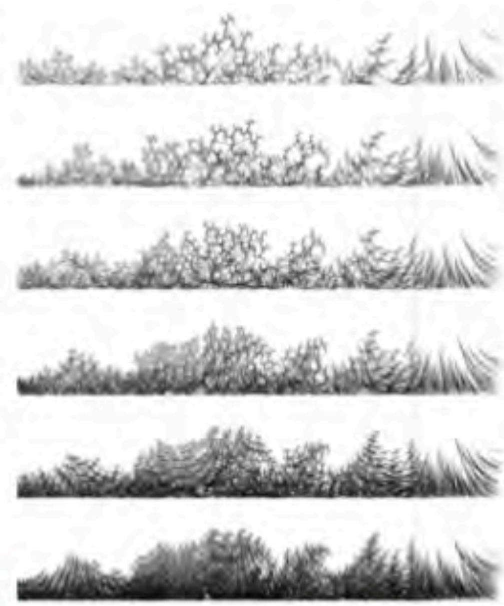


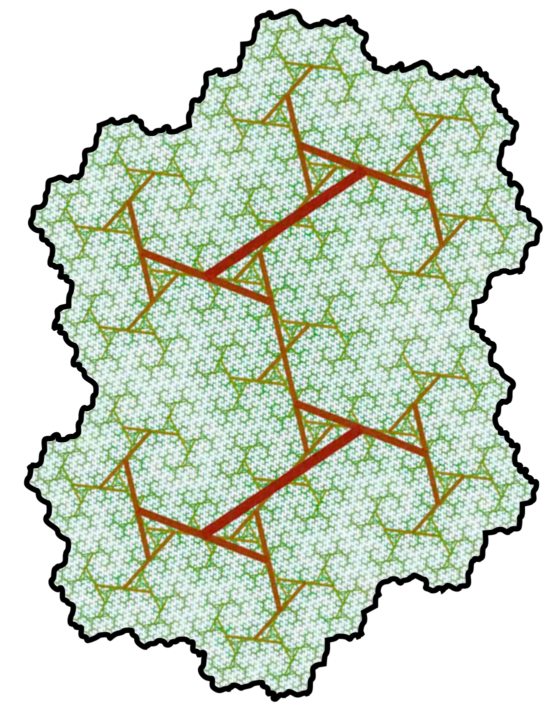


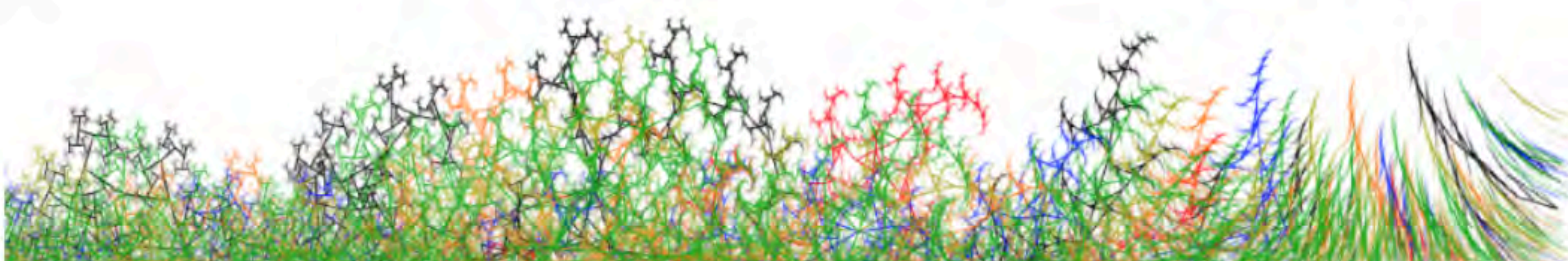
Bill Thurston presenting *M*, *M*_0, and the Thurston Set, Jackfest 2011

[Structure of Entropy: The hidden dimensions]

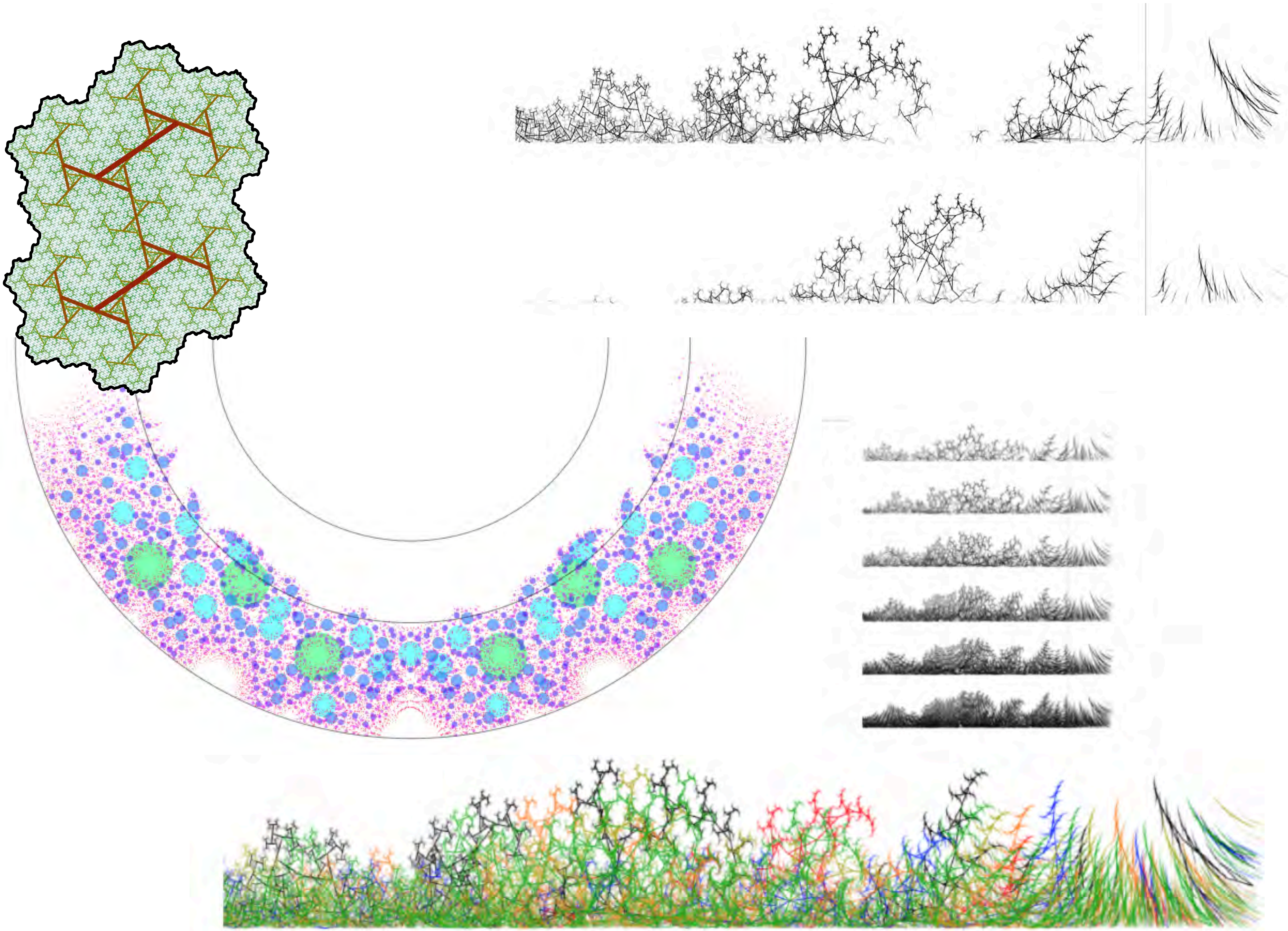




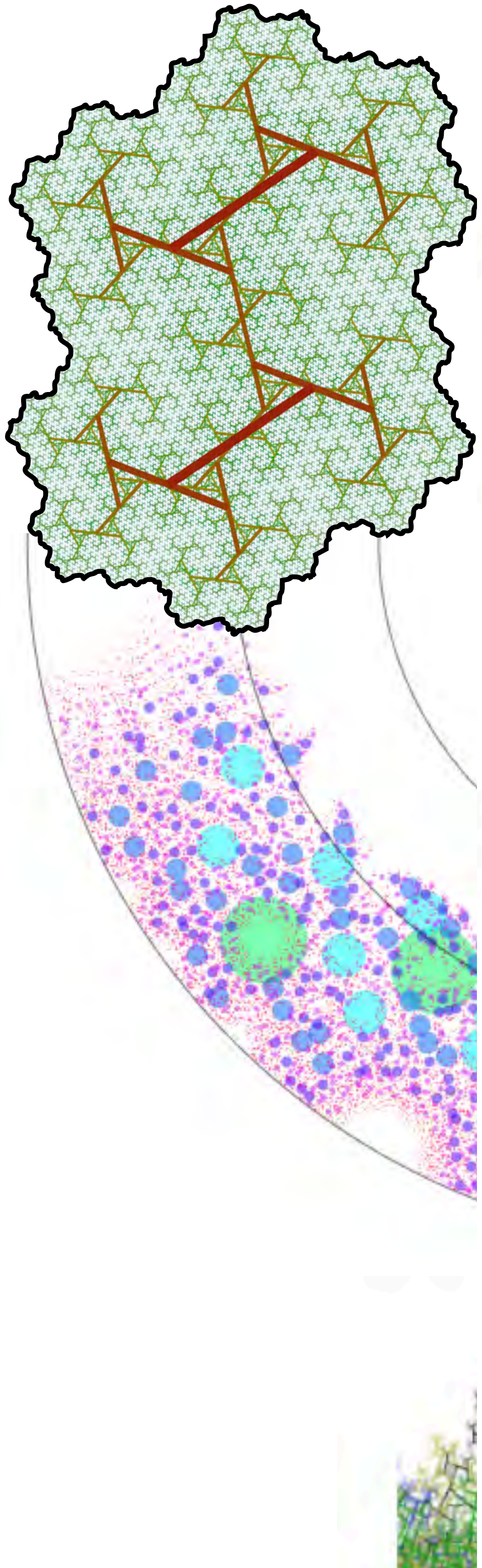




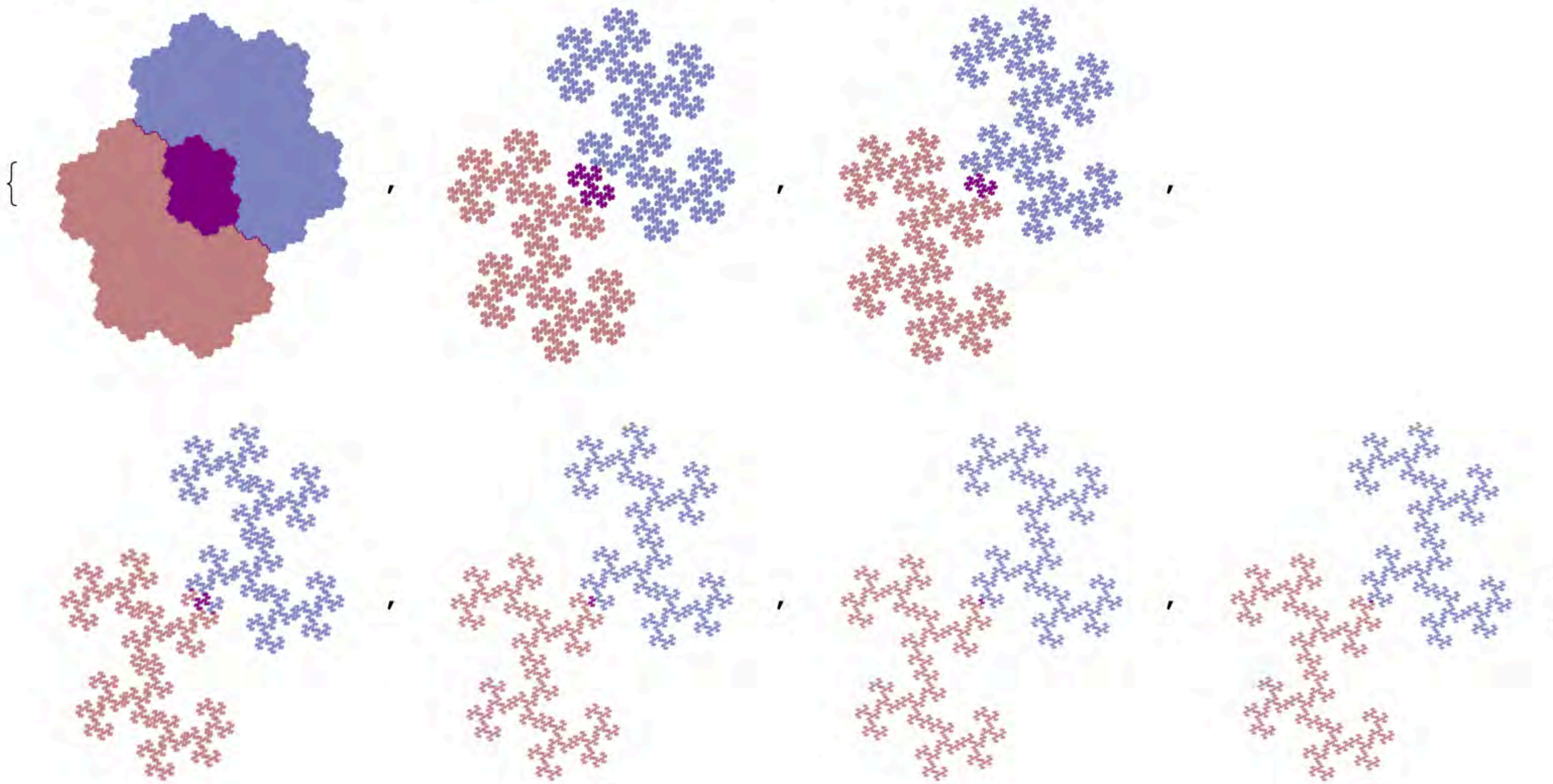
Internal structure of *M* illustrated by complex trees' piece-to-piece connectivity roots. 2018



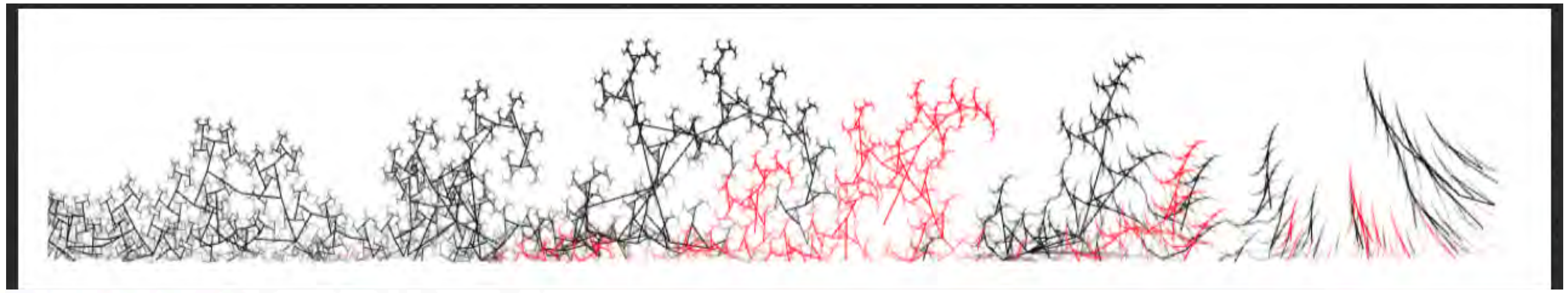
Internal structure of *M* illustrated by complex trees' piece-to-piece connectivity roots. 2018

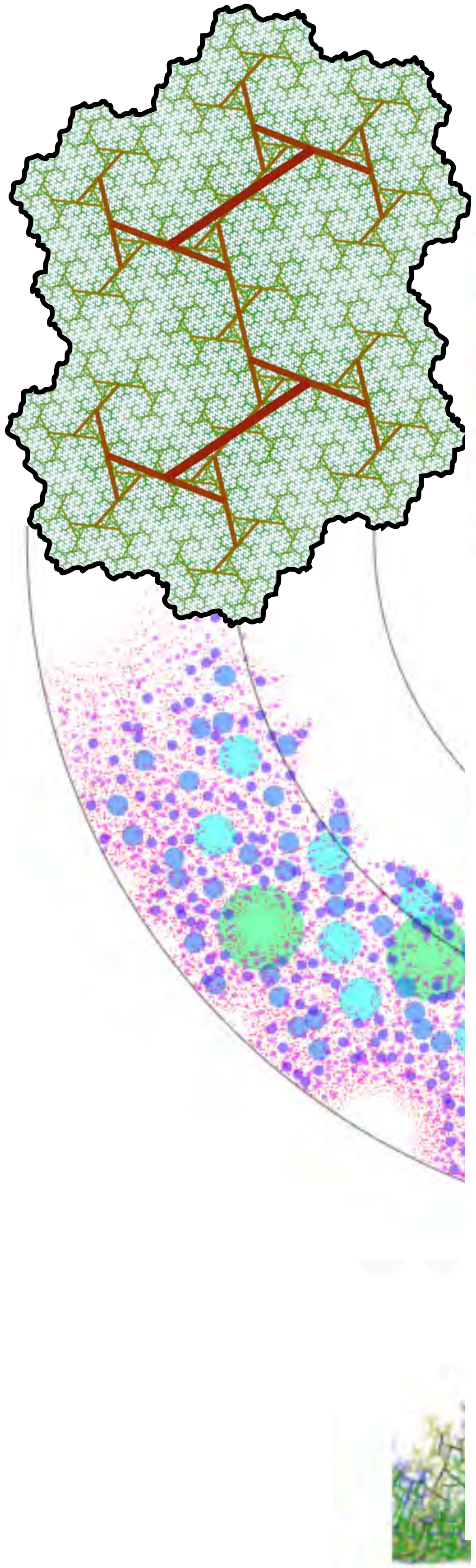


```
{aF[Root[1 - #1 + #1^2 + #1^3 &, 3] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 &, 4] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 + #1^5 &, 5] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 + #1^5 - #1^6 &, 6] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 + #1^5 - #1^6 - #1^7 &, 7] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 + #1^5 - #1^6 - #1^7 - #1^8 &, 8] // N, 500],
 aF[Root[1 - #1 + #1^2 + #1^3 + #1^4 + #1^5 - #1^6 - #1^7 - #1^8 - #1^9 &, 9] // N, 500]}
```

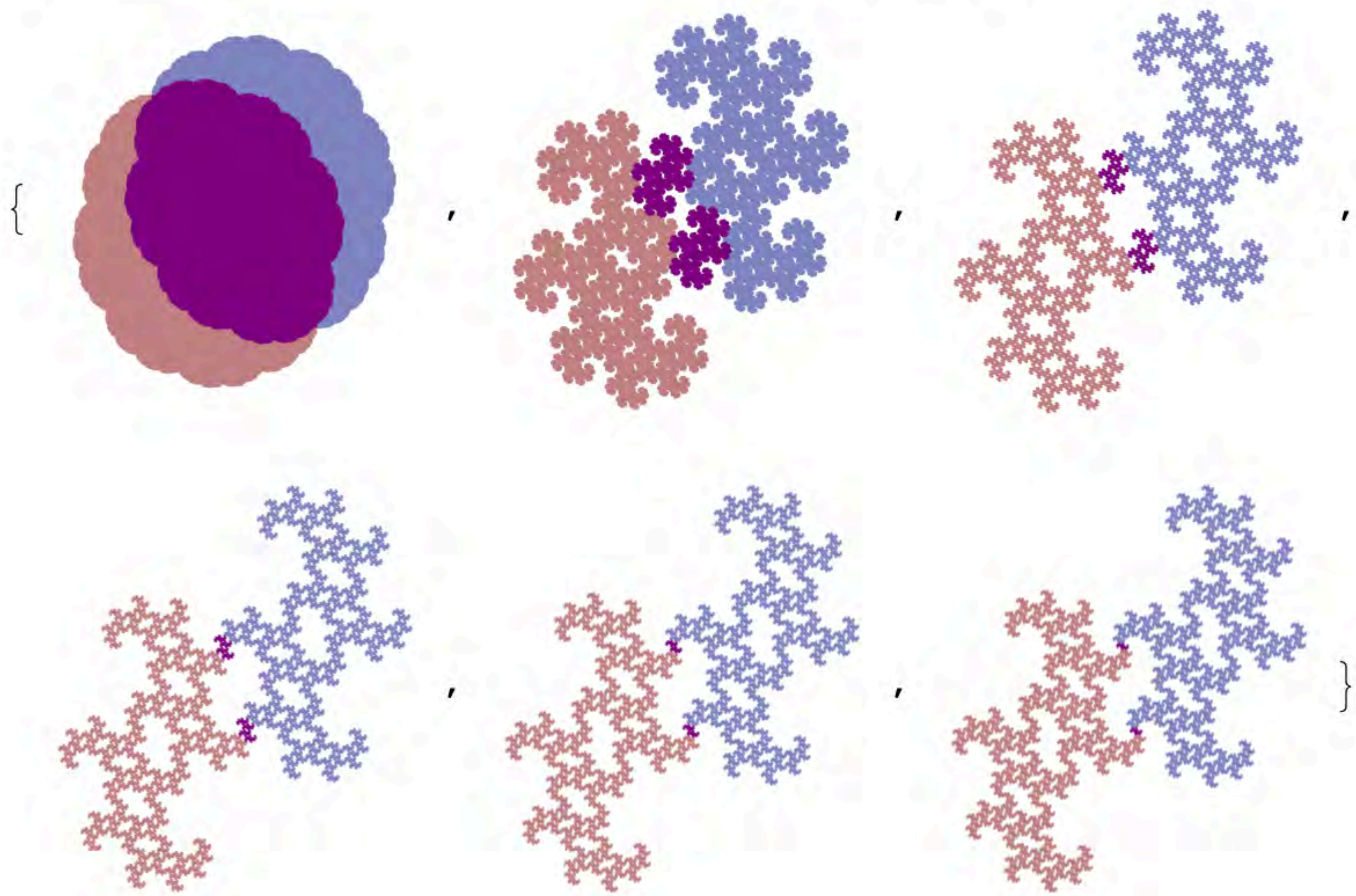


Internal structure of *M* illustrated by complex trees' piece-to-piece connectivity roots. 2018

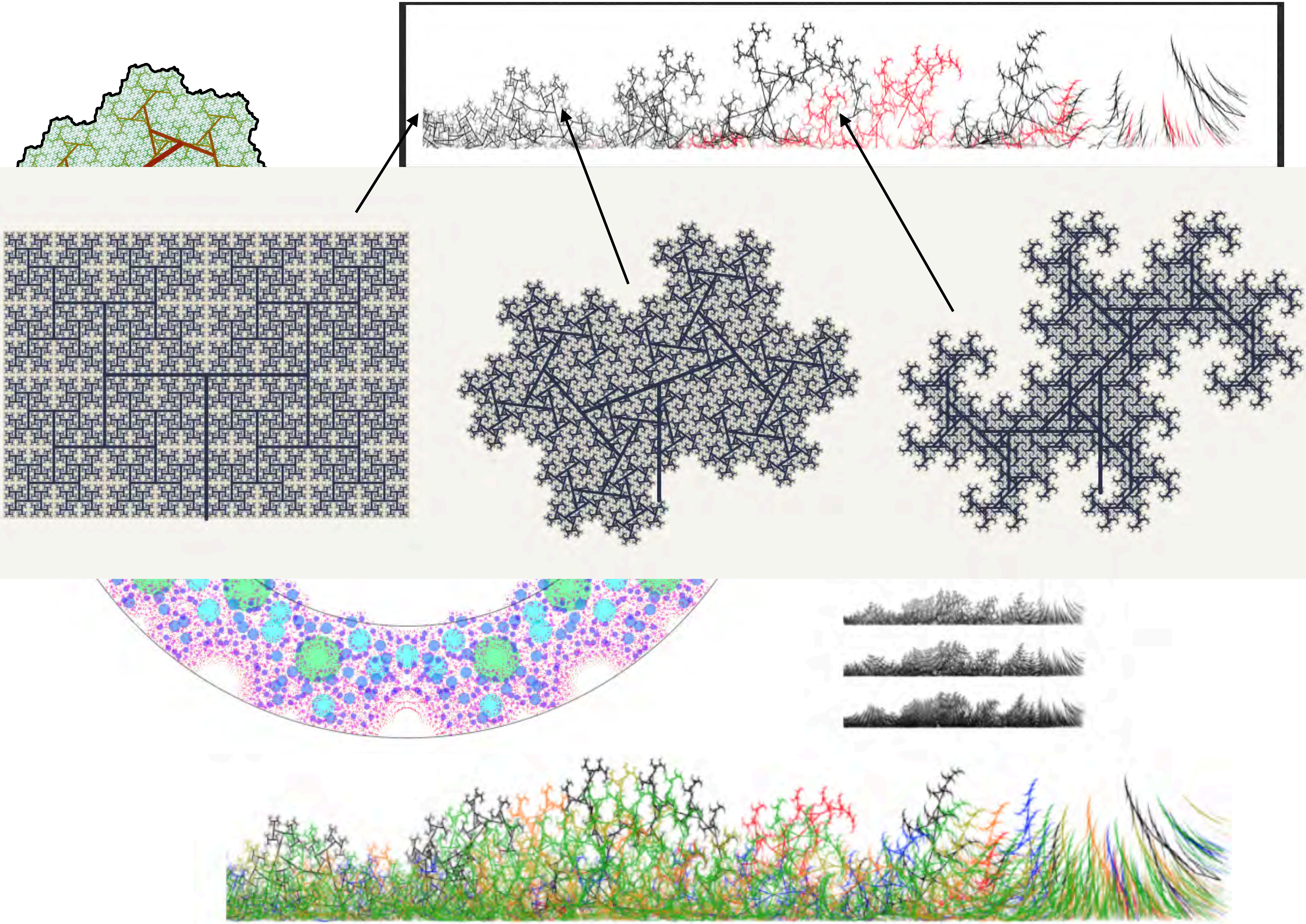




```
{aF[Root[1 - #1 + #1^2 + #1^3 &, 3] // N, 500],
{aF[Root[1 - #1 + #1^3 &, 3] // N, 500],
aF[Root[1 - #1 + #1^3 + #1^4 &, 4] // N, 500],
aF[Root[1 - #1 + #1^3 + #1^4 + #1^5 &, 5] // N, 500],
aF[Root[1 - #1 + #1^3 + #1^4 + #1^5 + #1^6 &, 6] // N, 500],
aF[Root[1 - #1 + #1^3 + #1^4 + #1^5 + #1^6 + #1^7 &, 7] // N, 500],
aF[Root[1 - #1 + #1^3 + #1^4 + #1^5 + #1^6 + #1^7 + #1^8 &, 8] // N, 500]}
```



Internal structure of *M* illustrated by complex trees' piece-to-piece connectivity roots. 2018



Internal structure of *M* illustrated by complex trees' piece-to-piece connectivity roots. 2018

$$p(x) = 1 + \sum_{k=1}^{m} \pm x^k = 1 \pm x \pm x^2 \pm \ldots \pm x^m$$

$1 + x$

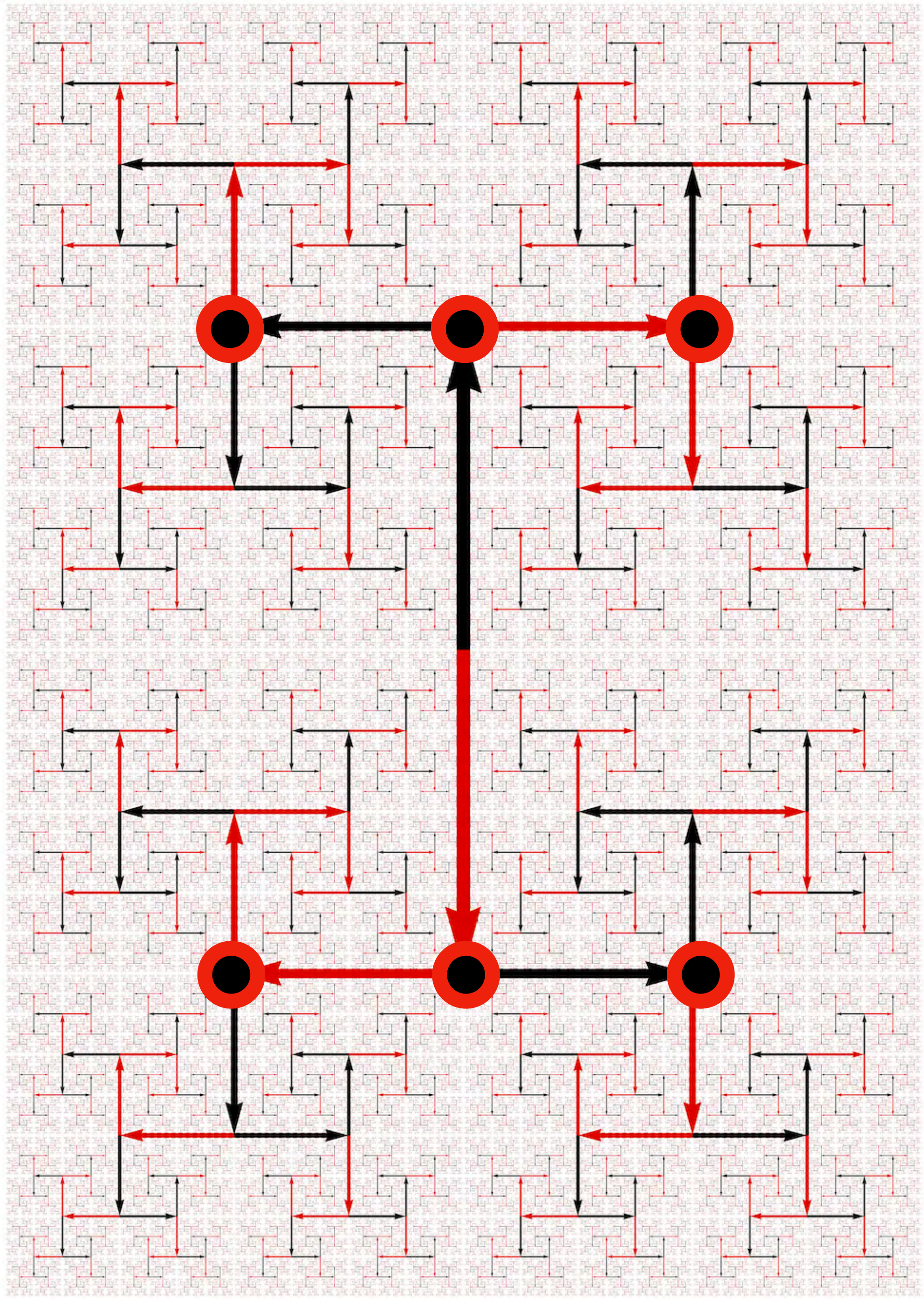
$1 - x$

$1 + x + x^2$

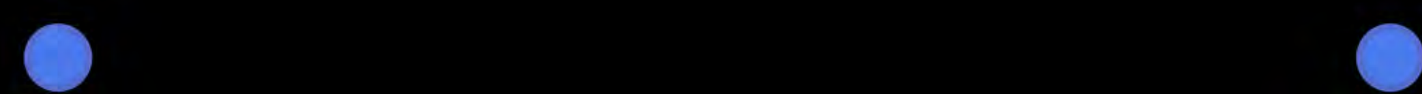
$1 + x - x^2$

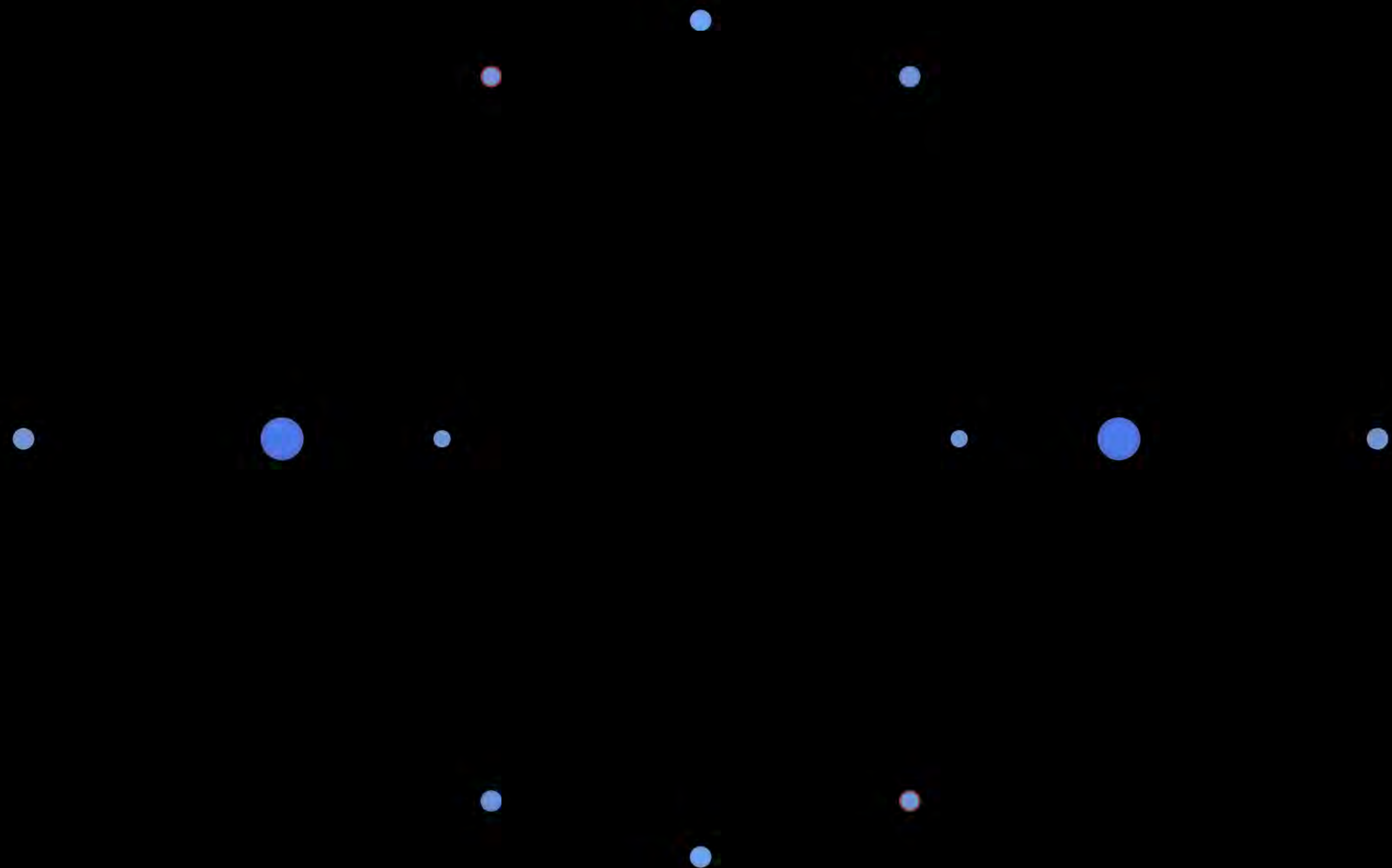
$1 - x + x^2$

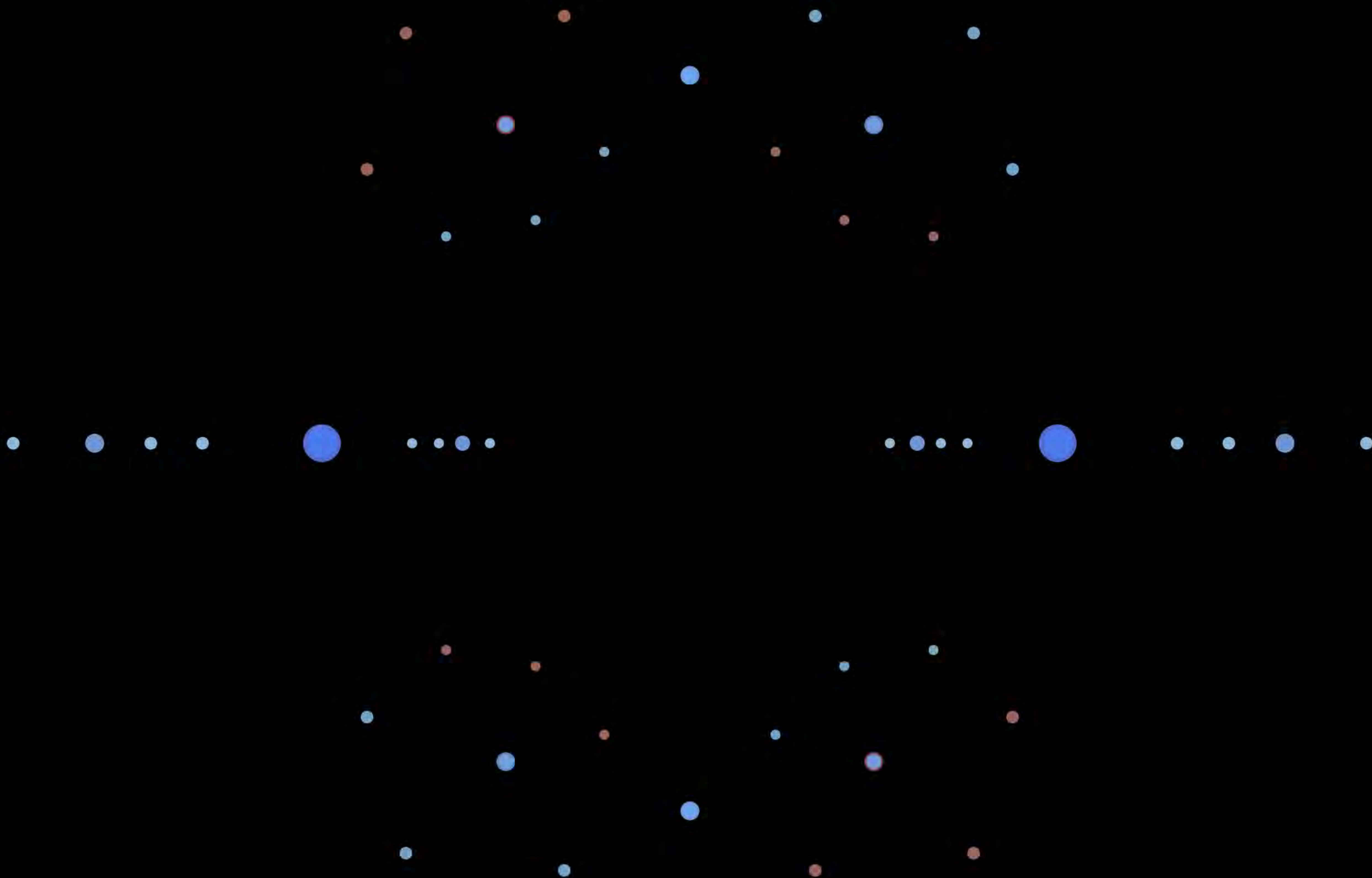
$1 - x - x^2$

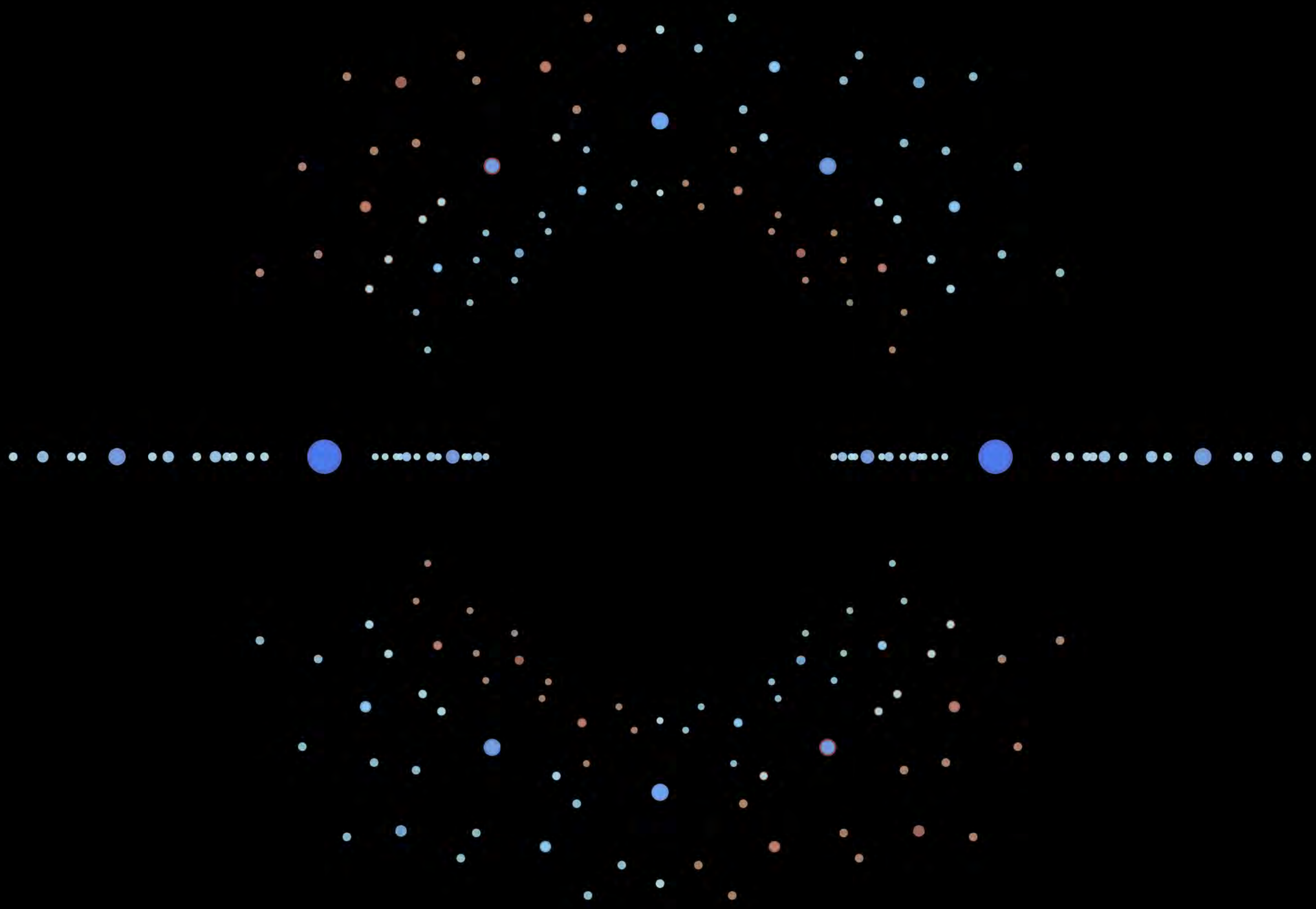


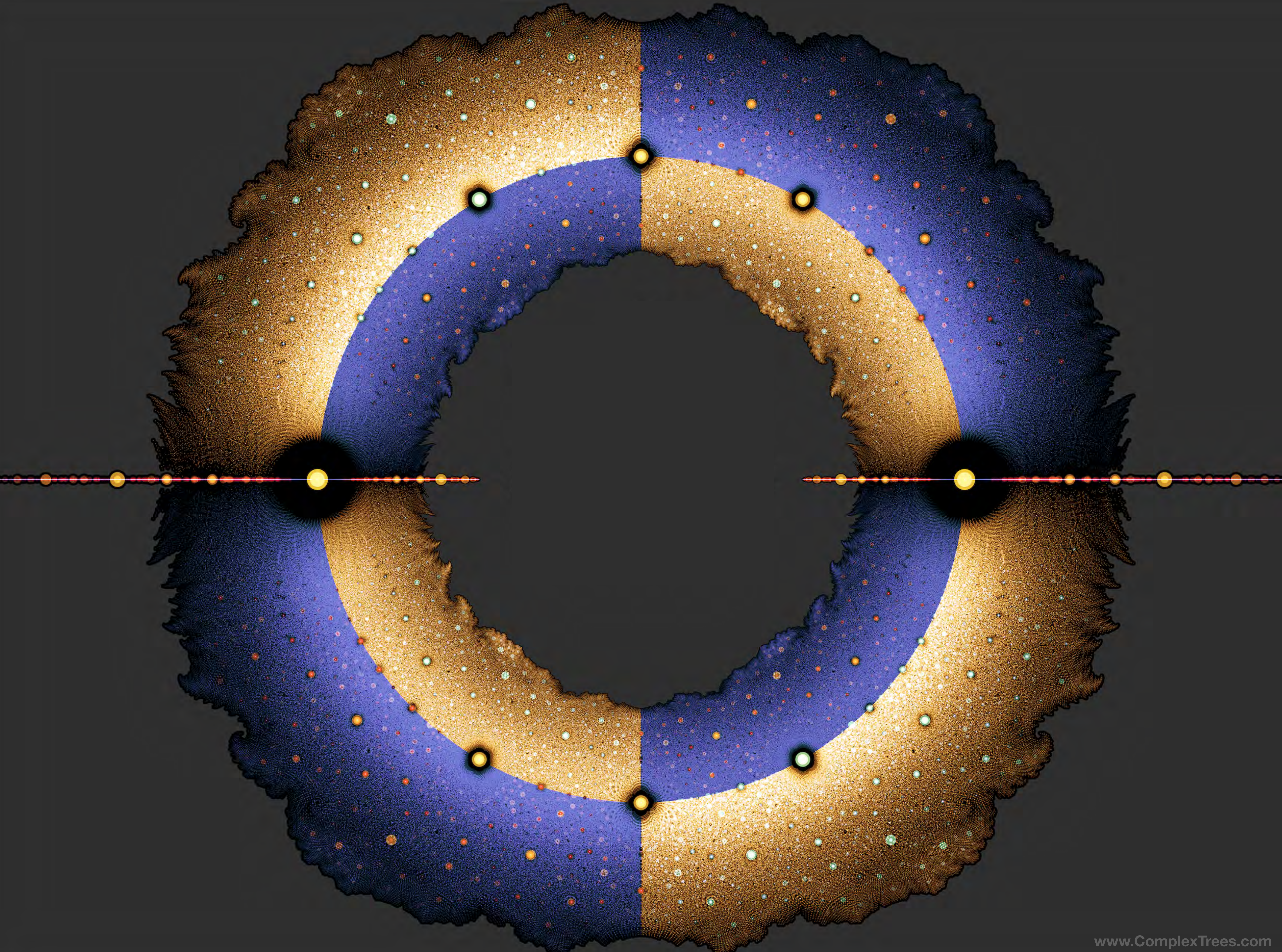
$x = i/\sqrt{2}$

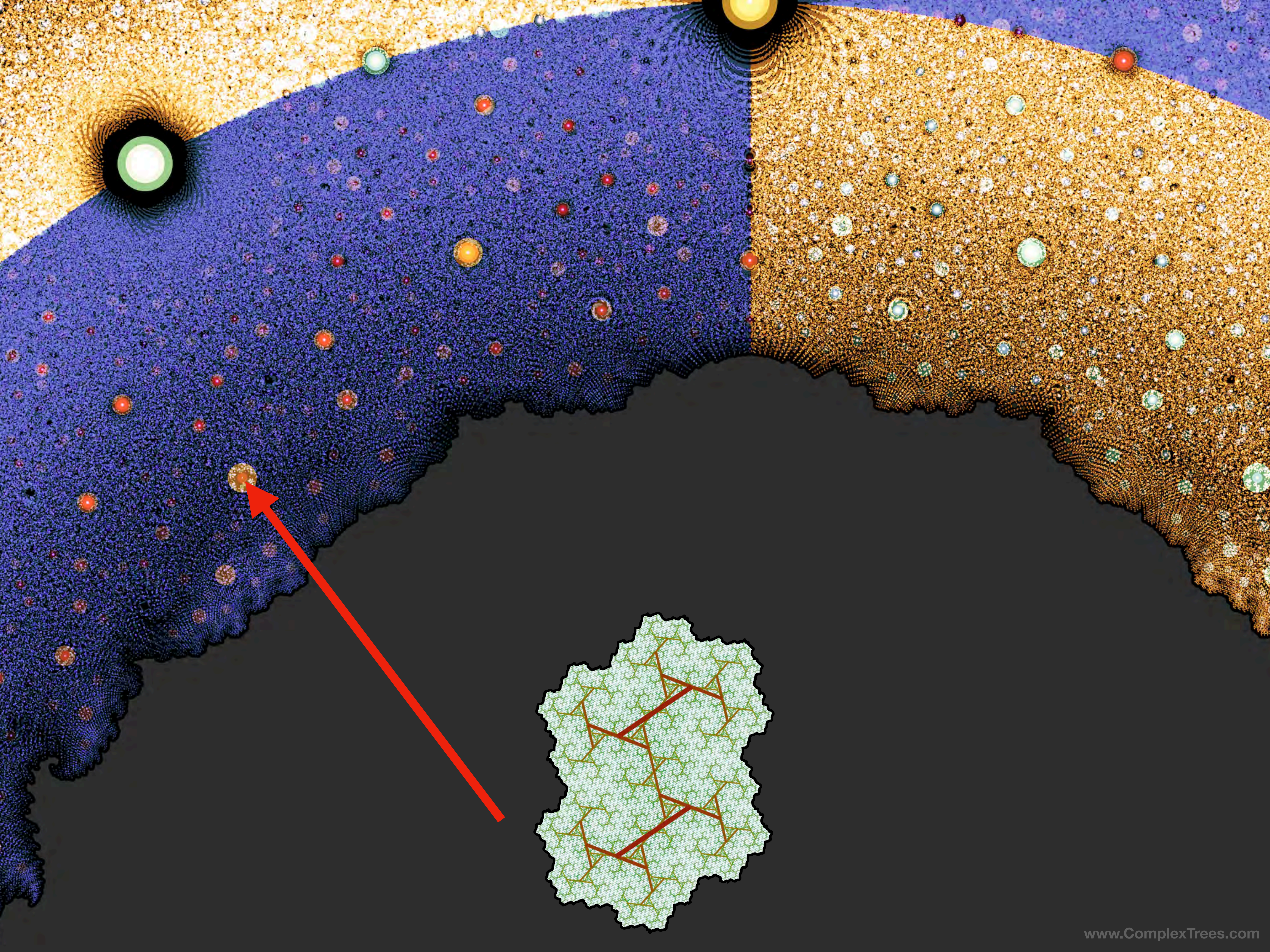


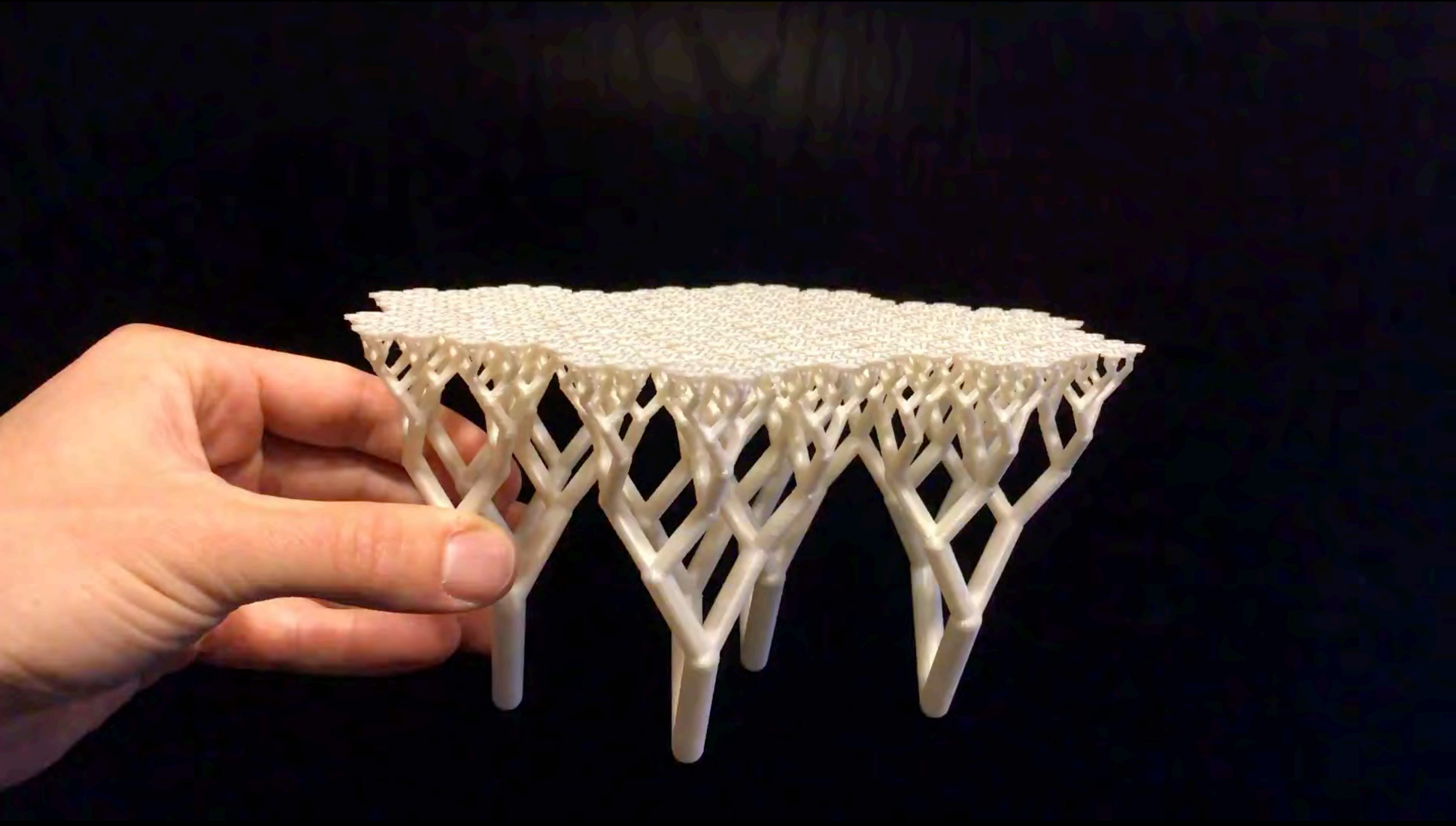









www.ComplexTrees.com



Definition (collinear digit set)

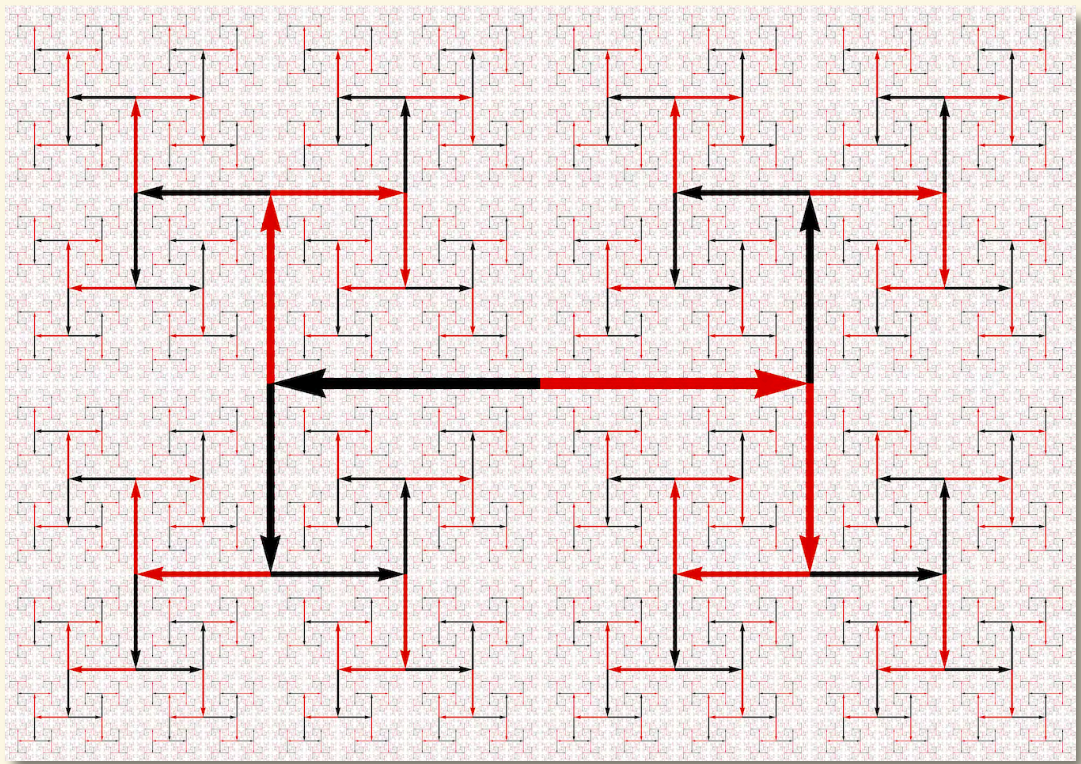
Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

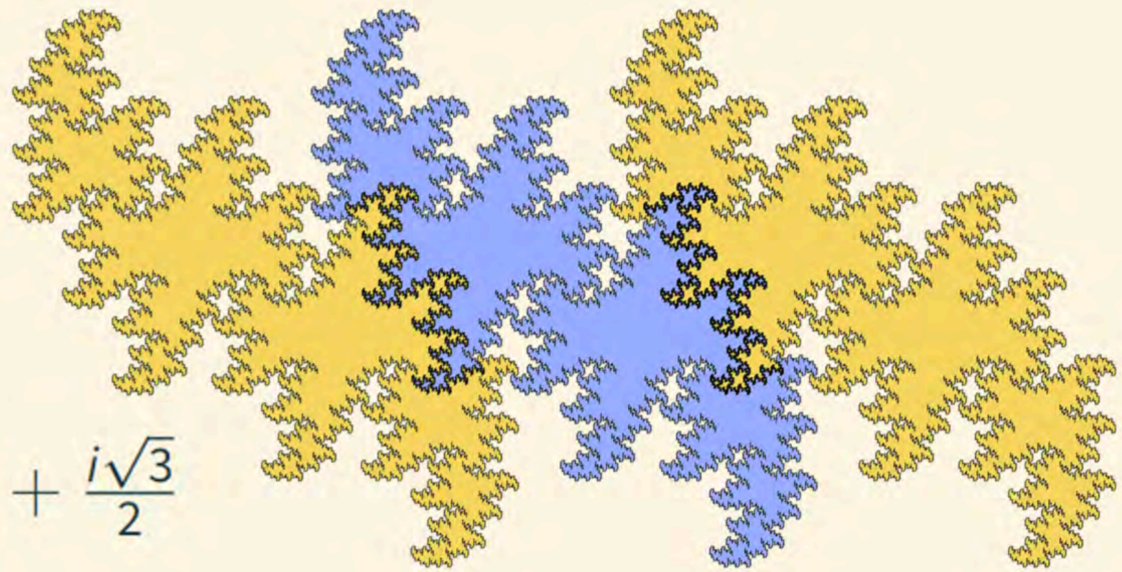
Definition (collinear fractal)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,
$\boldsymbol{E}(c,n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \mathcal{A}_n \right\}$.

$$p^{+}(c) = +1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$

$$p^{-}(c) = -1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$





$c = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

Example ($n = 3$)

$$\boldsymbol{E}(c,3) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \{-2,0,2\} \right\}$$
$$= \left(\frac{\boldsymbol{E}(c,3)}{c} - 2 \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} + 2 \right).$$

Joint work with my PhD supervisors, Joan Saldaña, and David Juher

Universitat de Girona

IFUdG 2022–2024 Santander

Joint work with my PhD supervisors,
Joan Saldaña, and
David Juher

Universitat de Girona

IFUdG 2022–2024 Santander

Definition (collinear digit set)

Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

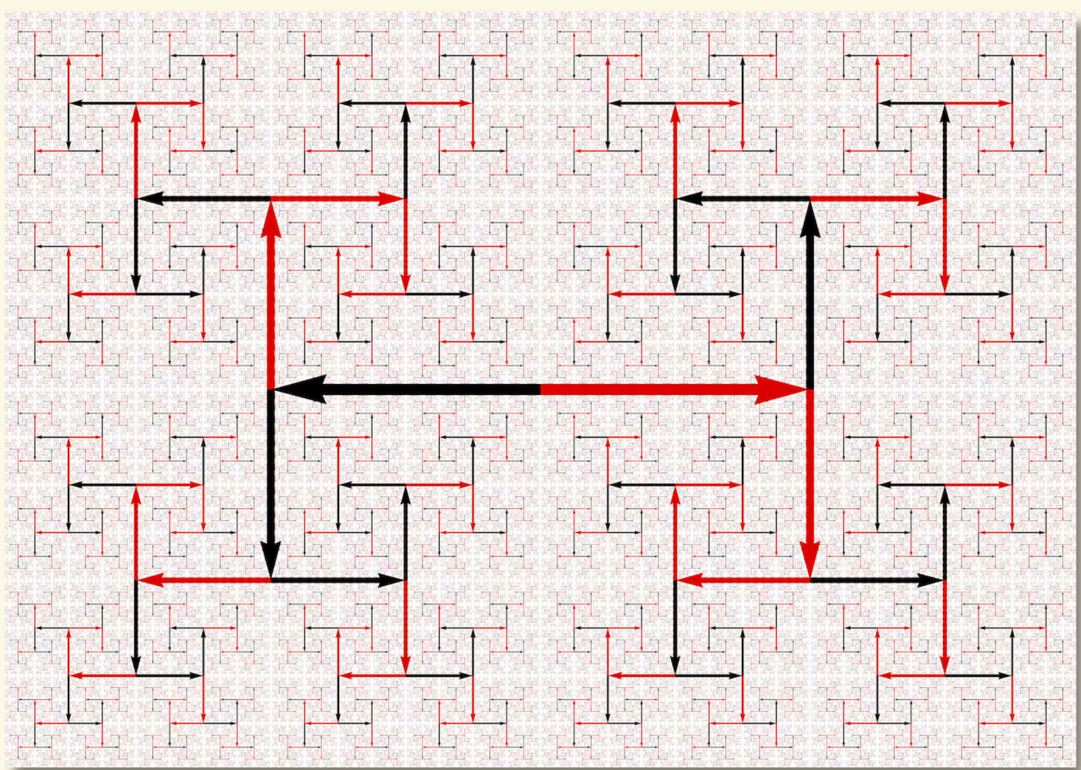
Definition (collinear fractal)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,

$$\boldsymbol{E}(c,n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \mathcal{A}_n \right\}.$$

$$p^{+}(c) = +1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$

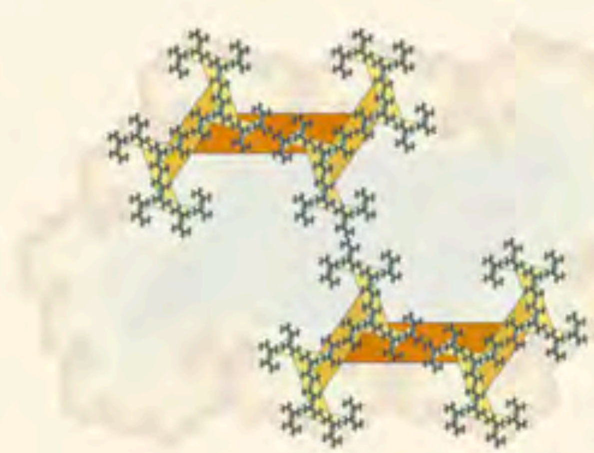
$$p^{-}(c) = -1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$



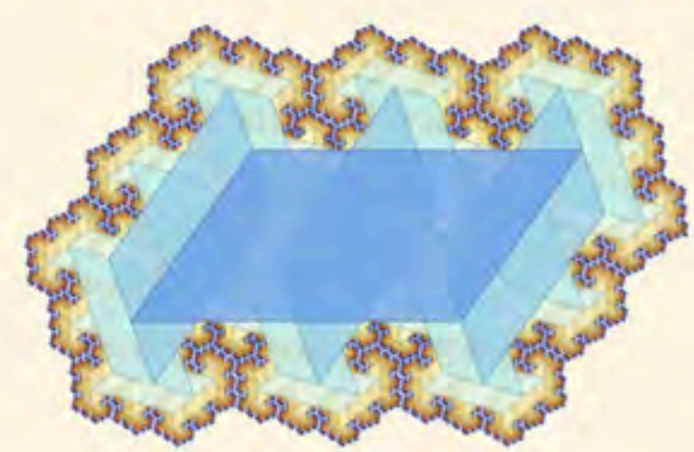
The Minkowski sum and geometric difference

$$\boldsymbol{E}(c, 2n-1) = \boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$$

$$= \boldsymbol{E}(c,n) \ominus \boldsymbol{E}(c,n)$$



$\boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$



$\boldsymbol{E}(c, 2n-1)$

Definition (collinear digit set)

Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

Definition (collinear fractal)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,
$\mathbf{E}(c,n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \mathcal{A}_n \right\}$.

Joint work with my PhD supervisors,
Joan Saldaña, and
David Juher

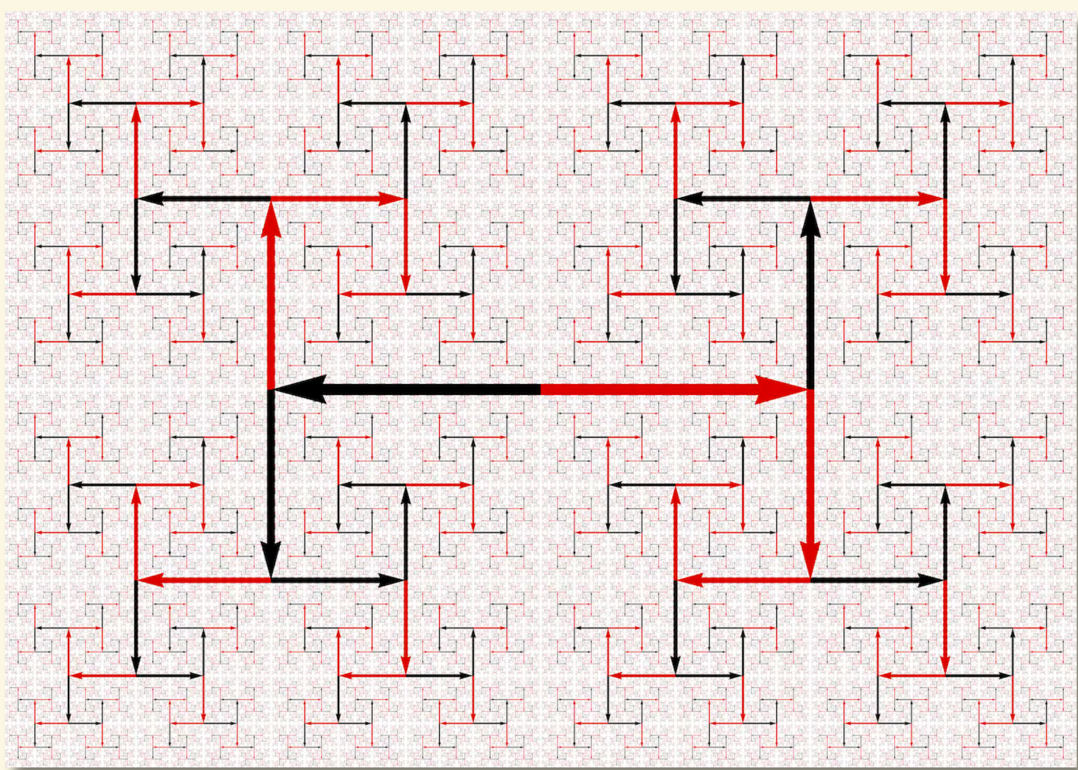
Universitat de Girona

IFUdG 2022–2024 Santander

$$\mathcal{M}_n = \left\{ c^{-1} \in \mathbb{D}^* : 2c \in \mathbf{E}(c, 2n-1) \right\}$$

$$p^+(c) = +1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$

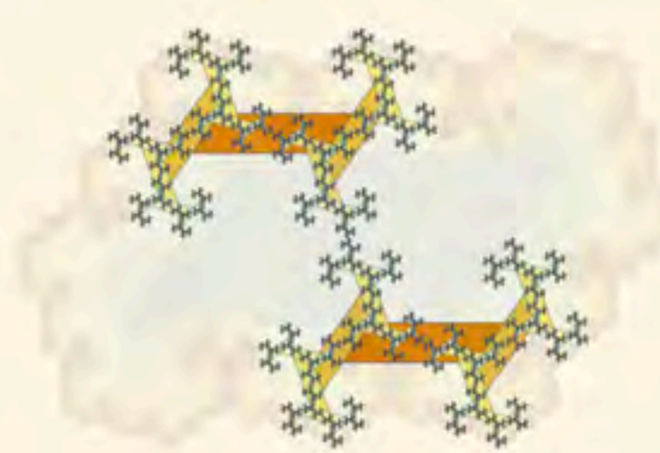
$$p^-(c) = -1 \pm c^{-1} \pm c^{-2} \pm \ldots \pm c^{-m}$$



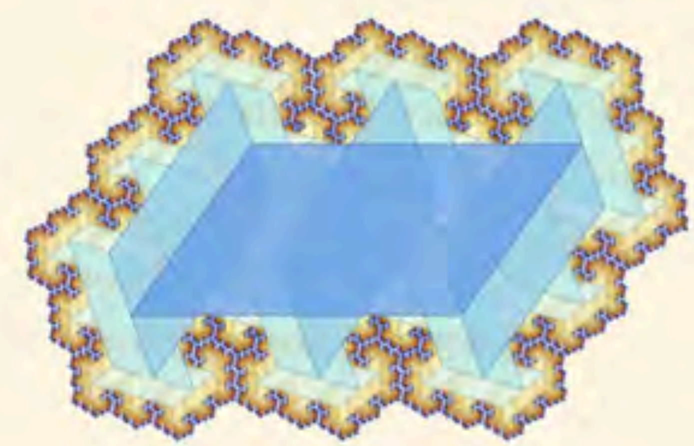
The Minkowski sum and geometric difference

$$\mathbf{E}(c, 2n-1) = \mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$$

$$= \mathbf{E}(c,n) \ominus \mathbf{E}(c,n)$$



$\mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$

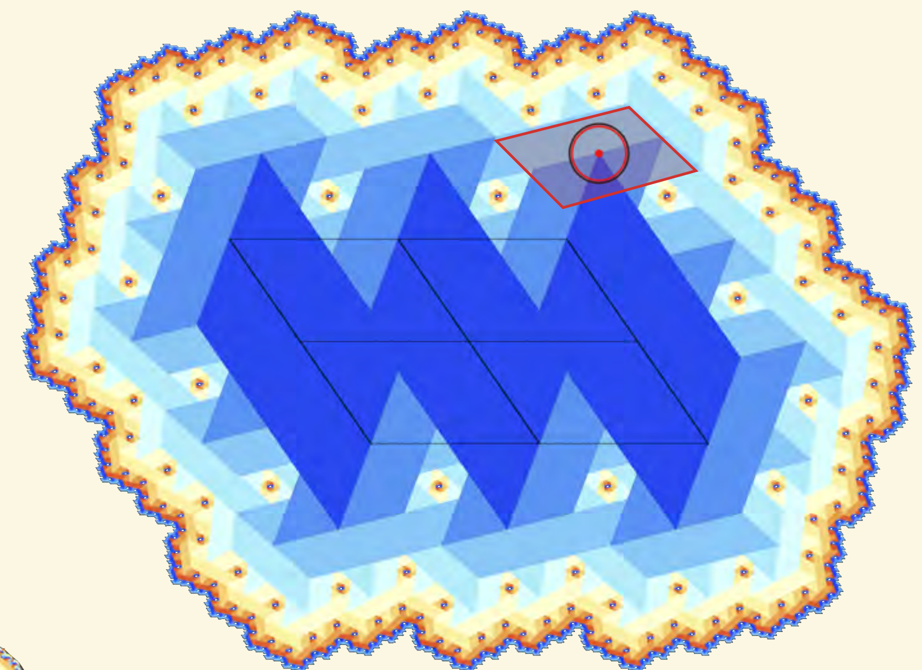


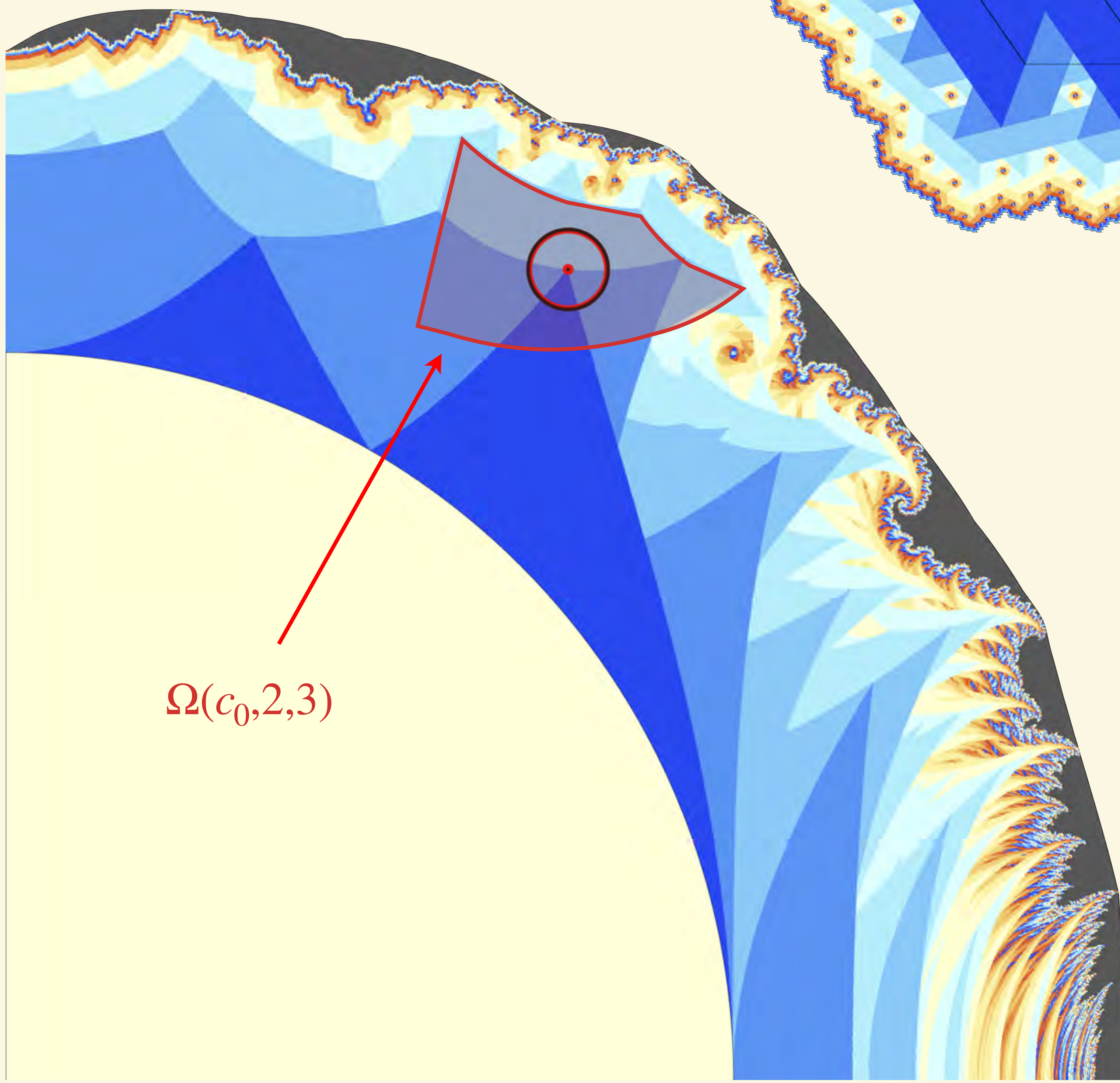
$\mathbf{E}(c, 2n-1)$

$c_0 = 0.7718 + i\,1.1151$

$c^3 - c^2 + c + 1 = 0$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$

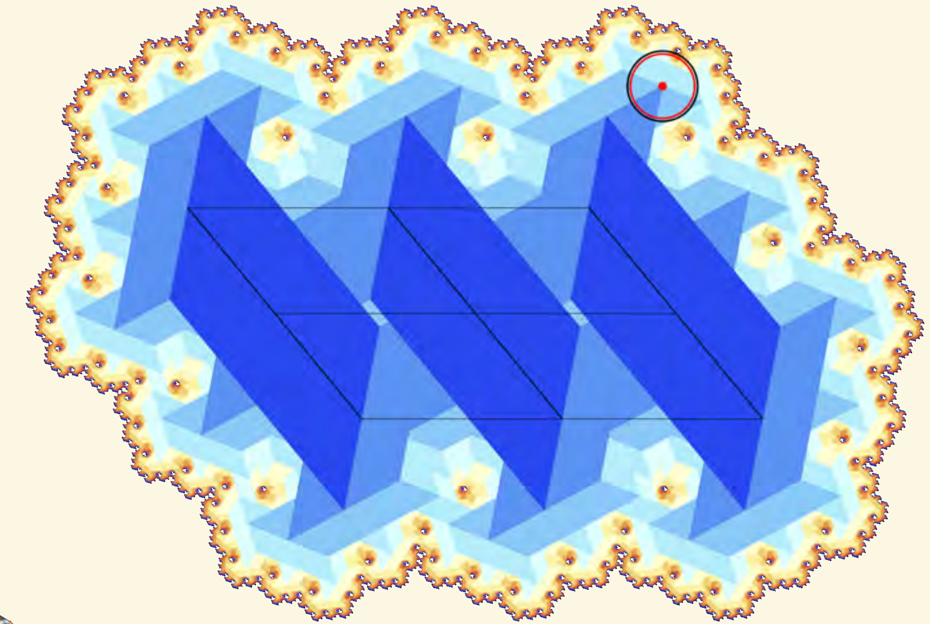


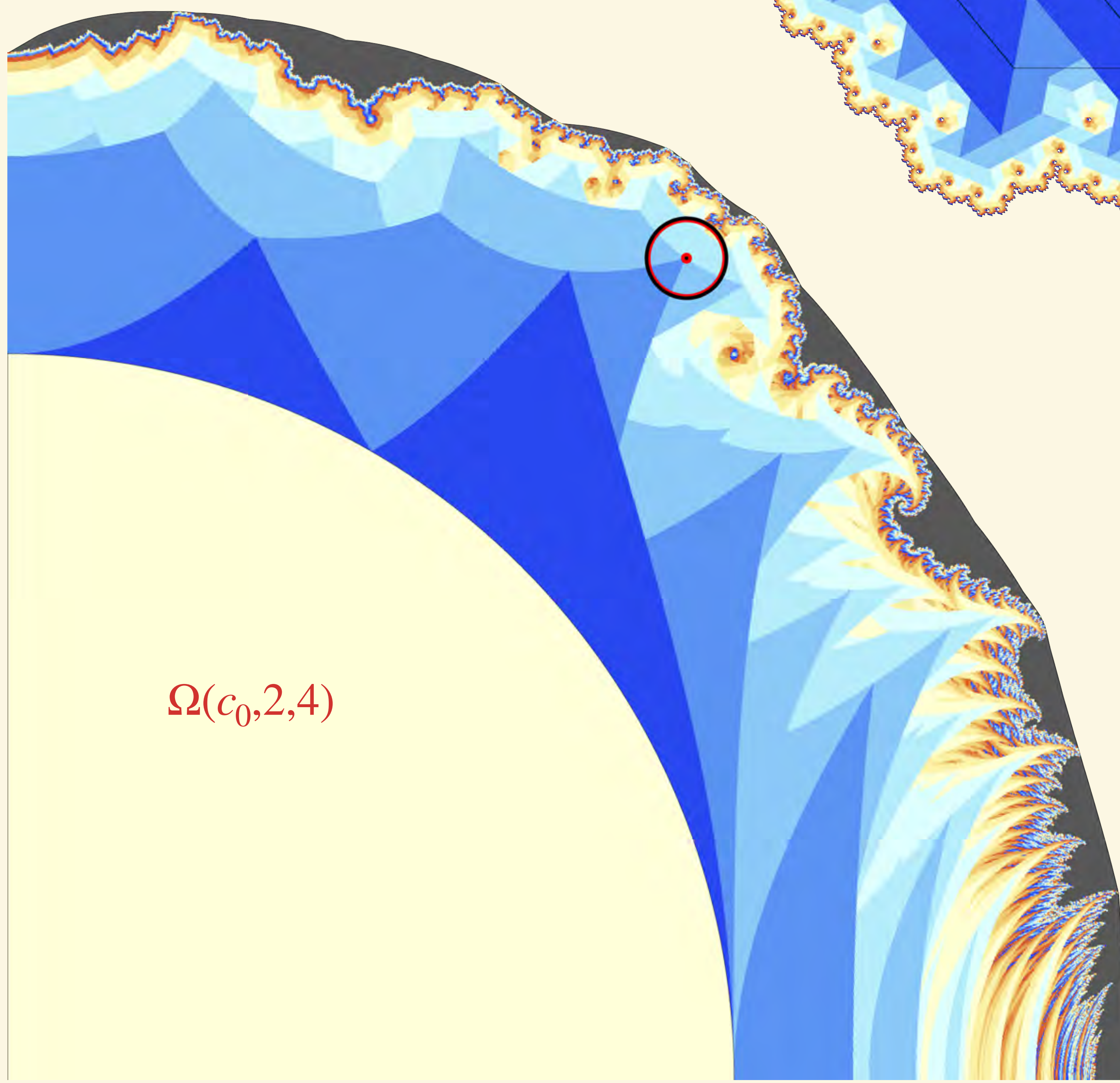


$c_0 = 0.9334 + i\,1.1325$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$

$$c^4 - c^3 + c^2 + c + 1 = 0$$

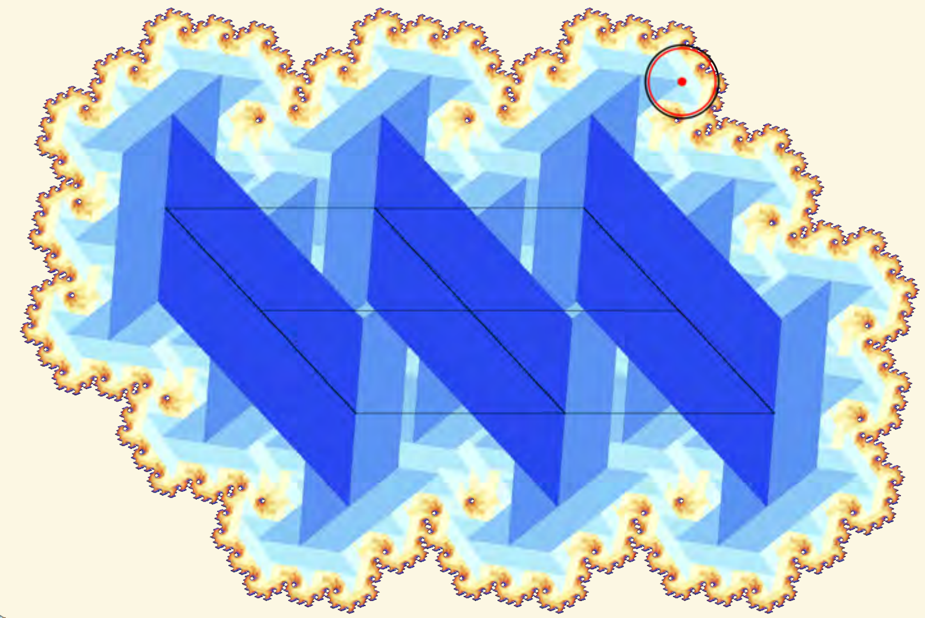


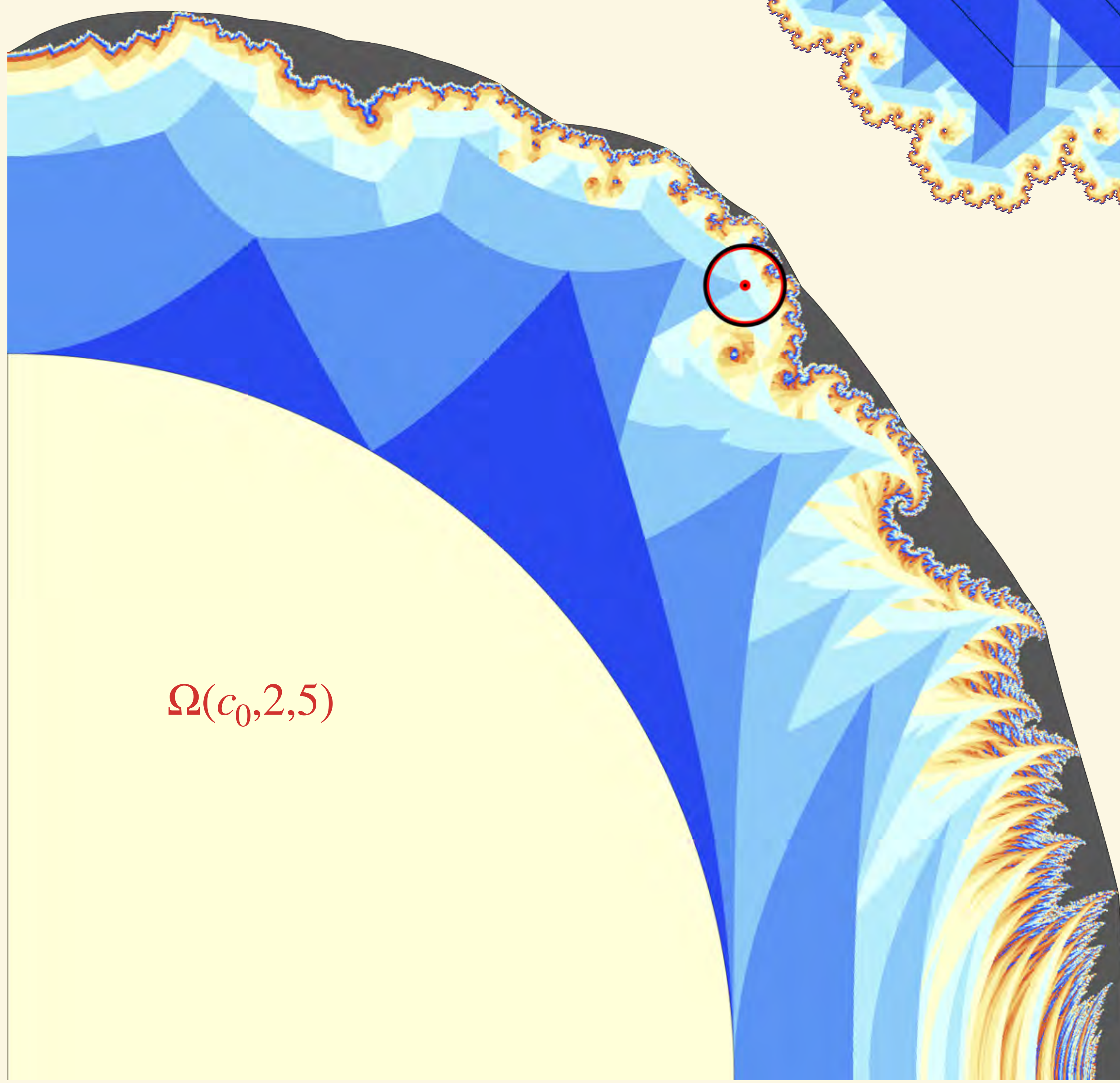


$c_0 = 1.0141 + i\,1.0951$

$$c^5 - c^4 + c^3 + c^2 + c + 1 = 0$$

$2c_0 \in \boldsymbol{E}(c, 2n-1)$



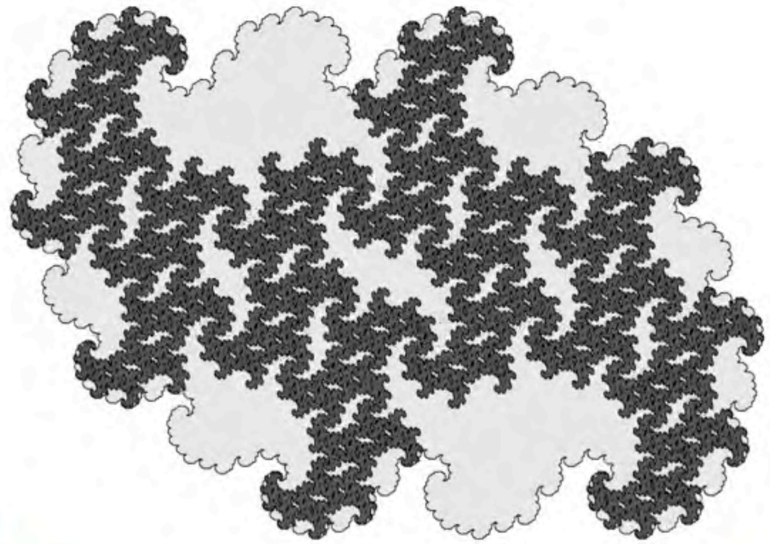


Animation: youtu.be/11NZDHNahJs

$c = 1.2345 + i\,0.7935$

$\mathrm{Arg}(c)=0.5713 = 32.73°$

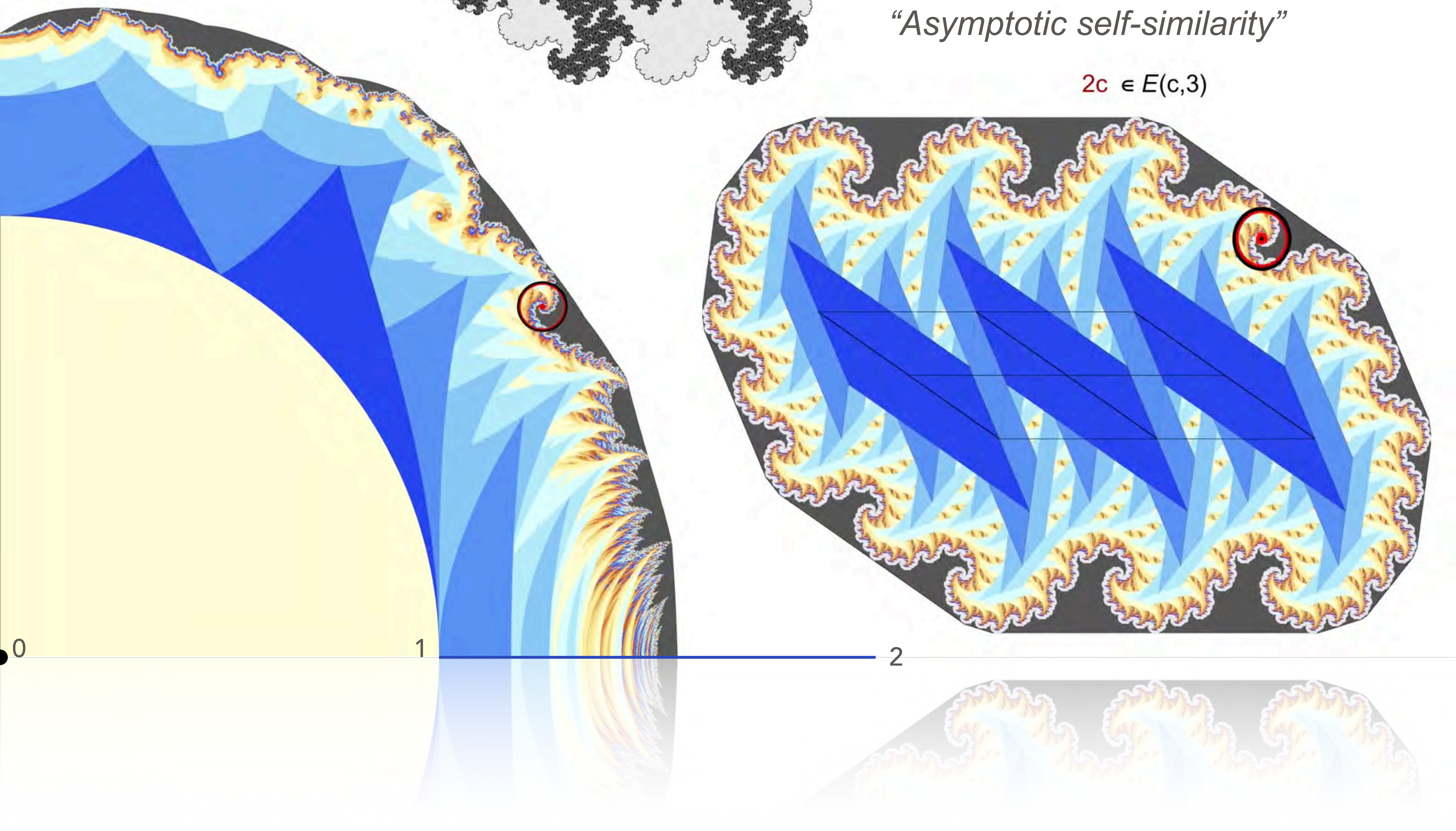
$r=|c^{-1}|=0.6814$



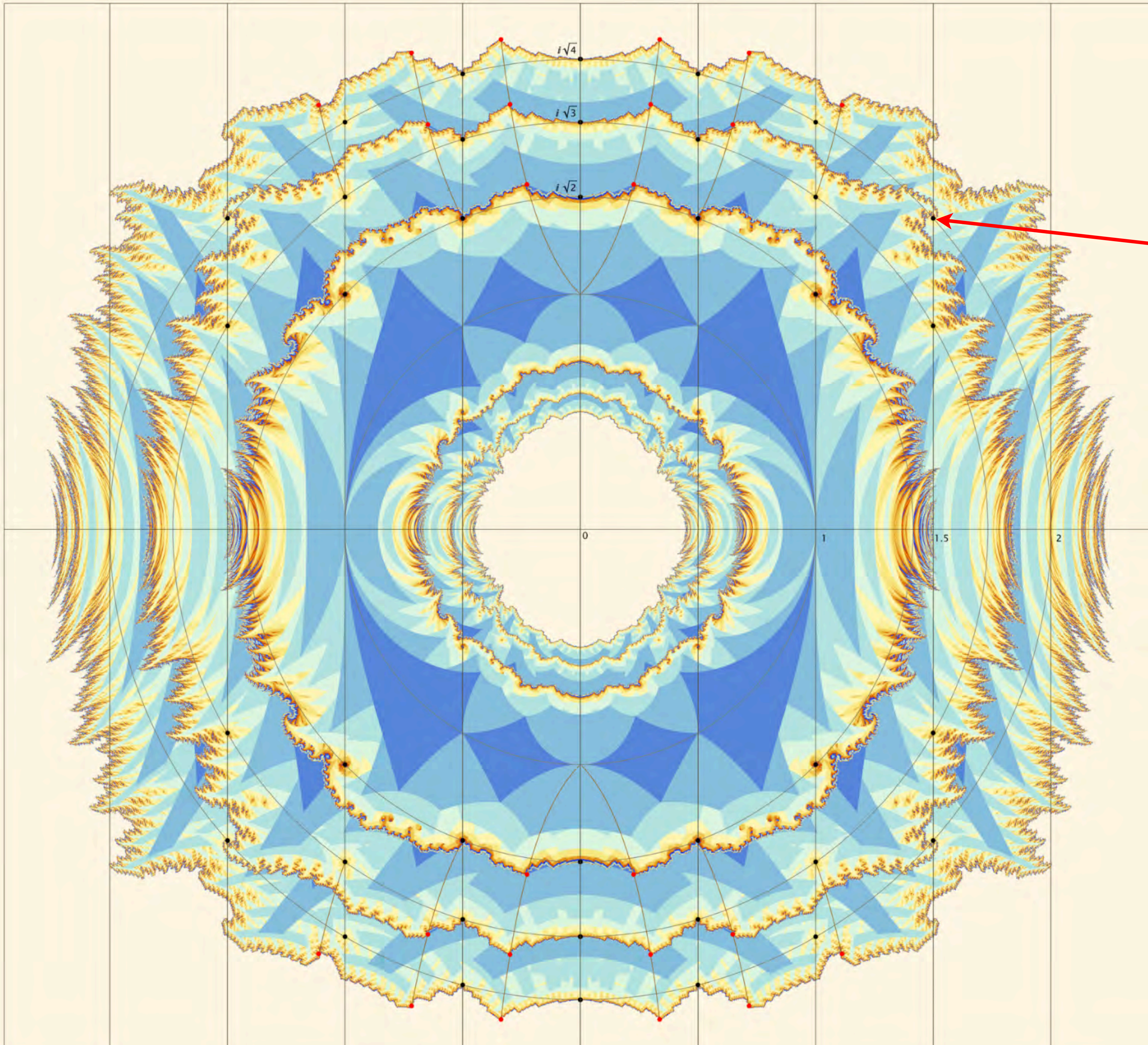
$$\dim_H E(c,2) = \frac{\log 2}{\log |c|} \approx 1.8071$$

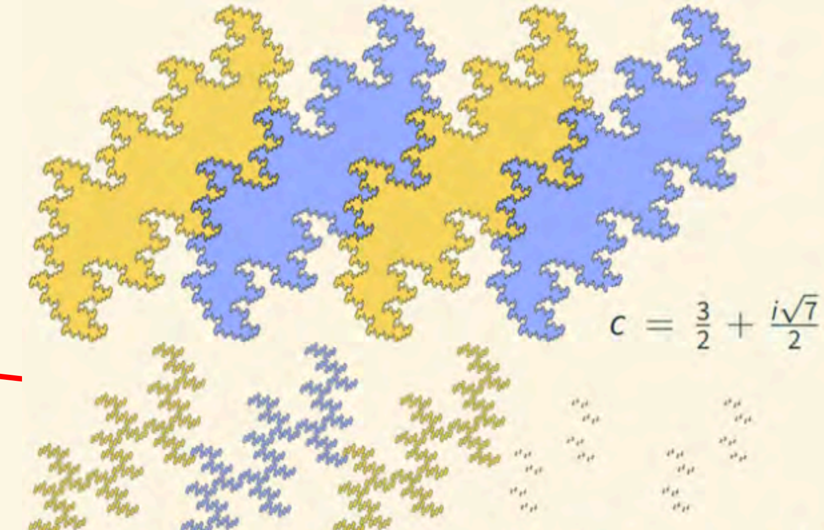
“Asymptotic self-similarity”

$2c \in E(c,3)$



$$\boldsymbol{E}(c, n+1) = \boldsymbol{E}(c, n) \oplus \boldsymbol{E}(c, 2)$$





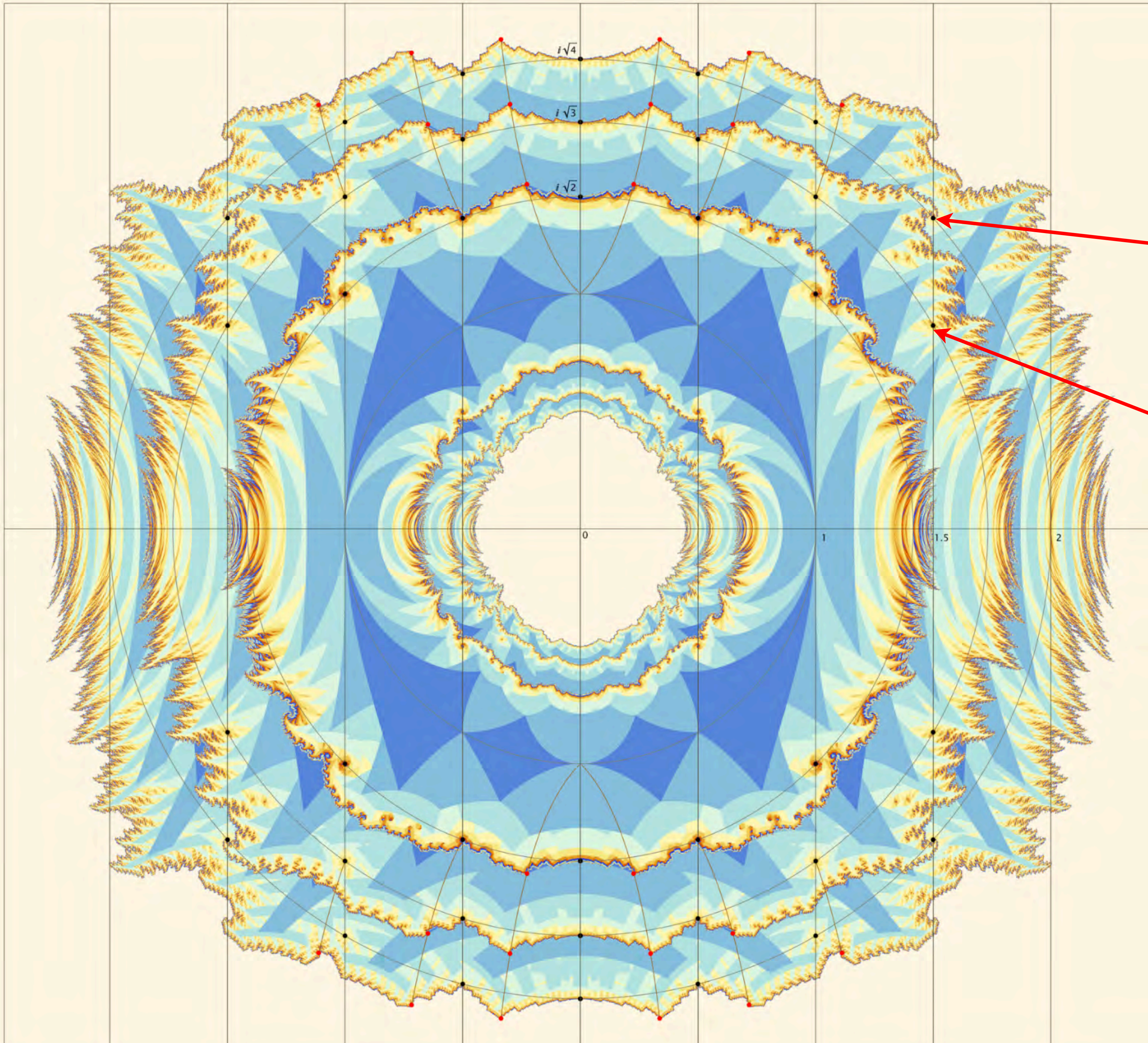
$$\boldsymbol{E}(c, 3) \oplus \boldsymbol{E}(c, 2) = \boldsymbol{E}(c, 4)$$

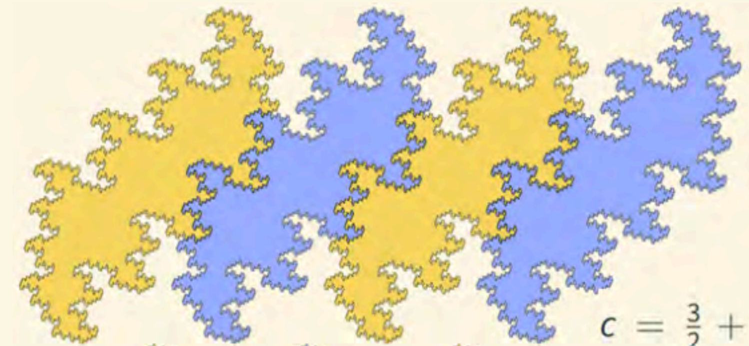
$$\mathcal{M}_n \subset \mathcal{M}_{n+1}$$

Proposition (nested components)

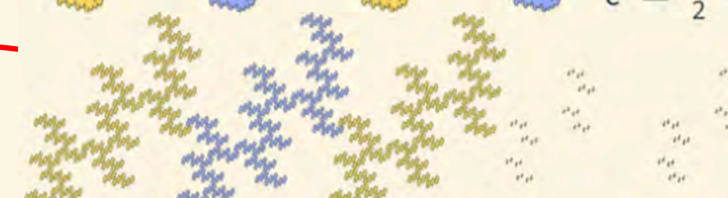
$$\boldsymbol{\Omega}(c_0, n, m) \subset \boldsymbol{\Omega}(c_0, n+1, m)$$

a=3

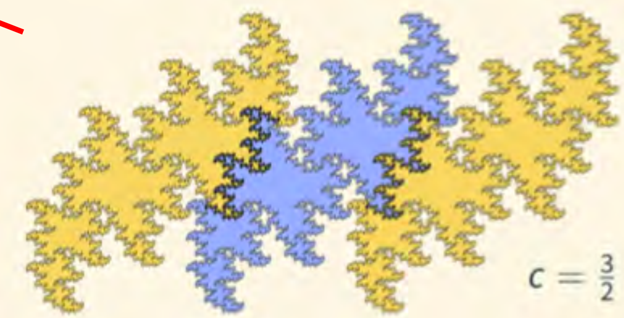




$c = \frac{3}{2} + \frac{i\sqrt{7}}{2}$



$\boldsymbol{E}(c,3) \oplus \boldsymbol{E}(c,2) = \boldsymbol{E}(c,4)$

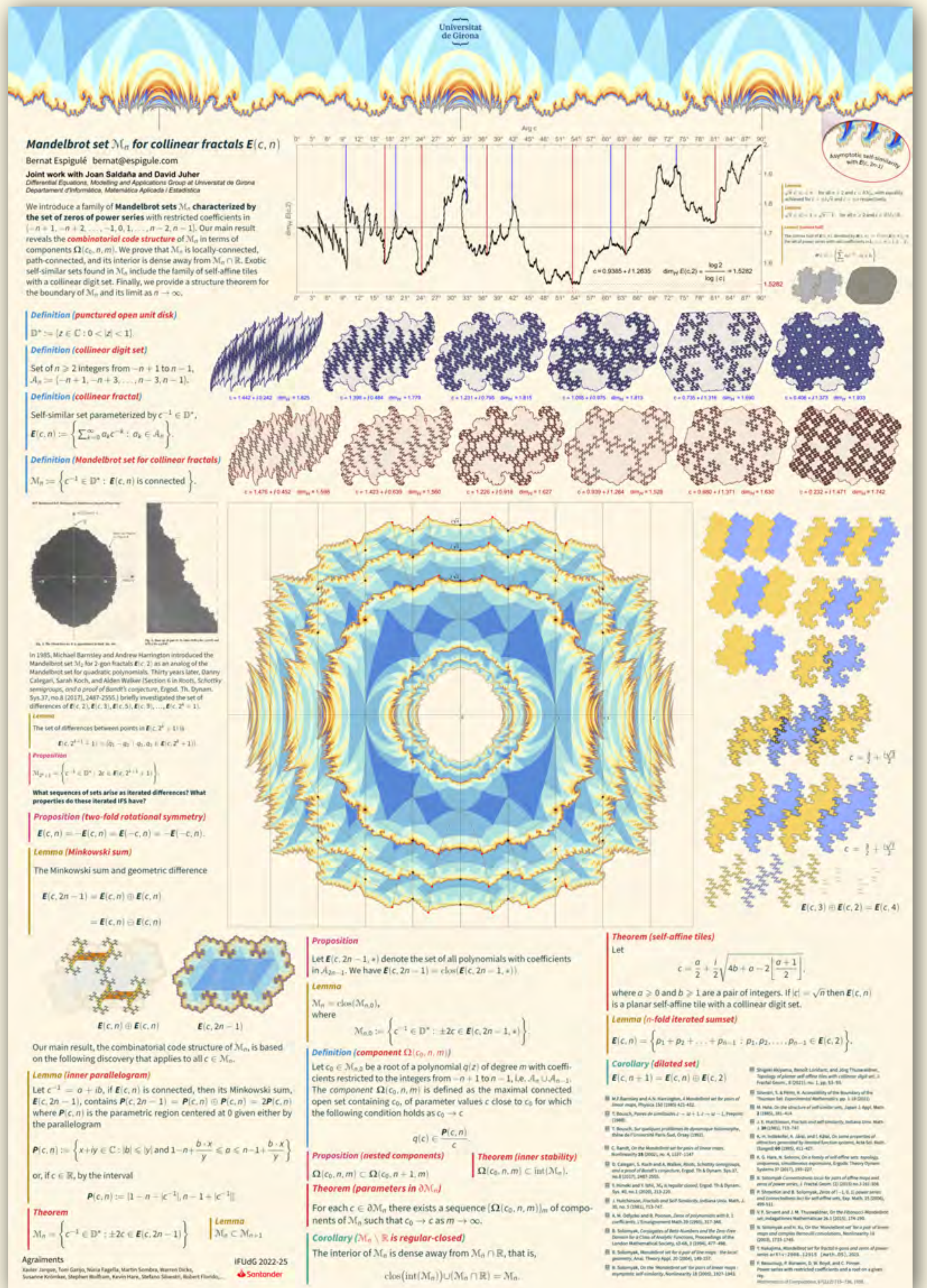


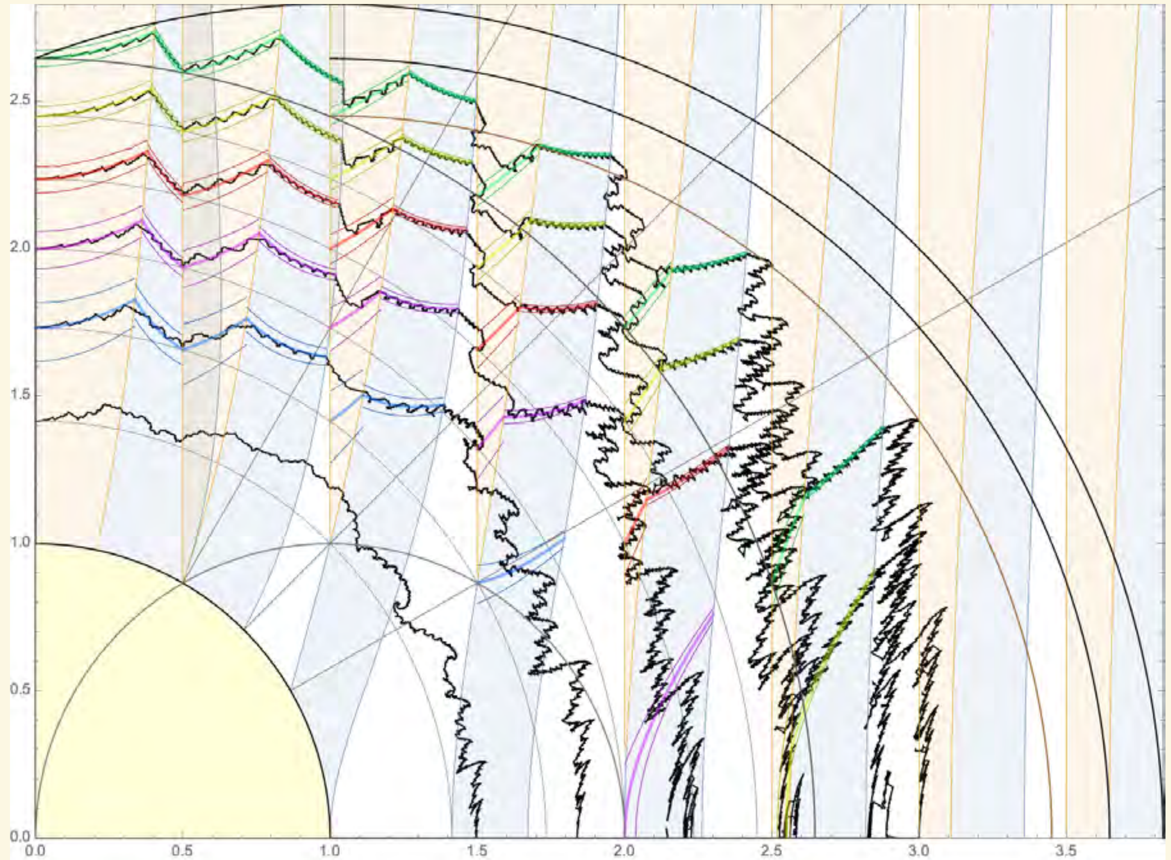
$c = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

Let

$$c = \frac{a}{2} + \frac{i}{2}\sqrt{4b + a - 2\left\lfloor \frac{a+1}{2} \right\rfloor},$$

where $a \geqslant 0$ and $b \geqslant 1$ are a pair of integers. If $|c| = \sqrt{n}$ then $\boldsymbol{E}(c,n)$ is a planar self-affine tile with a collinear digit set.





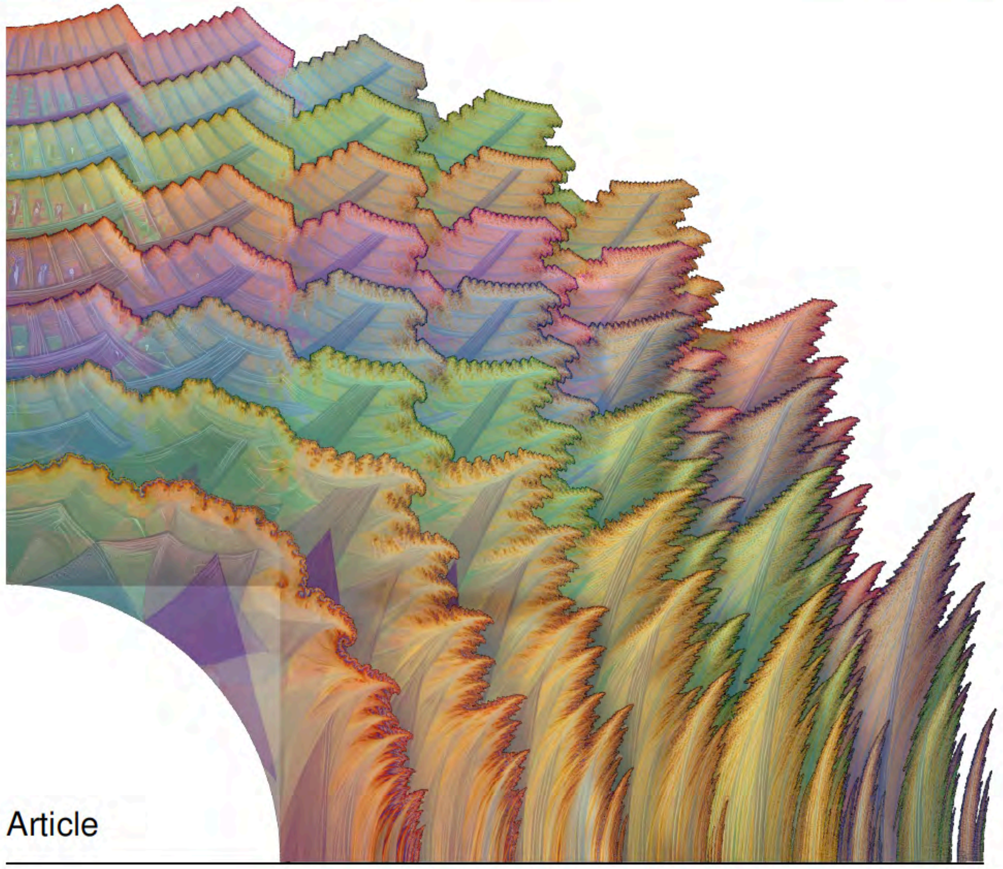
Our main contributions: combinatorial code structure, inner stability, components for any n>1, collinear tiles, structure theorem for the boundary of $\mathcal{M}_n$

- M.F.Barnsley and A.N. Harrington, *A Mandelbrot set for pairs of linear maps*, Physica 150 (1985) 421-432.
- T. Bousch, *Paires de similitudes* $z \to sz+1, z \to sz-1$, Preprint (1988).
- T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, thèse de l'Université Paris-Sud, Orsay (1992).
- C. Bandt, *On the Mandelbrot set for pairs of linear maps*, Nonlinearity **15** (2002), no. 4, 1127–1147
- D. Calegari, S. Koch and A. Walker, *Roots, Schottky semigroups, and a proof of Bandt's conjecture*, Ergod. Th & Dynam. Sys.37, no.8 (2017), 2487-2555.
- Y. Himeki and Y. Ishii, *$\mathcal{M}_4$ is regular closed*, Ergod. Th & Dynam. Sys. 40, no.1 (2020), 213-220.
- J. Hutchinson, *Fractals and Self-Similarity*, Indiana Univ. Math. J. 30, no. 5 (1981), 713-747.
- A. M. Odlyzko and B. Poonen, *Zeros of polynomials with* 0, 1 *coefficients*, L'Enseignement Math.39 (1993), 317-348.
- B. Solomyak, *Conjugates of Beta-Numbers and the Zero-Free Domain for a Class of Analytic Functions*, Proceedings of the London Mathematical Society, s3-68, 3 (1994), 477–498.
- B. Solomyak, *Mandelbrot set for a pair of line maps : the local geometry*, Anal. Theory Appl. 20 (2004), 149-157.
- B. Solomyak, *On the 'Mandelbrot set' for pairs of linear maps : asymptotic self-similarity*, Nonlinearity 18 (2005), 1927-1943.
- Silvestri, S. & Pérez, R. Accessibility of the Boundary of the Thurston Set. *Experimental Mathematics*. pp. 1-18 (2021)
- M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), 381–414.
- J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
- K.-H. Indlekofer, A. Járai, and I. Kátai, *On some properties of attractors generated by iterated function systems*, Acta Sci. Math. (Szeged) **60** (1995), 411–427.
- K. G. Hare, N. Sidorov, *On a family of self-affine sets: topology, uniqueness, simultaneous expansions*, Ergodic Theory Dynam. Systems 37 (2017), 193–227.
- B. Solomyak *Connectedness locus for pairs of affine maps and zeros of power series*, J. Fractal Geom. (2) (2015) no.3 281-308.
- P. Shmerkin and B. Solomyak, *Zeros of* $\{-1, 0, 1\}$ *power series and connectedness loci for self-affine sets*, Exp. Math. 15 (2006), 499-511.
- V. F. Sirvent and J. M. Thuswaldner, *On the Fibonacci-Mandelbrot set*, Indagationes Mathematicae 26.1 (2015), 174-190.
- B. Solomyak and H. Xu, *On the 'Mandelbrot set' for a pair of linear maps and complex Bernoulli convolutions*, Nonlinearity 16 (2003), 1733-1749.
- Y. Nakajima, *Mandelbrot set for fractal n-gons and zeros of power series* `arXiv:2008.12915 [math.DS]`, 2023.
- F. Beaucoup, P. Borwein, D. W. Boyd, and C. Pinner. Power series with restricted coefficients and a root on a given ray. *Mathematics of Computation*, 67(222):715–736, 1998.

fractal and fractional

IMPACT FACTOR 3.6

CITESCORE 4.6



Article

Collinear Fractals and Bandt's Conjecture

Bernat Espigule, David Juher and Joan Saldaña

MDPI

https://doi.org/10.3390/fractalfract8120725

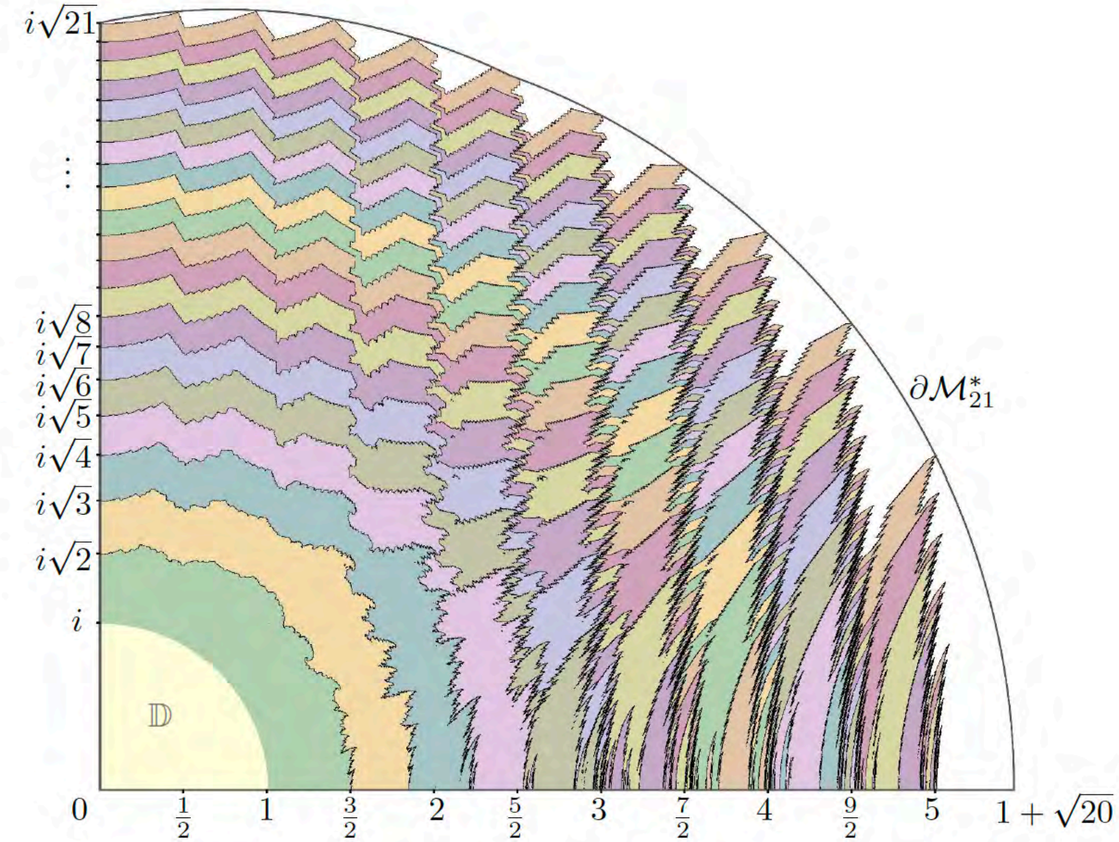


Figure 3. Superimposed arrangement of $\mathcal{M}_2$, $\mathcal{M}_3$, ..., $\mathcal{M}_{21}$ constrained within the upper-right section of the complex plane. From Proposition 3, we know that the connectedness loci are nested. The illustration suggests the existence of infinitely many holes, and that for $n \geq 4$, the intersection $\partial\mathcal{M}_n \cap \partial\mathcal{M}_{n+1} \setminus \mathbb{R}$ is nonempty.

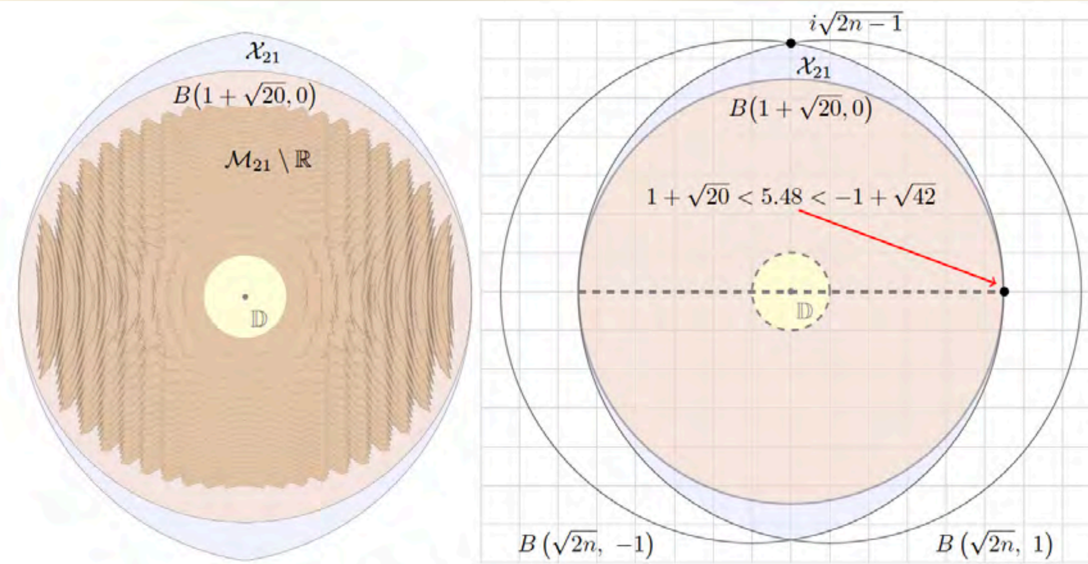
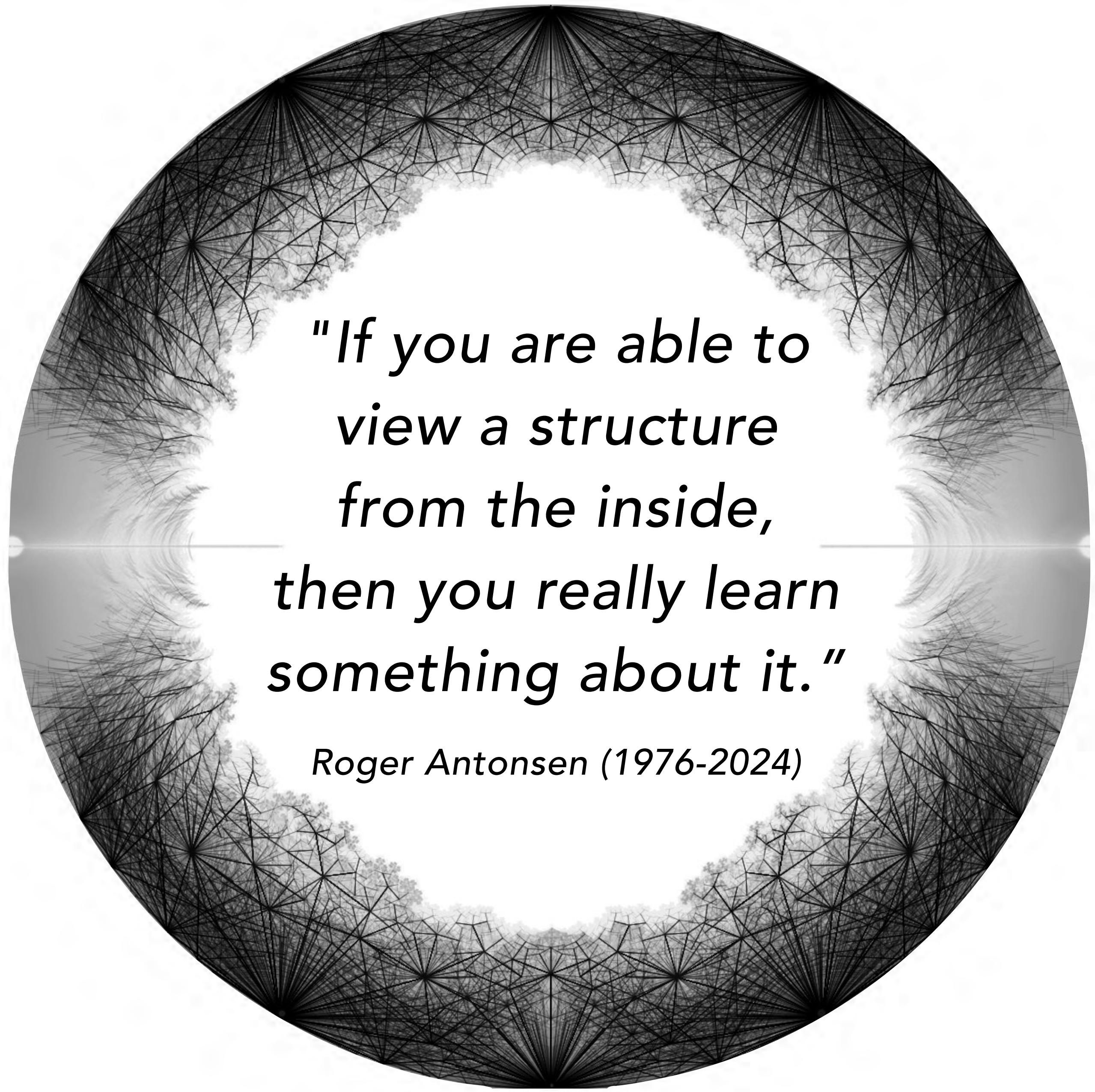
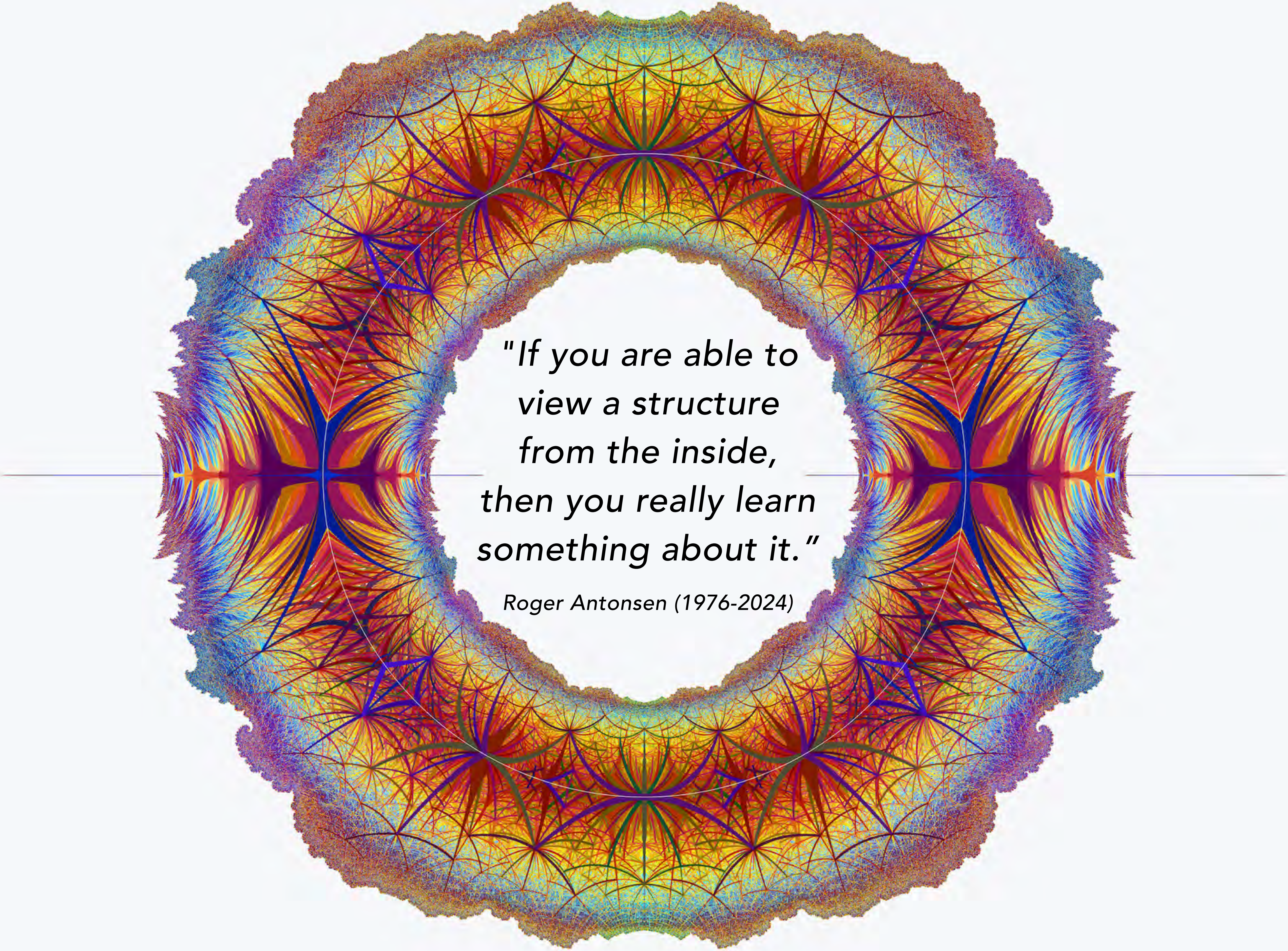


Figure 10. The set $\mathcal{M}_{21} \setminus \mathbb{R}$ contained in $\mathcal{X}_{21}$. Since $1+\sqrt{20} < -1+\sqrt{42}$, the region $\mathcal{X}_{21}$ contains $B\left(1+\sqrt{20},0\right) \setminus (\mathbb{R} \cup \mathbb{D})$ which in turn contains $\mathcal{M}_{21} \setminus \mathbb{R}$ by Proposition 4.

"If you are able to
view a structure
from the inside,
then you really learn
something about it."

Roger Antonsen (1976-2024)

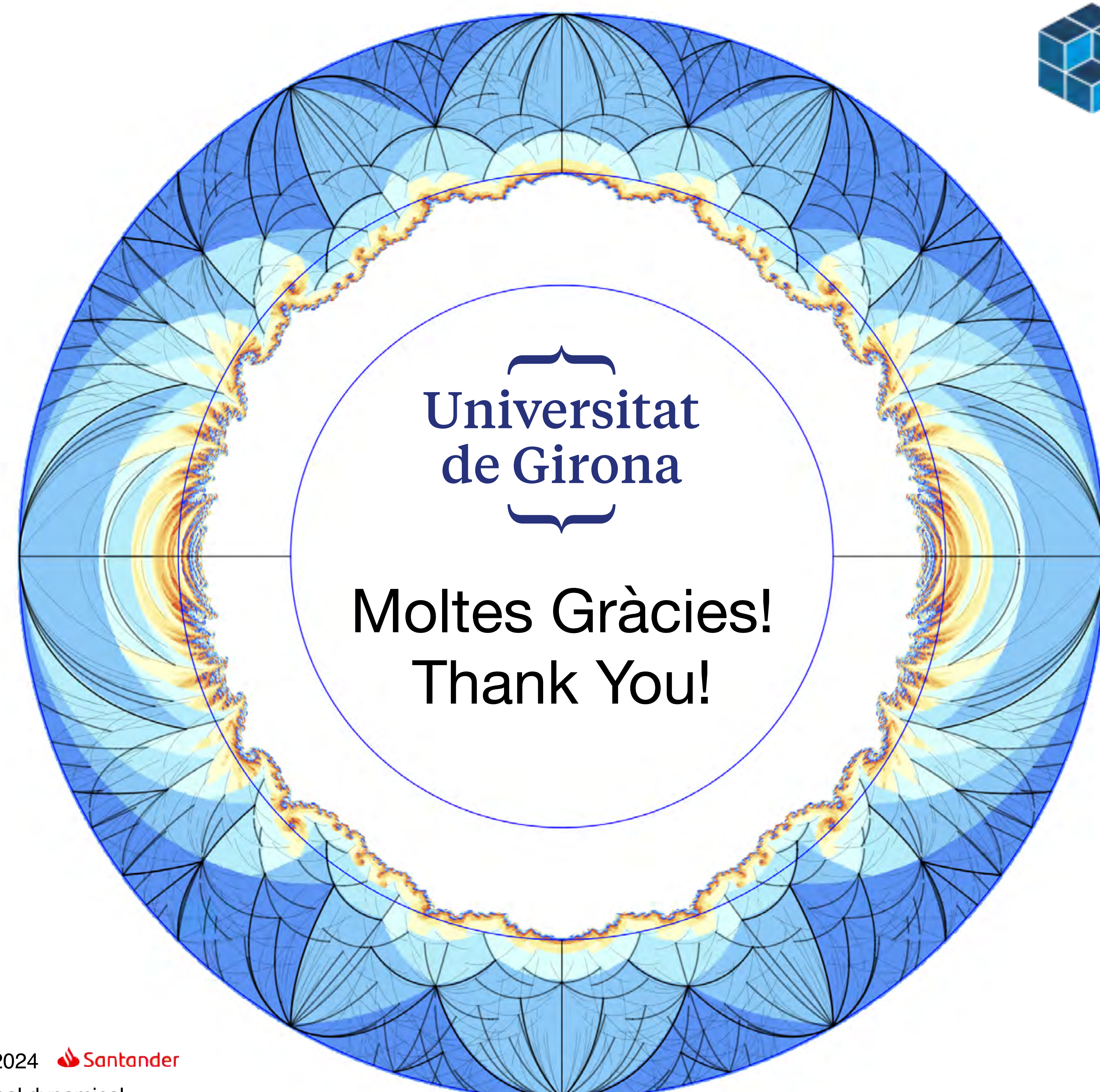

"If you are able to
view a structure
from the inside,
then you really learn
something about it."
Roger Antonsen (1976-2024)



"If you are able to view a structure from the inside, then you really learn something about it."

Roger Antonsen (1976-2024)

Joint work with my PhD supervisors, Joan Saldaña, and David Juher



ICERM
Fall 2019, Illustrating Mathematics

2017-2018
IMUB

WOLFRAM

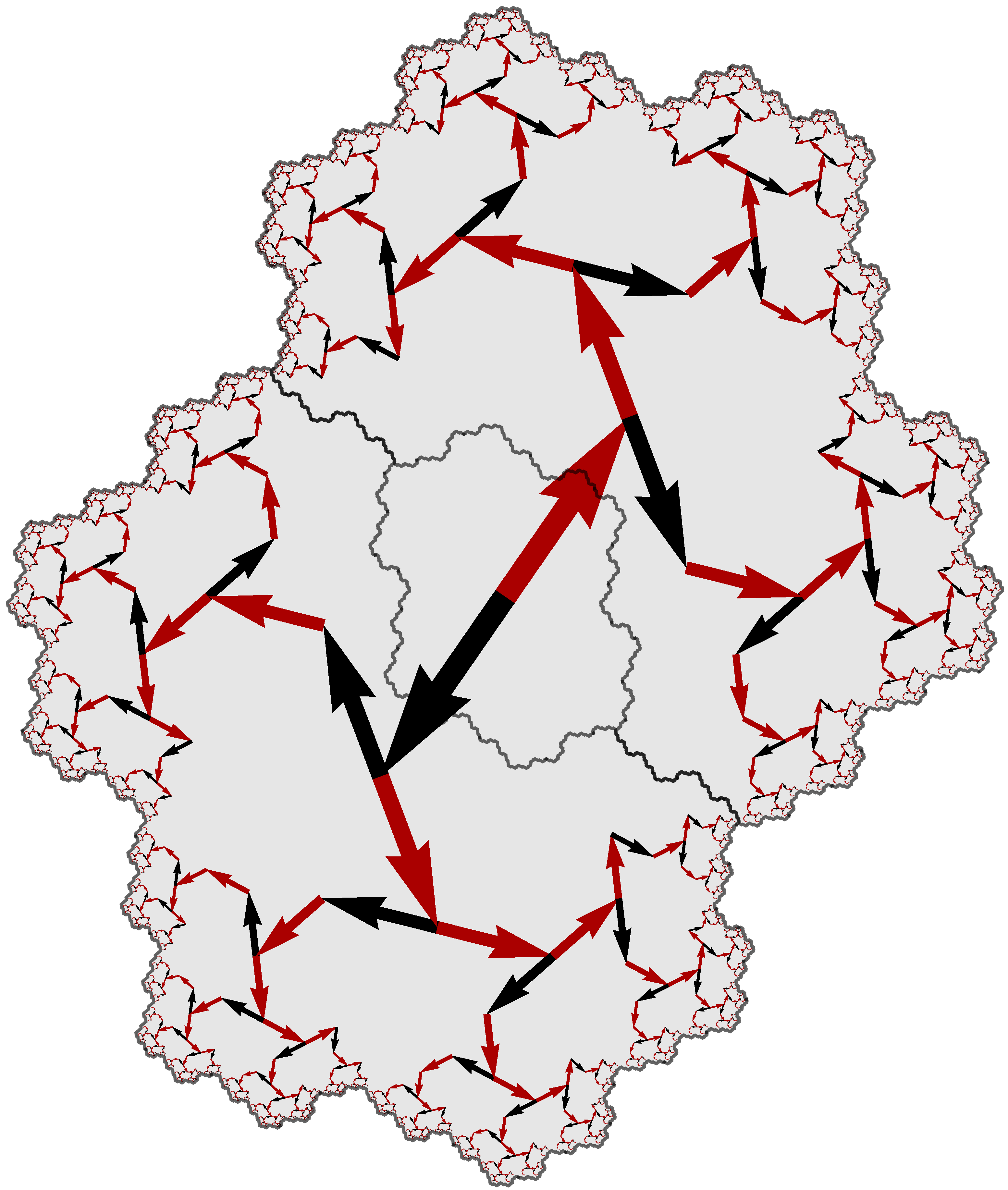
Stefano Silvestri,
Martin Sombra, Warren Dicks,
Susanne Krömker, Stephen Wolfram,
Kevin Hare, Christoph Bandt, …

Holomorphic Dynamics Group
Núria Fagella, Xavier Jarque,
Toni Garijo, Robert Florido, …

IFUdG 2022–2024 Santander
Low dimensional dynamical systems: topology, geometry, combinatorics and bifurcations
PID2023-146424NB-I00

Universitat de Girona *Departament d'Informàtica, Matemàtica Aplicada i Estadística*
Differential Equations, Modelling and Applications Group

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
AGENCIA ESTATAL DE INVESTIGACIÓN



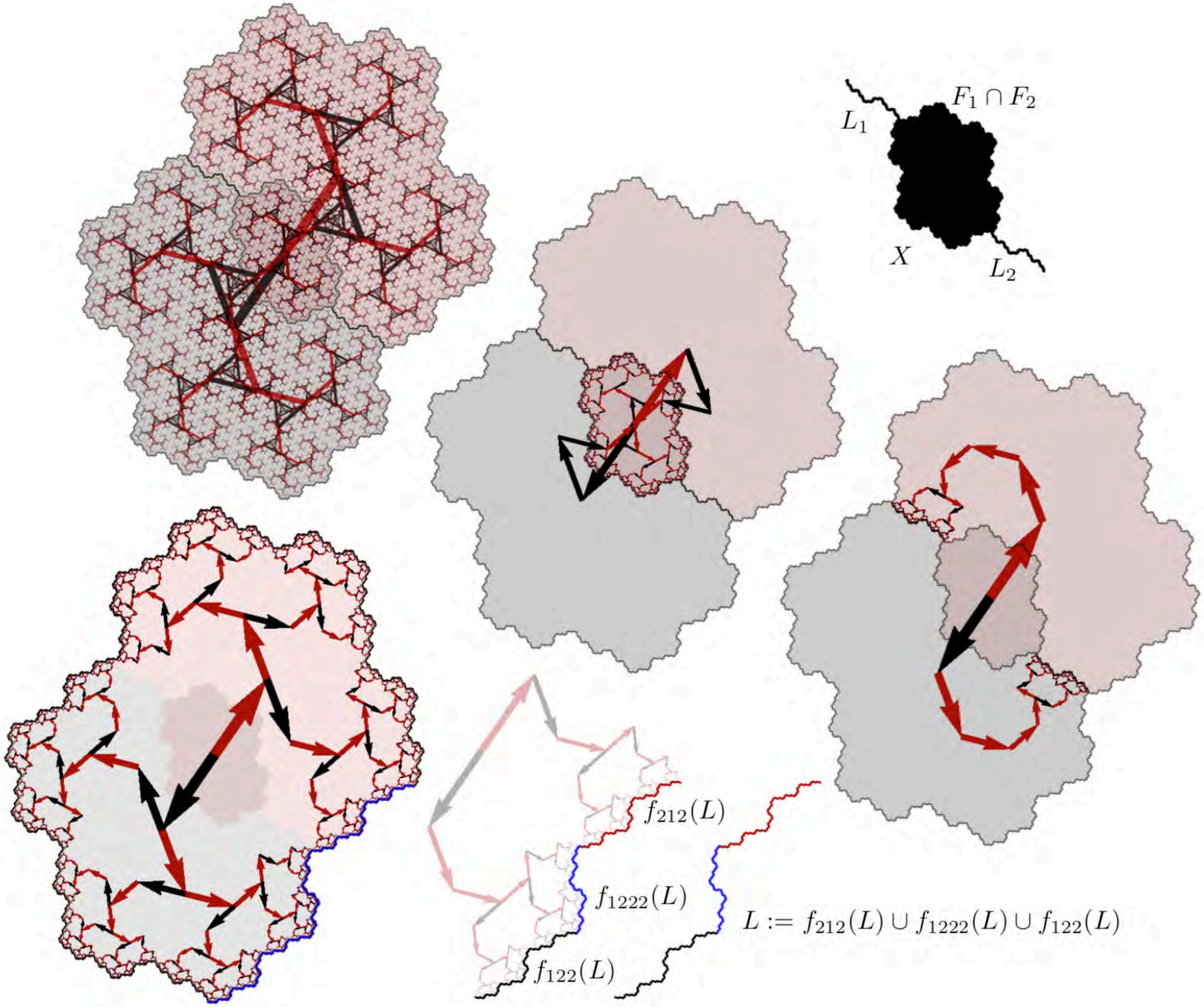


Figure 16: Binary complex tree $T\{c, -c\}$ where $c \approx 0.737e^{-2.176i}$ is a root of $1 + x + x^2 - x^3$. Since we have the following node-to-node intersections $\phi(11122) = \phi(21121)$, $\phi(11121) = \phi(21122)$, and $\phi(1112) = \phi(2112)$ theorem 6 tells us that the following piece-to-piece exact overlaps take place $F_{11122} = F_{21121}$, $F_{11121} = F_{21122}$. Moreover $F_{1112} = F_{11121} \cup F_{11122} = F_{21121} \cup F_{21122} = F_{2112}$, i.e. $F_{1112} = F_{2112}$. For this particular example the tipset $F\{c, -c\}$ is the classical Rauzy fractal shown in figure 20 with the particularity of having the first-level pieces F_1 and F_2 intersecting in an non-empty set $F_1 \cap F_2 = L_1 \cup X \cup L_2$ where $X = F_{1112} = F_{2112}$ and L_1 and L_2 are given in terms of the fractal curve $L := f_{212}(L) \cup f_{1222}(L) \cup f_{122}(L)$ as $L_1 = f_{22222}(L)$, $L_2 = f_{12222}(L)$.


ComplexTrees.com
ICERM
Bernat
Espigulé