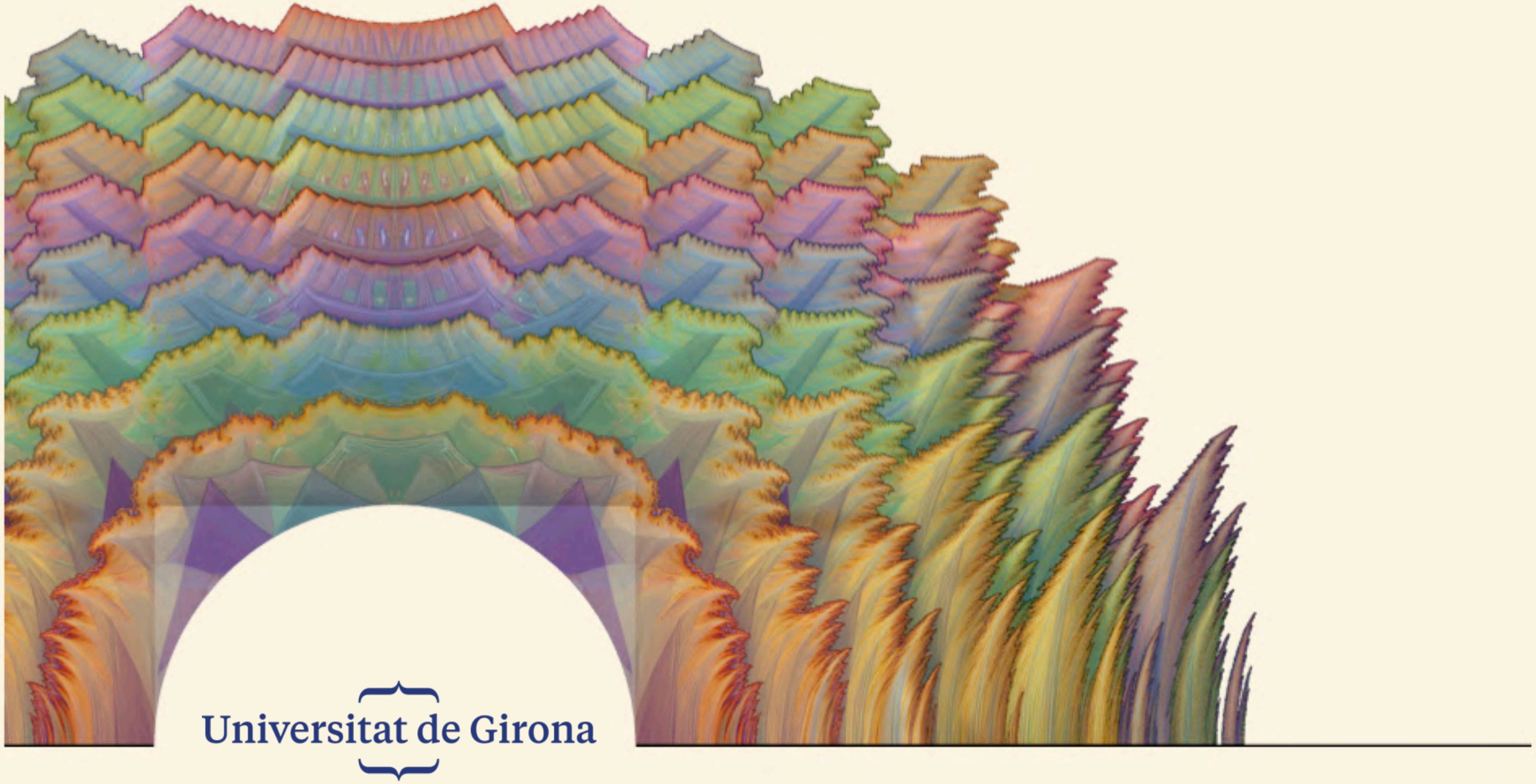


Mandelbrot set $\mathcal{M}_n$ ***for collinear fractals*** $\mathbf{E}(c,n)$

International Online GSDUAB Seminar 2024

Bernat Espigule

Universitat de Girona (Joint work with my PhD supervisors: David Juher and Joan Saldaña)

bernat@espigule.com

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought.

The Beauty of Roots

John C. Baez, J. Daniel Christensen, and Sam Derbyshire

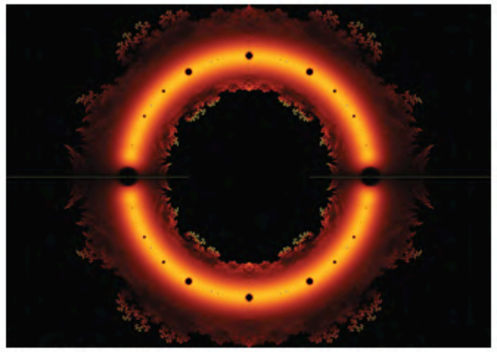


Figure 1. Roots of all polynomials of degree 23 whose coefficients are ± 1. The brightness shows the number of roots per pixel.

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought. For example, if we plot the roots of all polynomials of degree 23 whose coefficients are all 1 or -1, we get an astounding picture, shown in Figure 1.

More generally, define a **Littlewood polynomial** to be a polynomial $p(z) = \sum_{i=0}^{d} a_i z^i$ with each coefficient a_i equal to 1 or -1. Let $\mathbf{X}_n$ be the set of complex numbers that are roots of some Littlewood polynomial with n nonzero terms (and thus degree $n-1$). The 4-fold symmetry of Figure 1 comes from the fact that if $z \in \mathbf{X}_n$ so are $-z$ and $\overline{z}$. The set $\mathbf{X}_n$ is also invariant under the map $z \mapsto 1/z$, since if z is the root of some Littlewood polynomial then $1/z$ is a root of the polynomial with coefficients listed in the reverse order.

It turns out to be easier to study the set

$$\mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n = \{z \in \mathbb{C} | z \text{ is the root of some Littlewood polynomial}\}.$$

If n divides m then $\mathbf{X}_n \subseteq \mathbf{X}_m$, so $\mathbf{X}_n$ for a highly divisible number n can serve as an approximation to $\mathbf{X}$, and this is why we drew $\mathbf{X}_{24}$.

Some general properties of $\mathbf{X}$ are understood. It is easy to show that $\mathbf{X}$ is contained in the annulus $1/2 < |z| < 2$. On the other hand, Thierry Bousch showed [2] that the closure of $\mathbf{X}$ contains the annulus $2^{-1/4} \le |z| \le 2^{1/4}$. This means that the holes near roots of unity visible in the sets $\mathbf{X}_n$ must eventually fill in as we take the union over all

John C. Baez is a professor of mathematics at UC Riverside. His email address is john.baez@ucr.edu.

J. Daniel Christensen is a professor of mathematics at the University of Western Ontario. His email address is jdc@uwo.ca.

Sam Derbyshire is a Haskell consultant at Well-Typed. His email address is sam@well-typed.com.

DOI: https://doi.org/10.1090/noti2789

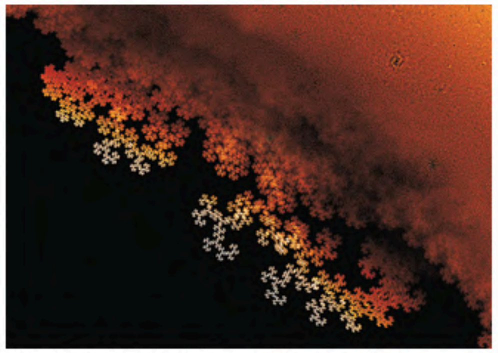


Figure 2. The region of $\mathbf{X}_{24}$ near the point $z = \frac{1}{2}e^{i/5}$.

n. More surprisingly, Bousch showed in 1993 that the closure $\overline{\mathbf{X}}$ is connected and locally path-connected [3]. It is worth comparing the work of Odlyzko and Poonen [7], who previously showed similar result for roots of polynomials whose coefficients are all 0 or 1.

The big challenge is to understand the diverse, complicated and beautiful patterns that appear in different regions of the set $\mathbf{X}$. There are websites that let you explore and zoom into this set online [4, 5, 8]. Different regions raise different questions.

For example, what is creating the fractal patterns in Figure 2 and elsewhere? An anonymous contributor suggested a fascinating line of attack which was further developed by Greg Egan [5]. Define two functions from the complex plane to itself, depending on a complex parameter q:

$$f_{+q}(z) = 1 + qz, \qquad f_{-q}(z) = 1 - qz.$$

When $|q| < 1$ these are both contraction mappings, so by a theorem of Hutchinson [6] there is a unique nonempty compact set $D_q \subseteq \mathbb{C}$ with

$$D_q = f_{+q}(D_q) \cup f_{-q}(D_q).$$

We call this set a **dragon**, or the **q-dragon** to be specific. And it seems that *for $|q| < 1$, the portion of the set $\mathbf{X}$ in a small neighborhood of the point q tends to look like a rotated version of D_q.*

Figure 3 shows some examples. To precisely describe what is going on, much less prove it, would take real work. We invite the reader to try. A heuristic explanation is known, which can serve as a starting point [1, 5]. Bousch [3] has also proved this related result:

Theorem. *For $q \in \mathbb{C}$ with $|q| < 1$, we have $q \in \overline{\mathbf{X}}$ if and only if $0 \in D_q$. When this holds, the set D_q is connected.*

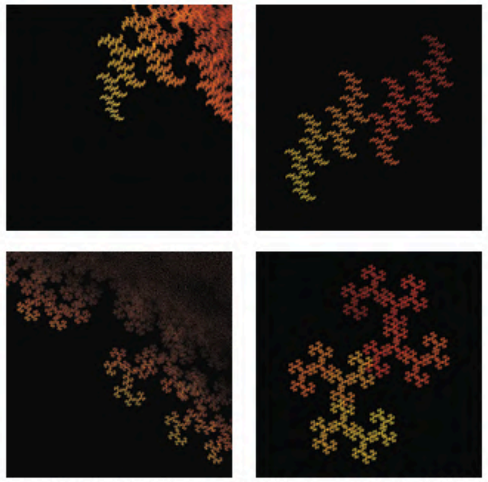


Figure 3. Top: the set $\mathbf{X}$ near $q = 0.594 + 0.254i$ at left, and the set D_q at right. Bottom: the set $\mathbf{X}$ near $q = 0.375453 + 0.544825i$ at left, and the set D_q at right.

References

[1] J. C. Baez, The beauty of roots. Available at http://math.ucr.edu/home/baez/roots.

[2] T. Bousch, Paires de similitudes $Z \to SZ + 1, Z \to SZ - 1$, January 1988. Available at https://www.imo.universite-paris-saclay.fr/~thierry.bousch/preprints/.

[3] T. Bousch, Connexité locale et par chemins hölderiens pour les systèmes itérés de fonctions, March 1993. Available at https://www.imo.universite-paris-saclay.fr/~thierry.bousch/preprints/.

[4] J. D. Christensen, Plots of roots of polynomials with integer coefficients. Available at http://jdc.math.uwo.ca/roots/.

[5] G. Egan, Littlewood applet. Available at http://www.gregegan.net/SCIENCE/Littlewood/Littlewood.html.

[6] J. E. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747. Also available at https://maths-people.anu.edu.au/~john/Assets/Research%20Papers/fractals_self-similarity.pdf.

[7] A. M. Odlyzko and B. Poonen, Zeros of polynomials with 0,1 coefficients, *L'Enseignement Math.* 39 (1993), 317–348. Also available at http://dx.doi.org/10.5169/seals-60430.

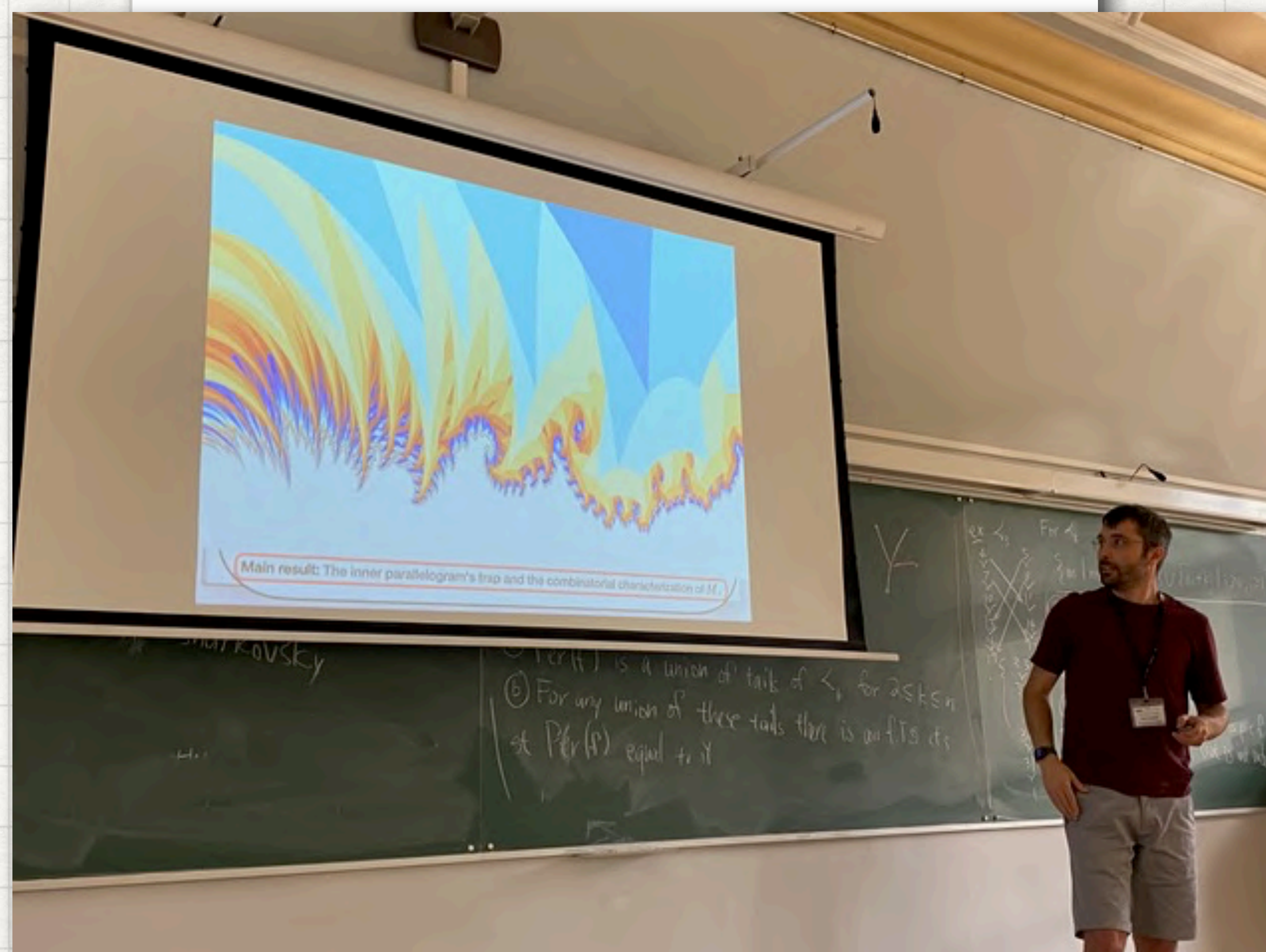
[8] R. Vanderbei, Roots of functions $F(z) = \sum_{j=0}^{n} \alpha_j f_j(z)$ where $\alpha_j \in \{-1, 1\}$. Available at https://vanderbei.princeton.edu/WebGL/roots_PlusMinusOne.html.

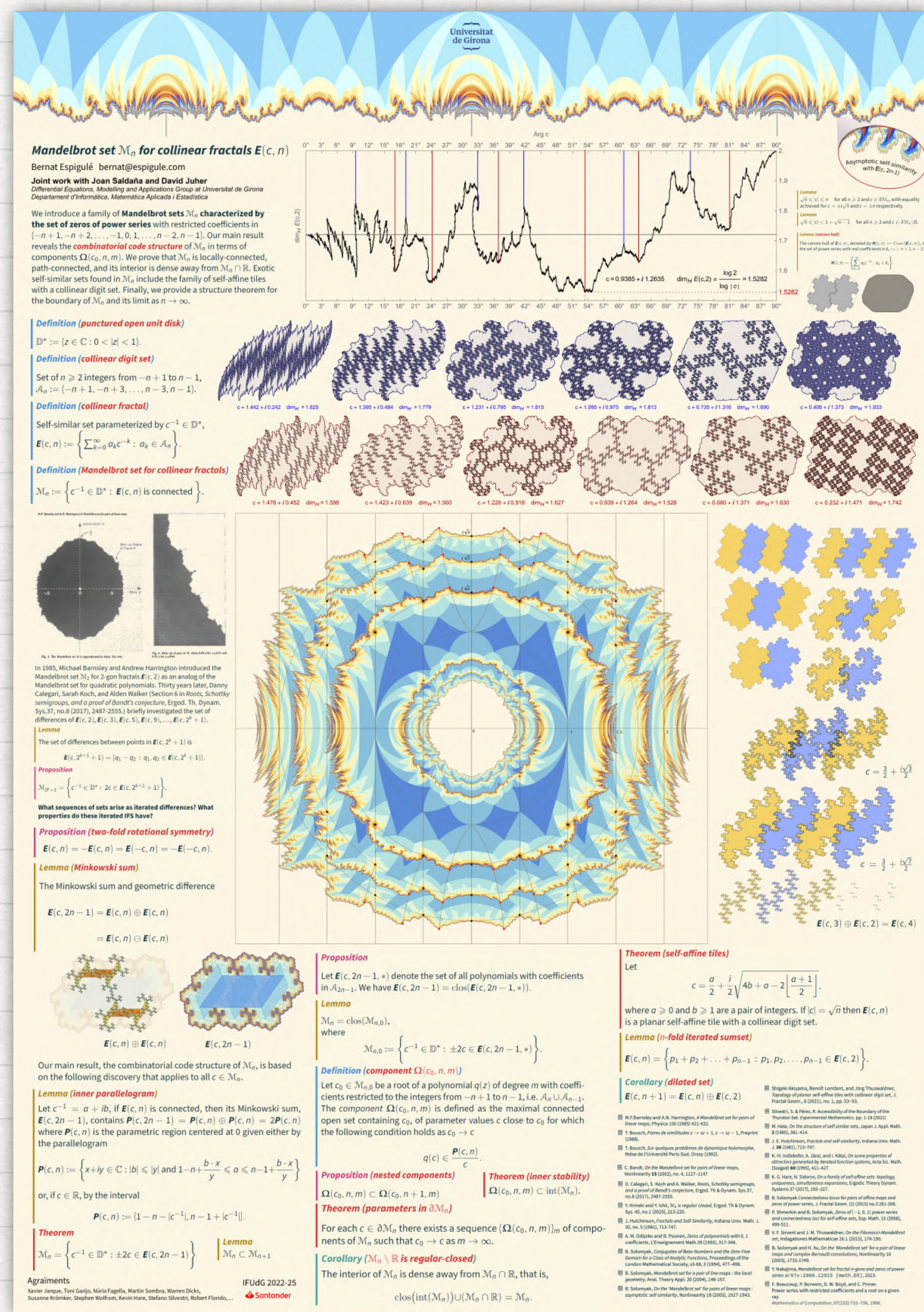
Complex trees and the internal structure of M_2

Bernat Espigulé

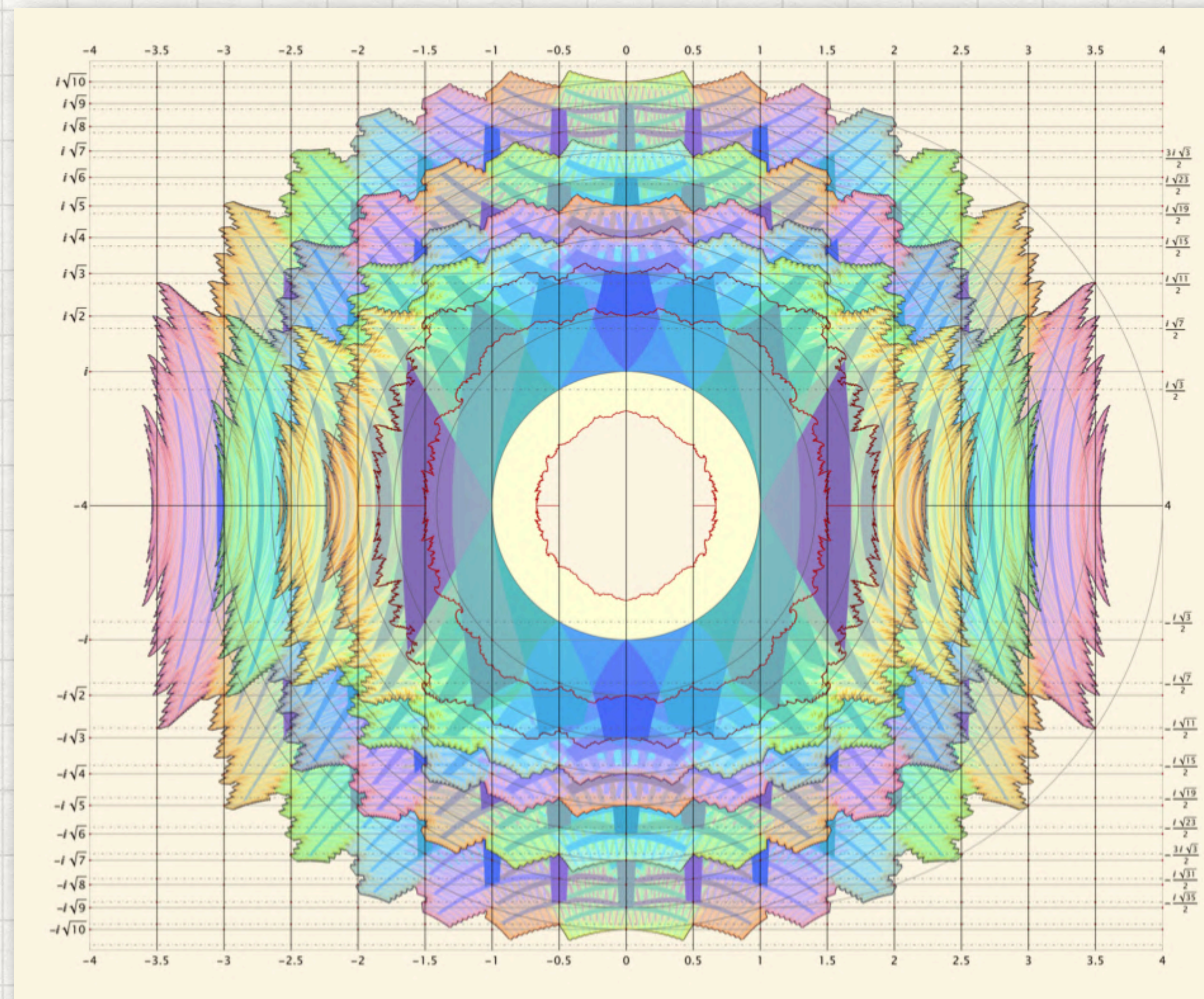
Universitat de Girona

In 1985, Barnsley and Harrington defined a "Mandelbrot Set" M_2 that can be identified with the closure of the set of roots of polynomials with coefficients in $\{-1, 0, 1\}$. In 2014 Calegari, Koch and Walker introduced the technique of *traps* in order to prove Bandt's conjecture that the interior points are dense away from the real axis. We now conjecture a much simpler condition to certify interior points of M_2 by exploiting a method based on the underlying complex tree structure of the limit set. Our main result is the hierarchical structure of the interior of M_2 defined by the algebraic curves that partially prove our conjecture.




Universitat de Girona
Mandelbrot set $\mathcal{M}_n$ for collinear fractals $E(c,n)$
Bernat Espigulé bernat@espigule.com
Joint work with Joan Saldaña and David Juher
Definition (punctured open unit disk)
Definition (collinear digit set)
Definition (collinear fractal)
Definition (Mandelbrot set for collinear fractals)
Proposition (two-fold rotational symmetry)
Lemma (Minkowski sum)
Lemma (inner parallelogram)
Theorem
Proposition
Definition (component $\Omega(c_0,n,m)$)
Proposition (nested components)
Theorem (inner stability)
Theorem (parameters in $\partial\mathcal{M}_n$)
Corollary ($\mathcal{M}_n \setminus \mathbb{R}$ is regular-closed)
Theorem (self-affine tiles)
Lemma (n-fold iterated sumset)
Corollary (dilated set)
Agraïments
IFUdG 2022-25


Centro di Ricerca Matematica Ennio De Giorgi
weekly calendar
RSS
Parameter spaces in complex dynamics and related topics
27 May 2024 – 31 May 2024



mathoverflow

Questions Tags Users Badges Una

Perron number distribution

60

18

A *Perron number* is a real algebraic integer λ that is larger than the absolute value of any of its Galois conjugates. The Perron-Frobenius theorem says that any non-negative integer matrix M such that some power of M is strictly positive has a unique positive eigenvector whose eigenvalue is a Perron number. Doug Lind proved the converse: given a Perron number λ, there exists such a matrix, perhaps in dimension much higher than the degree of λ. Perron numbers come up frequently in many places, especially in dynamical systems.

My question:

> What is the limiting distribution of Galois conjugates of Perron numbers λ in some bounded interval, as the degree goes to infinity?

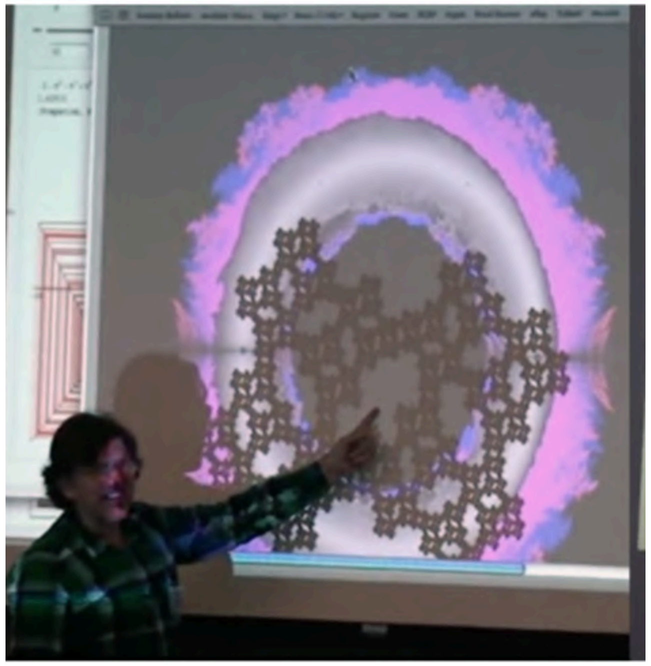
share cite improve this question

edited Jan 11 '11 at 4:04

asked Jan 11 '11 at 3:50
Bill Thurston
21.4k ● 11 ● 82 ● 113

2 Should be related to the distribution at math.ucr.edu/home/baez/roots ; there are references at that link, I think. – Qiaochu Yuan Jan 11 '11 at 4:20

2 @Qiaochu Yuan: Thanks for bringing it up. I actually intended to check out and point to those references, until my question got too long. I was trying to take a slice of things in a way that eliminates the fractal distribution of roots of polynomials with bounded coefficients. My motivation for this question originated in trying understand topological entropy for postcritically finite iterated polynomials, where a Mandelbrot-like distribution comes up that is very related to those exhibited by Baez (and others). – Bill Thurston Jan 11 '11 at 4:41



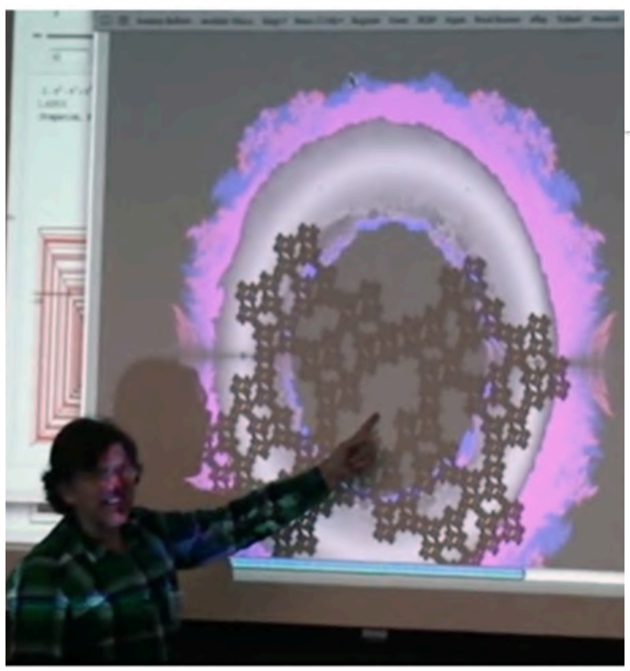
Bill Thurston

(1946 - 2012)

Thurston's last paper

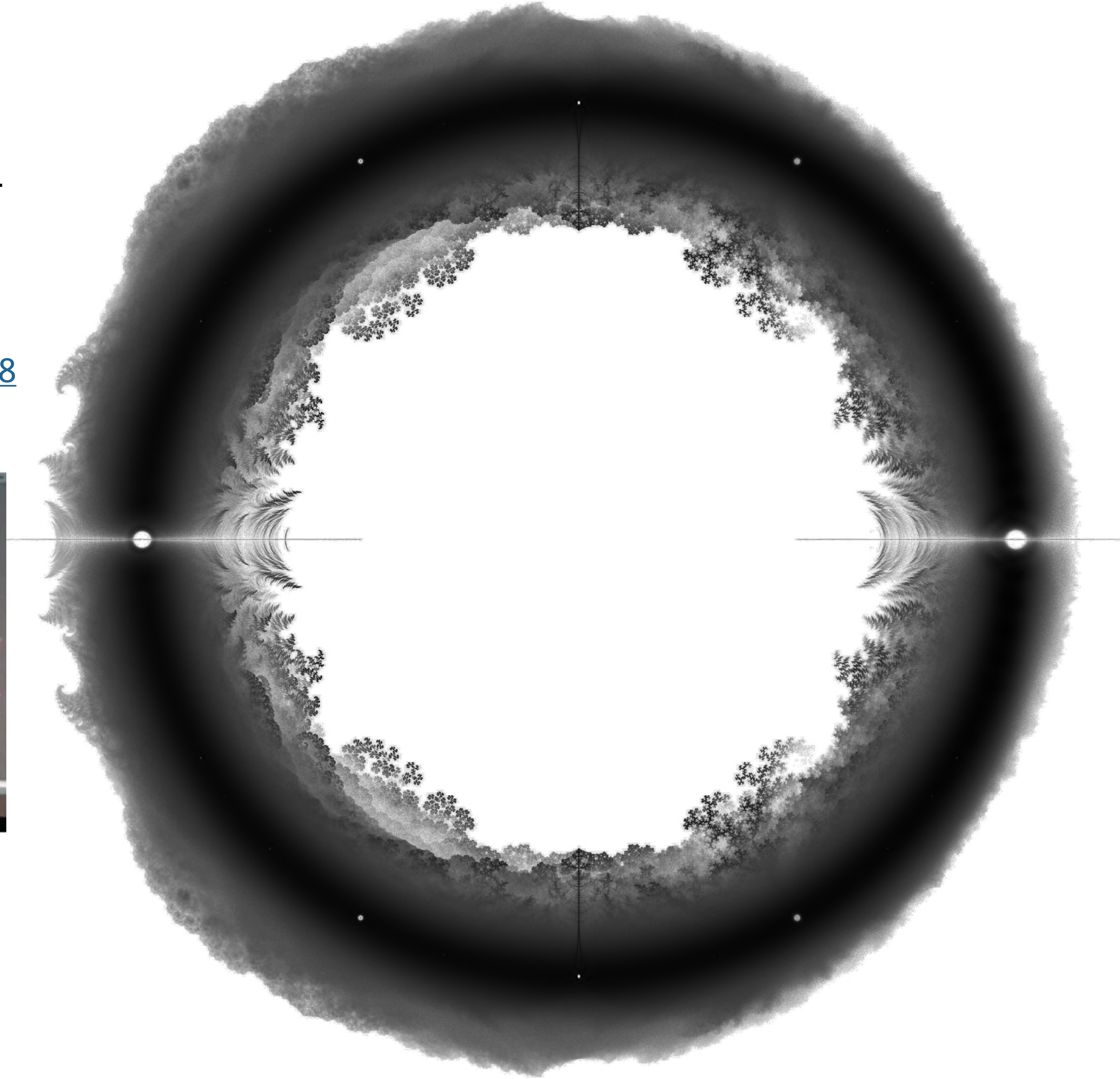
ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



Bill Thurston

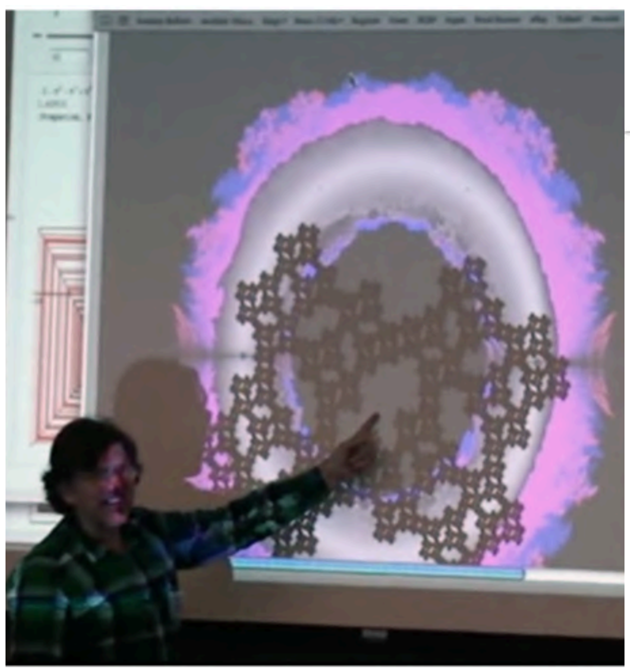
(1946 – 2012)



Thurston's last paper

ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



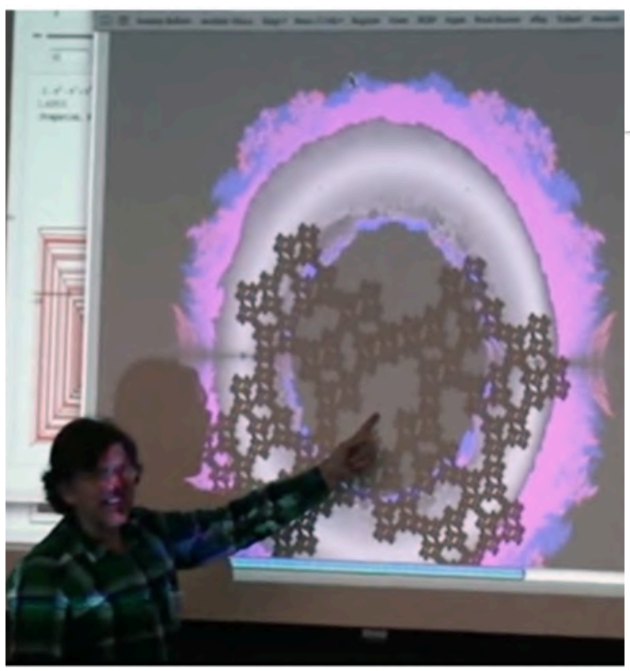
Bill Thurston

(1946 – 2012)

Thurston’s last paper

ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008



Bill Thurston

(1946 - 2012)

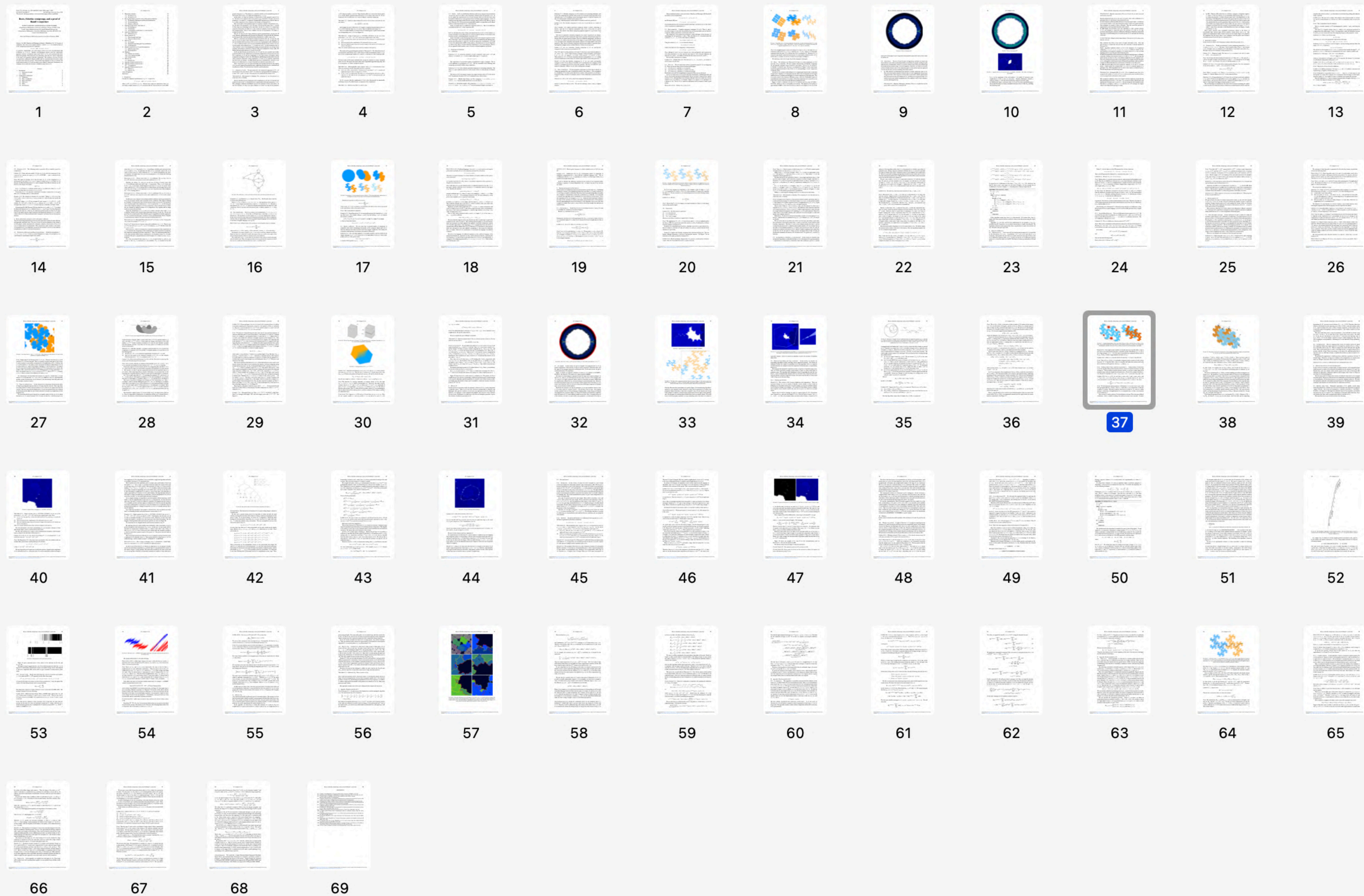
Ergod. Th. & Dynam. Sys. (First published online 2016), page 1 of 69*
doi:10.1017/etds.2016.17
*Provisional—final page numbers to be inserted when paper edition is published

Roots, Schottky semigroups, and a proof of Bandt's conjecture

DANNY CALEGARI†, SARAH KOCH‡ and ALDEN WALKER†

† *Department of Mathematics, University of Chicago, Chicago, IL 60637, USA*
(e-mail: dannyc@math.uchicago.edu, akwalker@math.uchicago.edu)
‡ *Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA*
(e-mail: kochsc@umich.edu)

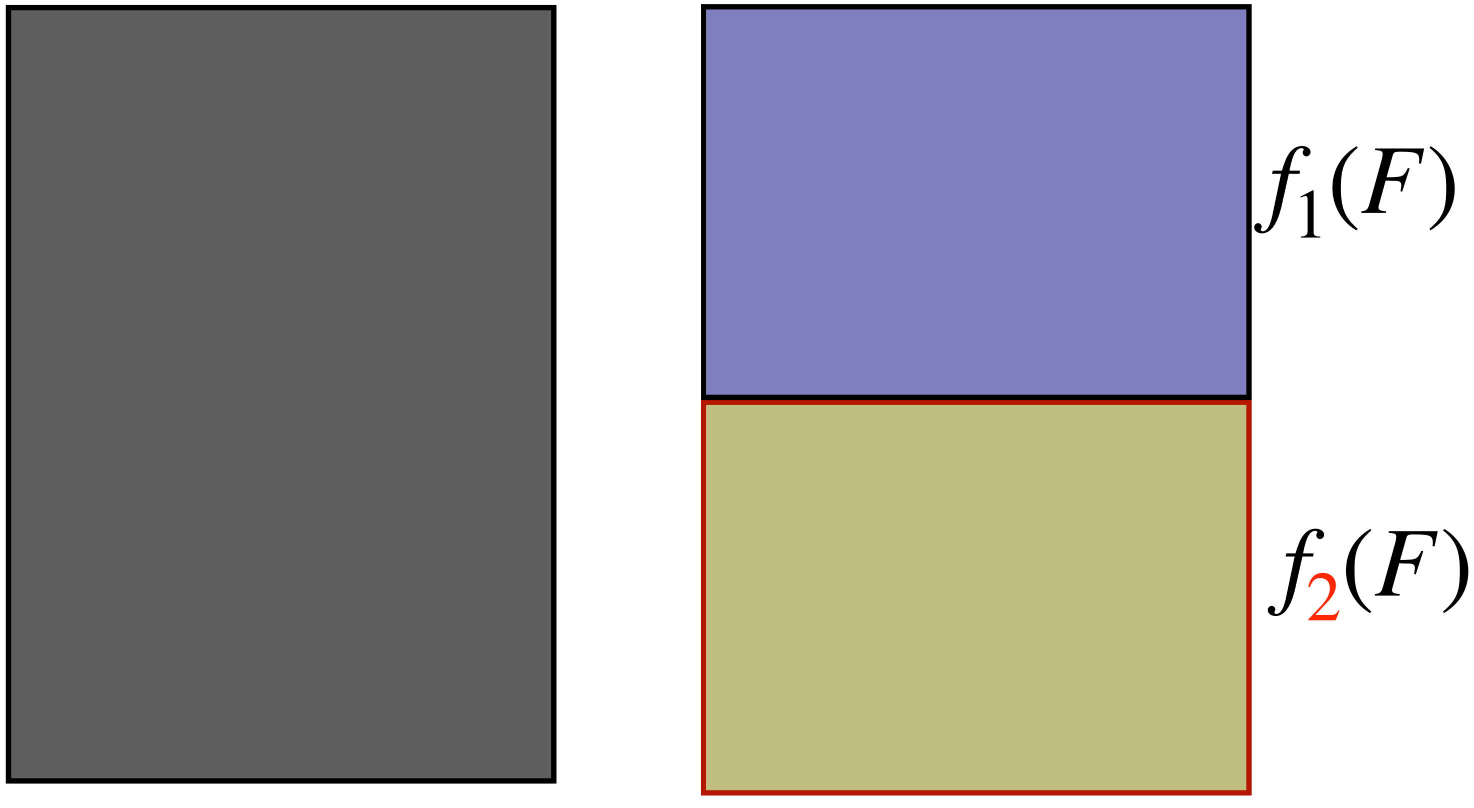
(*Received* 10 *January* 2015 *and accepted in revised form* 25 *January* 2016)


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60
61
62
63
64
65
66
67
68
69

self-similar sets

$$F = f_1(F) \cup f_2(F)$$

$$f_1(z) := 1 + c_1 \cdot z$$
$$f_2(z) := 1 + c_2 \cdot z$$

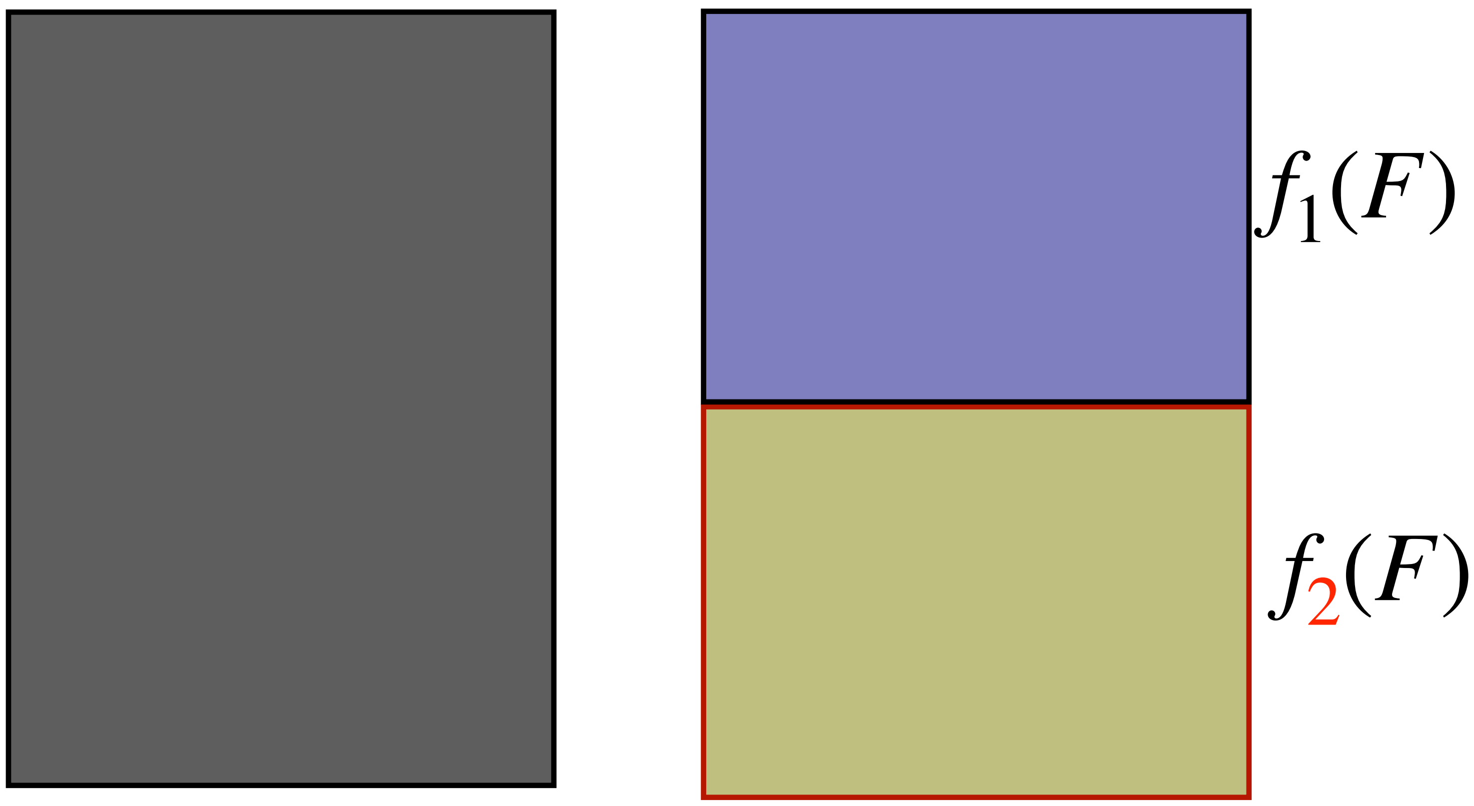


$$c_1 = i/\sqrt{2} \qquad c_2 = -i/\sqrt{2}$$

self-similar sets

$$F = f_1(F) \cup f_2(F)$$

$$f_1(z) := 1 + c_1 \cdot z$$
$$f_2(z) := 1 + c_2 \cdot z$$



$$A := \{c_1, c_2\}$$

binary alphabet

$$c_1 = i/\sqrt{2} \qquad c_2 = -i/\sqrt{2}$$

$$F := f_1(F) \cup f_2(F)$$

self-similar set

Complex-parametric families of self-similar sets

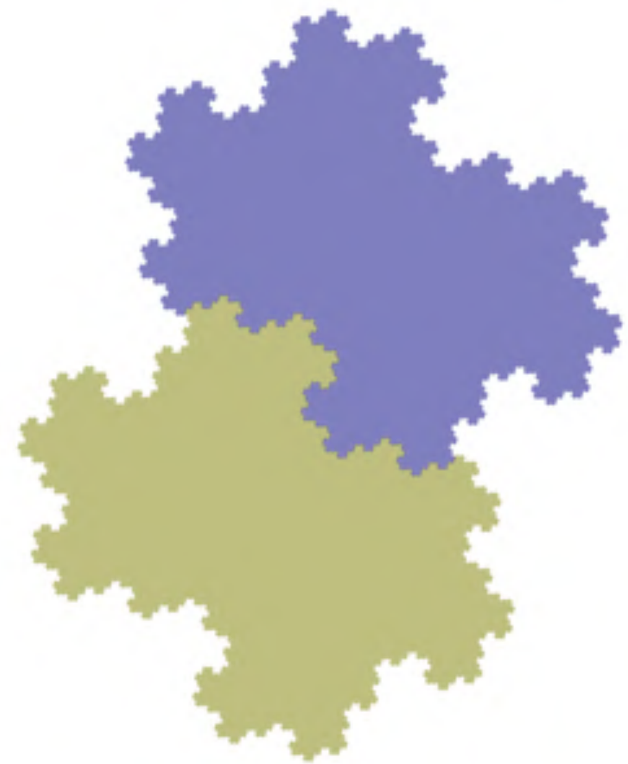
$$f_1(z) := 1 + c_1(c) \cdot z$$
$$f_2(z) := 1 + c_2(c) \cdot z$$

$$A(c) := \{c_1(c), c_2(c)\}$$

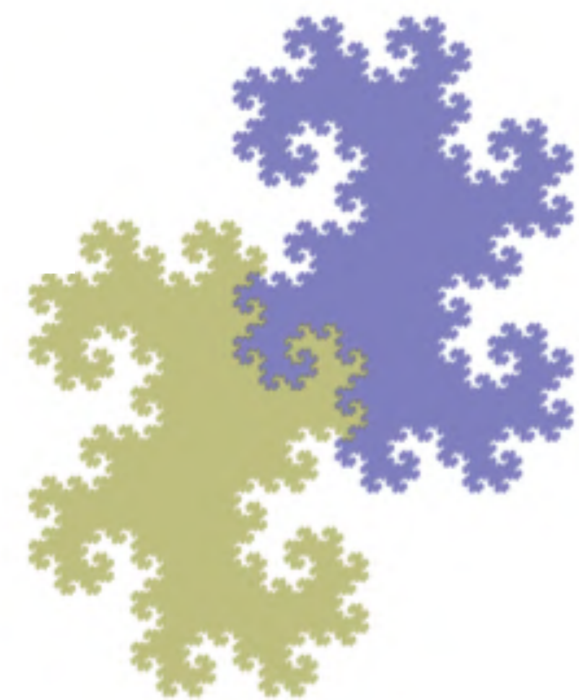
$$c_1(c) := c$$
$$c_2(c) := -c$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$

self-similar set



$c = (1 + i\sqrt{7})/4$



$F\{c, -c\}$

$c = (1 + i)/2$

Complex-parametric families of self-similar sets

$$f_1(z) := 1 + c_1(c) \cdot z$$
$$f_2(z) := 1 + c_2(c) \cdot z$$

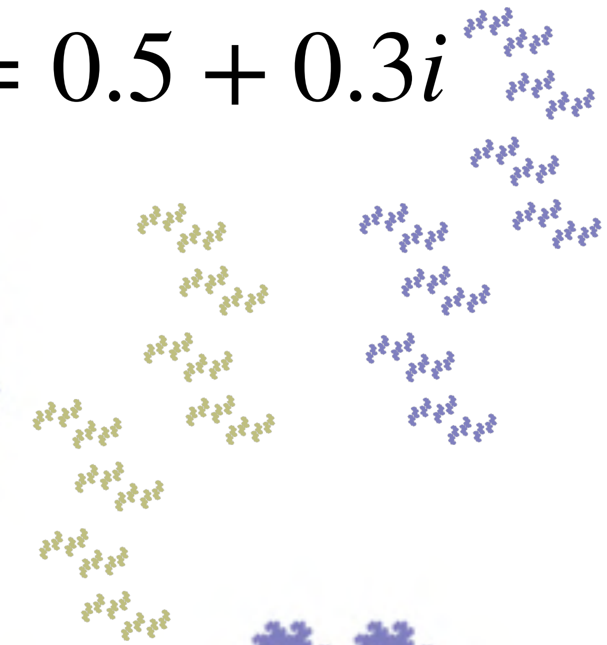
$$A(c) := \{c_1(c), c_2(c)\}$$

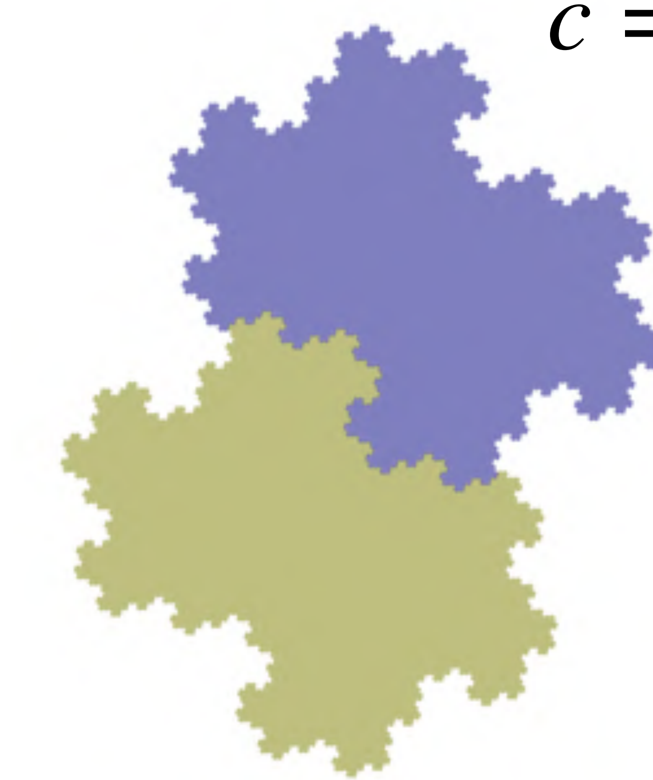
$$c_1(c) := c$$
$$c_2(c) := -c$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$

self-similar set

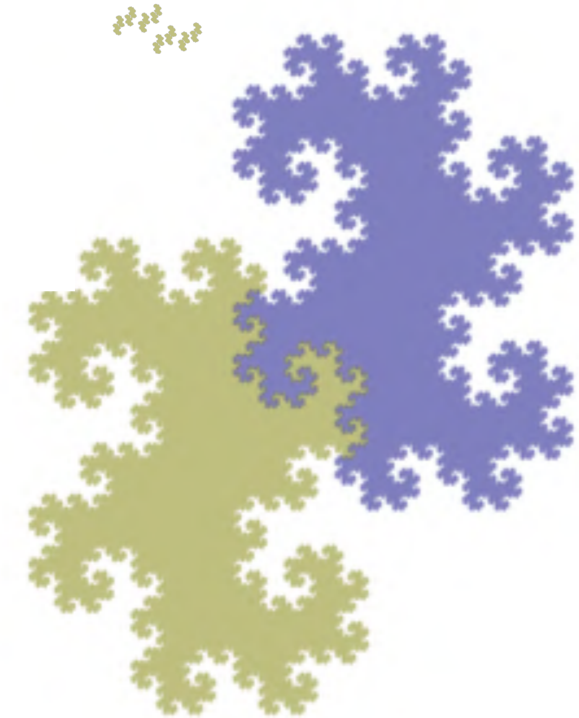
$c = 0.5 + 0.3i$





$c = (1 + i\sqrt{7})/4$

$F\{c, -c\}$



$c = (1 + i)/2$

Physica 15D (1985) 421–432
North-Holland, Amsterdam

A MANDELBROT SET FOR PAIRS OF LINEAR MAPS†

M.F. BARNSLEY
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

and

A.N. HARRINGTON
Department of Mathematical Sciences, Loyola University, 6525 N. Sheridan Road, Chicago, Illinois 60626, USA

Received 26 July 1984

The set of points $A(s) = \{\pm 1 \pm s \pm s^2 \pm s^3 \pm \ldots$ for all sequences of $+$ and $-\}$, where $s \in \mathbf{C}$ and $|s| < 1$, is generically a fractal. For example $A(1/3)$ is the classical Cantor set and $A(\frac{1}{2} + i/2)$ is a dragon curve. This set of fractals can be classified in terms of an associated Mandelbrot set $D = \{s \in \mathbf{C}: |s| < 1,\ A(s) \text{ is disconnected}\}$. The structure of D and its boundary are investigated.

1. Introduction

The discovery [1] of the Mandelbrot set M for the iterated complex polynomial $z^2 - s$ has generated considerable research activity [2, 3], especially because of its relation to cascades of bifurcations and universal phenomena [4]. In this paper, we describe an analogous set associated with linear mappings.

Our motivations are best explained starting from M, and so we characterize the Julia set [5] $J(s)$ for $z^2 - s$. For almost any $s \in \mathbf{C}$, if we start at almost any point $z \in \mathbf{C}$ and calculate the limits of all sequences of compositions of the two maps

$$w_+(z) = \sqrt{z+s},$$
$$w_-(z) = -\sqrt{z+s},$$

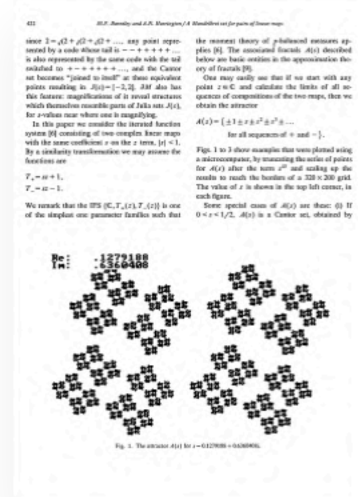
then we obtain the attractor

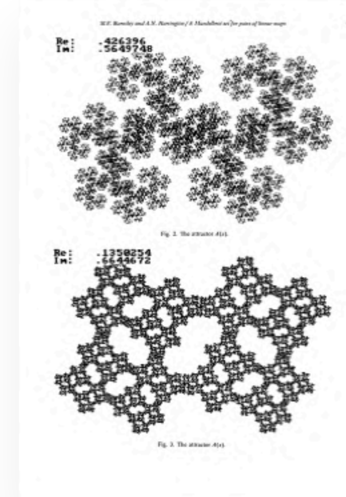
$$J(s) = \{\pm\sqrt{(s \pm \sqrt{(s \pm \sqrt{(s \pm \ldots}}} \quad \text{for all sequences of } + \text{ and } -\}.$$

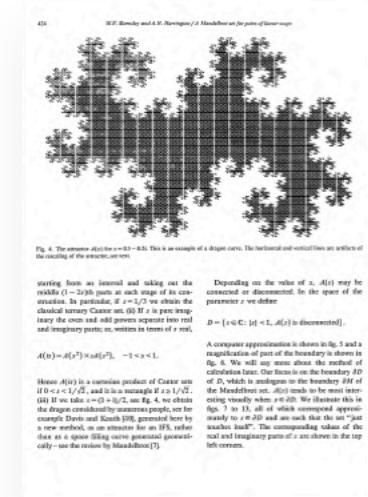
†Supported in part by NSF grant DMS-8401609.

In the nomenclature of Barnsley and Demko [6], $\{\mathbf{C} \cup \{\infty\}, w_+(z), w_-(z)\}$ is an iterated function system (IFS). $J(s)$ is an attractor for the IFS.

M is the set of parameter values $s \in \mathbf{C}$ such that $J(s)$ is connected. Since $J(s)$ is either connected or totally disconnected, M is the complement of the set of parameter values $s \in \mathbf{C}$ such that $J(s)$ is disconnected. Parameter space, with M marked on it, is a map of fractals [7] in the sense that each point in the space corresponds to a single Julia set $J(s)$; and knowing where one is on this map provides good information about the structure of the corresponding $J(s)$. The boundary ∂M of M is of special interest; for example, its intersection with the real line gives the parameter values at which bifurcations occur for the iterated real map $x^2 - s$, s real. ∂M appears to contain all of the values of s such that $J(s)$ is a fractal tree (i.e., such that $J(s)$ is connected and $\bar{\mathbf{C}} \setminus J(s)$ has only one component); it is connected but may not be locally connected [12]; and it represents the set on which $J(s)$ is "just connected." We illustrate the latter statement. For $s > 2$ $J(s)$ is a Cantor set contained in the real line [8]. However, when $s = 2$,


422


423


424

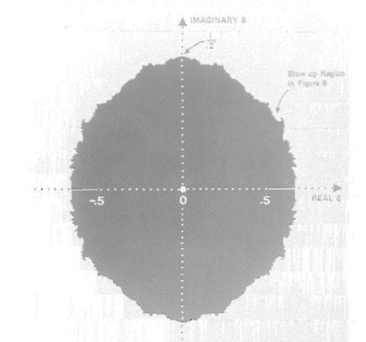


Fig. 5. The Mandelbrot set D is approximated in black. See text.

2. Some analysis of $A(s)$ and ∂D

We formalize the notation for points in $A(s)$. Let Ω be the space of all sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ where $\omega_j = +1$ or -1 (abbreviated $+, -$). Let $g_s: \Omega \to A(s)$ with

$$g_s(\omega) = \omega_0 + \omega_1 s + \omega_2 s^2 + \cdots.$$

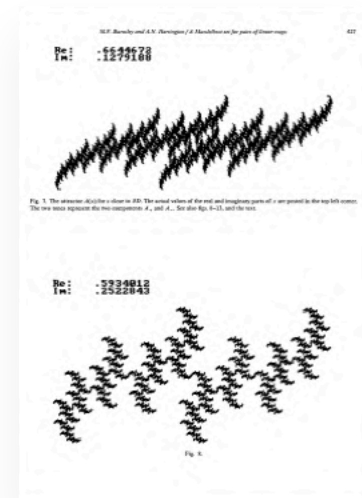
When s is fixed we will shorten our notation to g and A. With the usual cylinder set topology on Ω, $g: \Omega \to A$ is continuous [6].

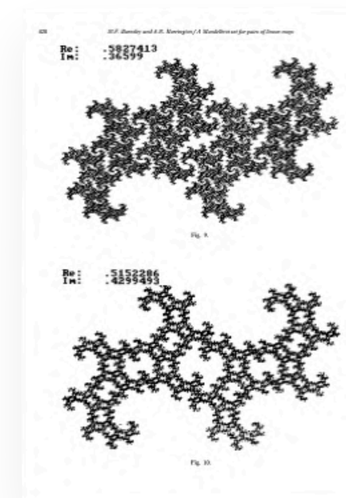
To better understand the geometry of $A(s)$ it is instructive to consider the generation of the set in relation to the fixed points and cycles of $T_\pm$. The fixed points of $T_\pm$ are $\pm 1/(1-s)$. One can imagine shrinking the attractor $A(s)$ by a factor s (denoting rescaling by $|s|$ and rotation by $\arg(s)$) about either fixed point. We denote the images by $A_+ = T_+(A)$ and $A_- = T_-(A)$. Then

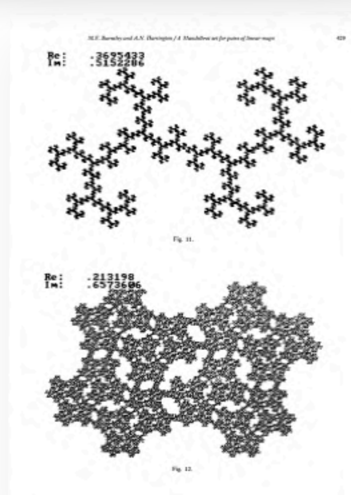
$$A = A_+ \cup A_-$$

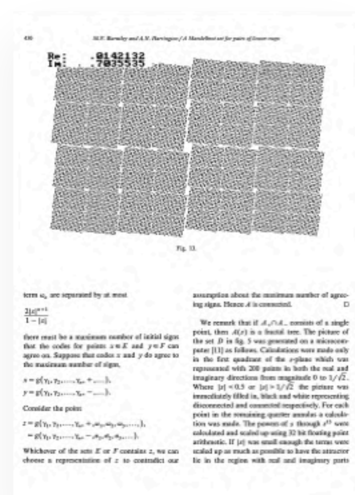
Thus, A is generally a fractal, as defined by


426


427


428


429


430


431


432

Physica 15D (1985) 421–432
North-Holland, Amsterdam

A MANDELBROT SET FOR PAIRS OF LINEAR MAPS†

M.F. BARNSLEY
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

and

A.N. HARRINGTON
Department of Mathematical Sciences, Loyola University, 6525 N. Sheridan Road, Chicago, Illinois 60626, USA

Received 26 July 1984

The set of points $A(s) = \{\pm 1 \pm s \pm s^2 \pm s^3 \pm \ldots$ for all sequences of $+$ and $-\}$, where $s \in \mathbb{C}$ and $|s| < 1$, is generically a fractal. For example $A(1/3)$ is the classical Cantor set and $A(\frac{1}{2} + i/2)$ is a dragon curve. This set of fractals can be classified in terms of an associated Mandelbrot set $D = \{s \in \mathbb{C}: |s| < 1, A(s)$ is disconnected$\}$. The structure of D and its boundary are investigated.

1. Introduction

The discovery [1] of the Mandelbrot set M for the iterated complex polynomial $z^2 - s$ has generated considerable research activity [2, 3], especially because of its relation to cascades of bifurcations and universal phenomena [4]. In this paper, we describe an analogous set associated with linear mappings.

Our motivations are best explained starting from M, and so we characterize the Julia set [5] $J(s)$ for $z^2 - s$. For almost any $s \in \mathbb{C}$, if we start at almost any point $z \in \mathbb{C}$ and calculate the limits of all sequences of compositions of the two maps

$$w_+(z) = \sqrt{z+s},$$
$$w_-(z) = -\sqrt{z+s},$$

then we obtain the attractor

$$J(s) = \{\pm\sqrt{(s \pm \sqrt{(s \pm \sqrt{(s \pm \ldots}}} \quad \text{for all sequences of } + \text{ and } -\}.$$

†Supported in part by NSF grant DMS-8401609.

In the nomenclature of Barnsley and Demko [6], $\{\mathbb{C} \cup \{\infty\}, w_+(z), w_-(z)\}$ is an iterated function system (IFS). $J(s)$ is an attractor for the IFS.

M is the set of parameter values $s \in \mathbb{C}$ such that $J(s)$ is connected. Since $J(s)$ is either connected or totally disconnected, M is the complement of the set of parameter values $s \in \mathbb{C}$ such that $J(s)$ is disconnected. Parameter space, with M marked on it, is a map of fractals [7] in the sense that each point in the space corresponds to a single Julia set $J(s)$; and knowing where one is on this map provides good information about the structure of the corresponding $J(s)$. The boundary ∂M of M is of special interest; for example, its intersection with the real line gives the parameter values at which bifurcations occur for the iterated real map $x^2 - s$, s real. ∂M appears to contain all of the values of s such that $J(s)$ is a fractal tree (i.e., such that $J(s)$ is connected and $\bar{\mathbb{C}} \setminus J(s)$ has only one component); it is connected but may not be locally connected [12]; and it represents the set on which $J(s)$ is "just connected." We illustrate the latter statement. For $s > 2$ $J(s)$ is a Cantor set contained in the real line [8]. However, when $s = 2$,

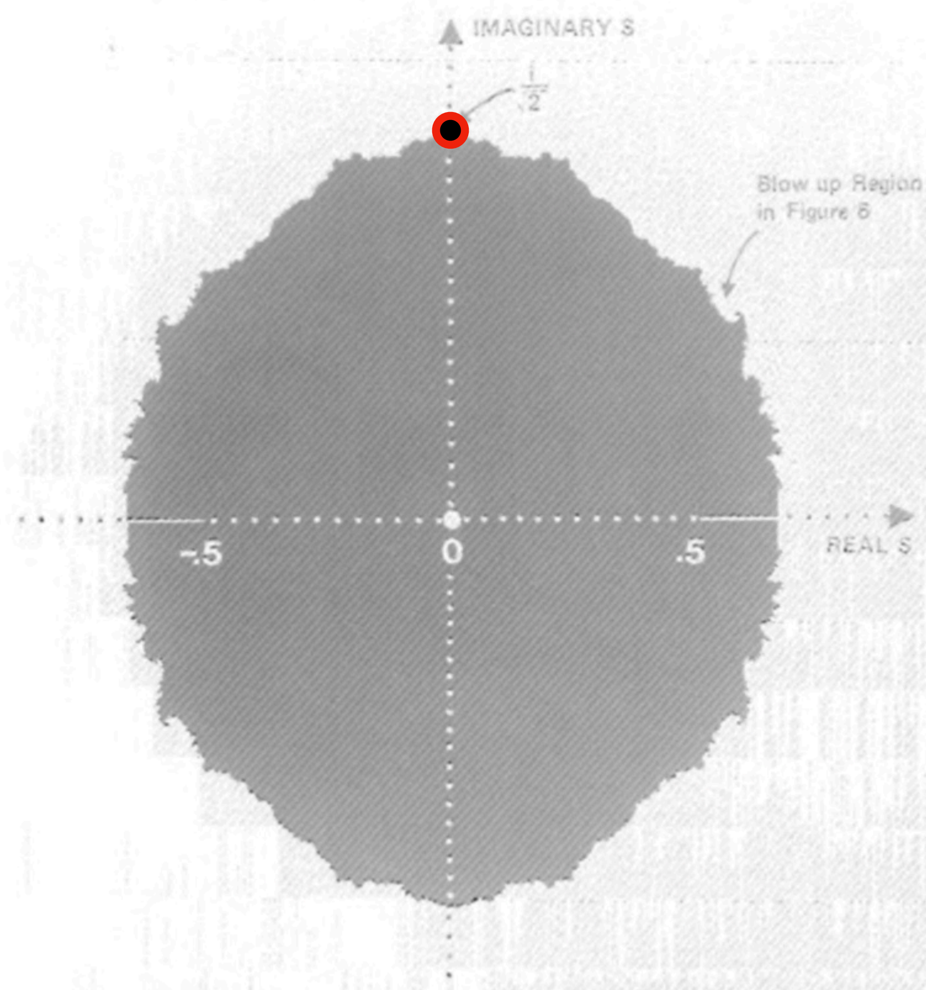


Fig. 5. The Mandelbrot set D is approximated in black. See text.

2. Some analysis of $A(s)$ and ∂D

We formalize the notation for points in $A(s)$. Let Ω be the space of all sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ where $\omega_j = +1$ or -1 (abbreviated $+, -$). Let g_s: $\Omega \to A(s)$ with

$$g_s(\omega) = \omega_0 + \omega_1 s + \omega_2 s^2 + \cdots.$$

When s is fixed we will shorten our notation to g and A. With the usual cylinder set topology on Ω, g: $\Omega \to A$ is continuous [6].

To better understand the geometry of $A(s)$ it is instructive to consider the generation of the set in relation to the fixed points and cycles of $T_\pm$. The fixed points of $T_\pm$ are $\pm 1/(1-s)$. One can imagine shrinking the attractor $A(s)$ by a factor s (denoting rescaling by $|s|$ and rotation by $\arg(s)$) about either fixed point. We denote the images by $A_+ = T_+(A)$ and $A_- = T_-(A)$. Then

$$A = A_+ \cup A_-$$

Thus, A is generally a fractal, as defined by

Complex-parametric families of self-similar sets

$$f_1(z) := 1 + c_1(c) \cdot z$$

$$f_2(z) := 1 + c_2(c) \cdot z$$

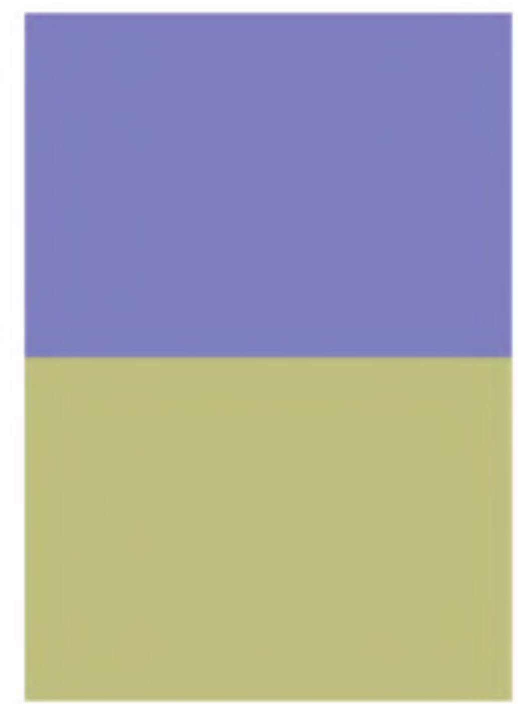
$$A(c) := \{c_1(c), c_2(c)\}$$

$$c_1(c) := c$$

$$c_2(c) := -c$$

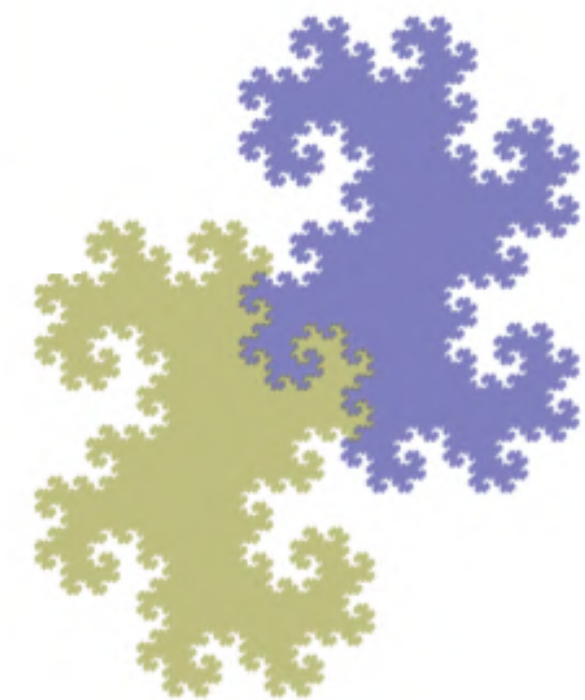
$$F_A = f_1(F_A) \cup f_2(F_A)$$

self-similar set



$$c = i/\sqrt{2}$$

$$F\{c, -c\}$$



$$c = (1+i)/2$$

Complex-parametric families of self-similar sets

$$f_1(z) := 1 + c_1(c) \cdot z$$

$$f_2(z) := 1 + c_2(c) \cdot z$$

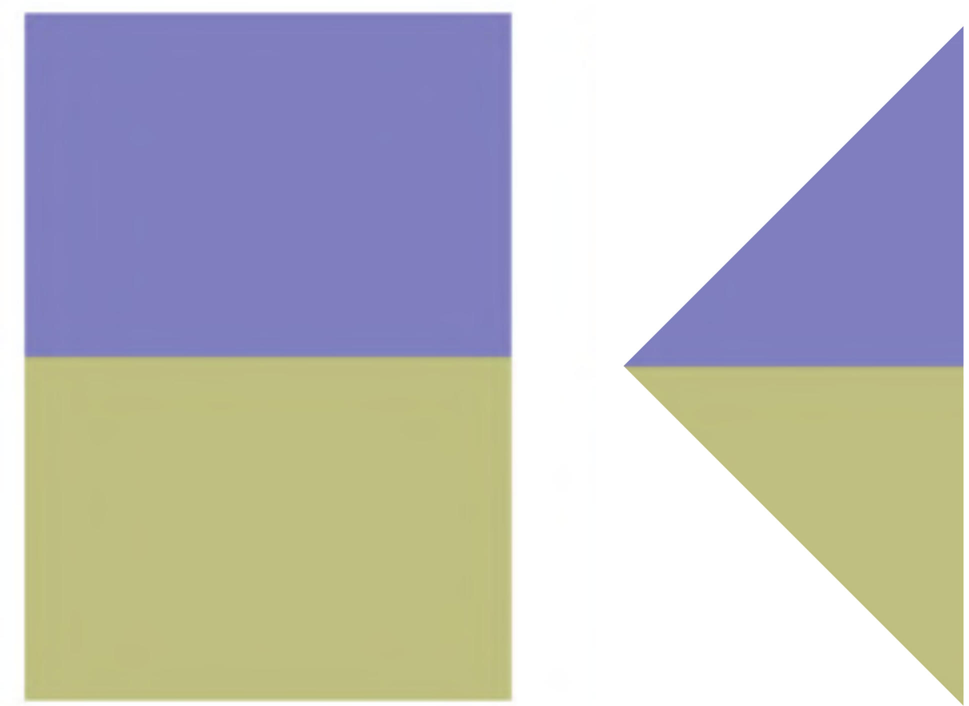
$$A(c) := \{c_1(c), c_2(c)\}$$

$$c_1(c) := c$$

$$c_2(c) := c^*$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$

self-similar set



$c = i/\sqrt{2}$ $\qquad$ $c = (i-1)/2$

$F\{c, c^*\}$

$F\{c,$ ~~$-c$~~$\}$

$c := x + iy$

$c^* := x - iy$

HYPERBOLIC ITERATED FUNCTION SYSTEMS AND APPLICATIONS

A THESIS

Presented to

The Faculty of the Division of Graduate Studies

by

Douglas Patten Hardin

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mathematics

Georgia Institute of Technology

December, 1985

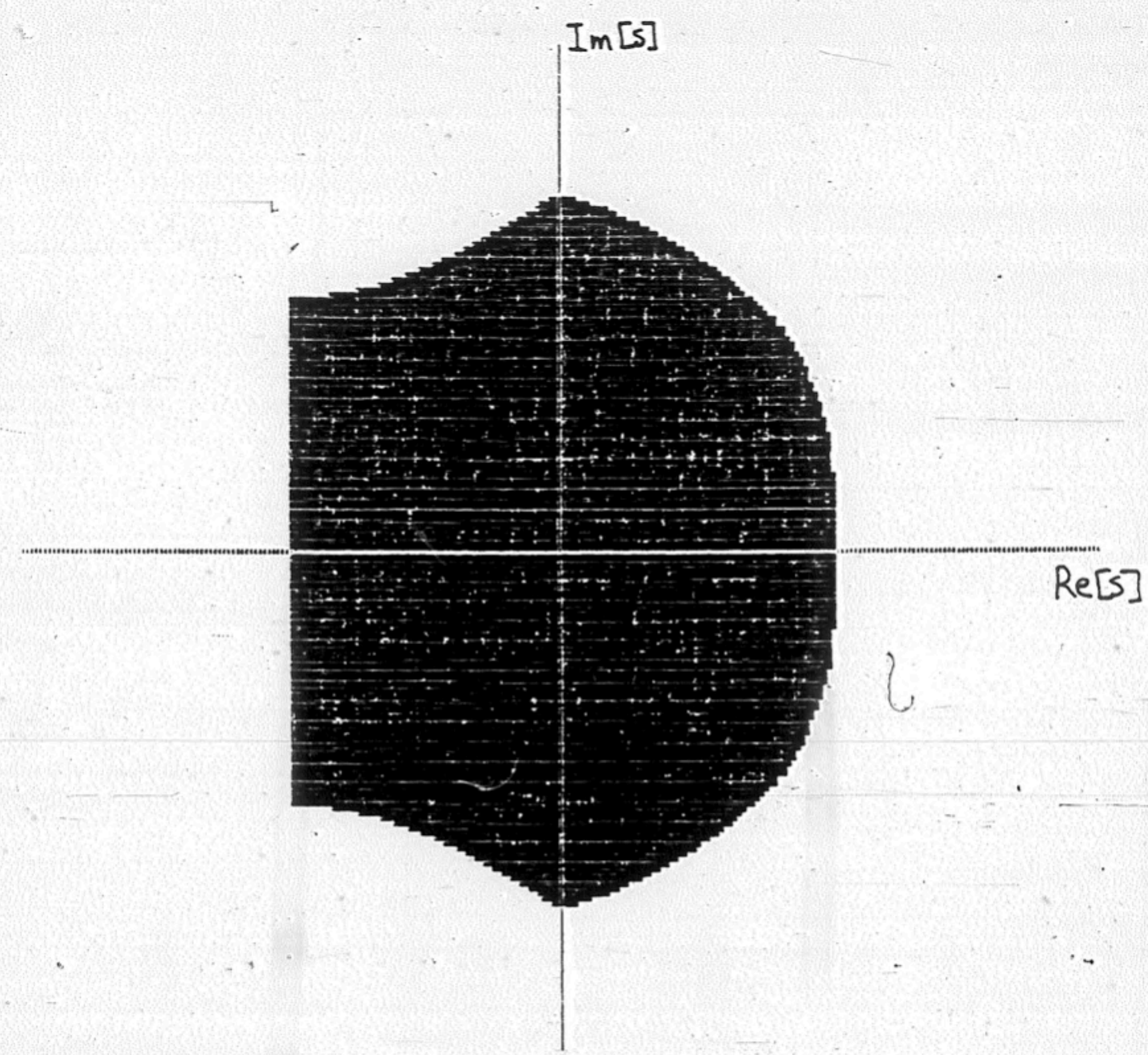


Figure 2.8. The Mandelbrot Set G.

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 2, October 1989

A MANDELBROT SET WHOSE BOUNDARY IS PIECEWISE SMOOTH

M. F. BARNSLEY AND D. P. HARDIN

ABSTRACT. It is proved that the Mandelbrot set associated with the pair of maps $w_{1,2}: \mathbf{C} \to \mathbf{C}$, $w_1(z) = sz + 1$, $w_2(z) = s^* z - 1$, with parameter $s \in \mathbf{C}$, is connected and has piecewise smooth boundary.

INTRODUCTION

The discovery [1] of the Mandelbrot set M for the iterated complex polynomial $z^2 + s$ has generated considerable research activity [2, 3], especially because of its relation to cascades of bifurcations and universal phenomena [4].

The Mandelbrot set M consists of those values of $s \in \mathbf{C}$ such that the Julia set $J(s)$ for $z^2 - s$ is connected. Barnsley and Harrington [5] considered an analogous Mandelbrot set D associated with the two affine maps $T_{1,2}: \mathbf{C} \to \mathbf{C}$ defined by

$$T_1(z) = sz + 1, \quad T_2(z) = sz - 1$$

for $s \in \mathbf{C}$ and $|s| < 1$. There is a unique nonempty compact set $A(s)$ which is invariant under T_1 and T_2 (i.e., $T_1(A(s)) \cup T_2(A(s) = A(s))$ [5, 6]. Generically, $A(s)$ is a fractal. D is defined to be the set of $s \in \mathbf{C}$, $|s| < 1$ for which $A(s)$ is disconnected. The boundary of D contains self-similar structures (see Figure 2) and appears to be a fractal. It is not known whether D is connected; however, new pictures of this set presented here indicate that it is.

In this paper we study the Mandelbrot set G associated with the two affine maps $w_{1,2}: \mathbf{C} \to \mathbf{C}$ defined by

$$w_1(z) = sz + 1, \quad w_2(z) = s^* z - 1$$

for $s \in \mathbf{C}$ with $|s| < 1$. (Here s^* denotes the conjugate of s.) As in the previous case, there is a unique invariant compact set $A(s)$ which is generically a fractal. Despite the apparent similarity between the two pairs of maps, G is easier to analyze than D. We will show among other things that G is connected and, remarkably, has a piecewise smooth boundary. Pictures of the associated fractals as one travels around the boundary of G are given.

Received by the editors November 23, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 40A99, 51M99.
Research by M. F. Barnsley supported in part by DARPA "Image Compression Project," ONR contract N00014-86-C-0446.

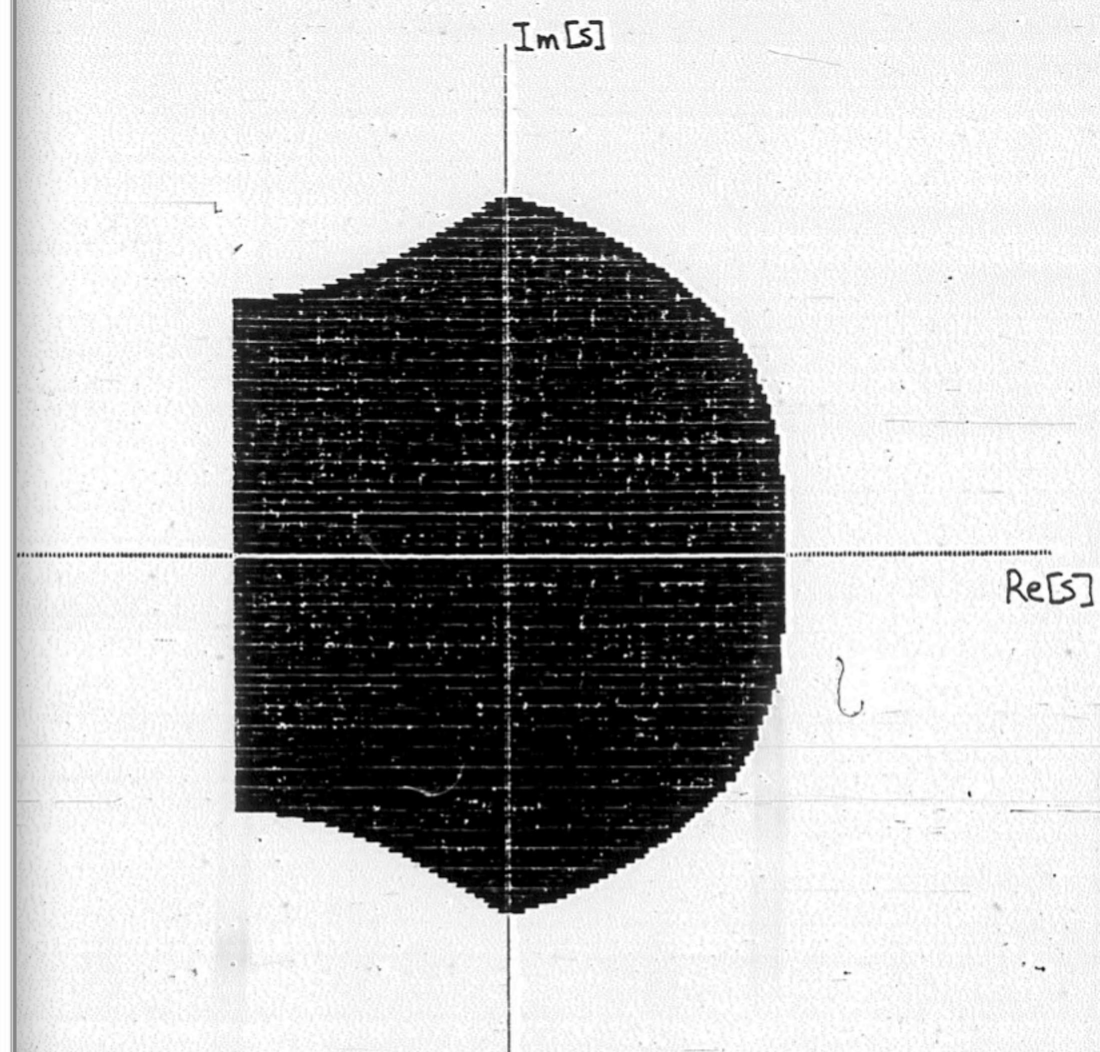


Figure 2.8. The Mandelbrot Set G.

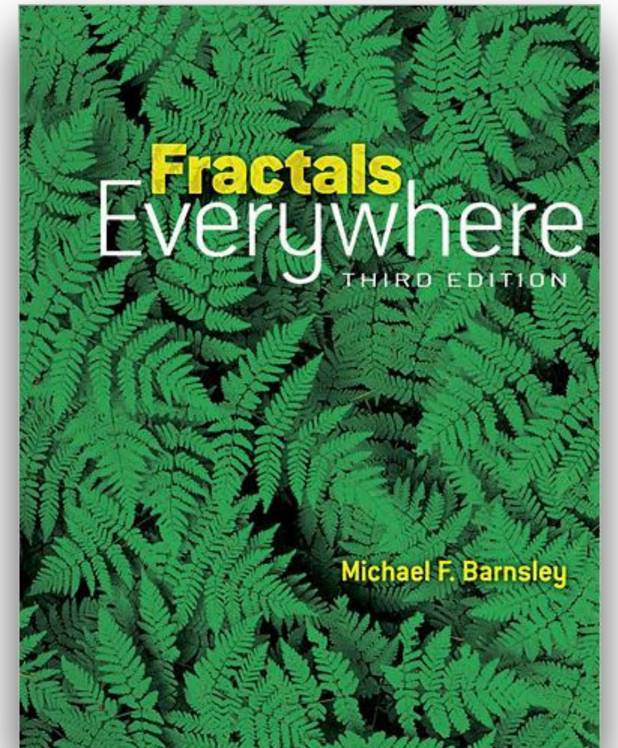
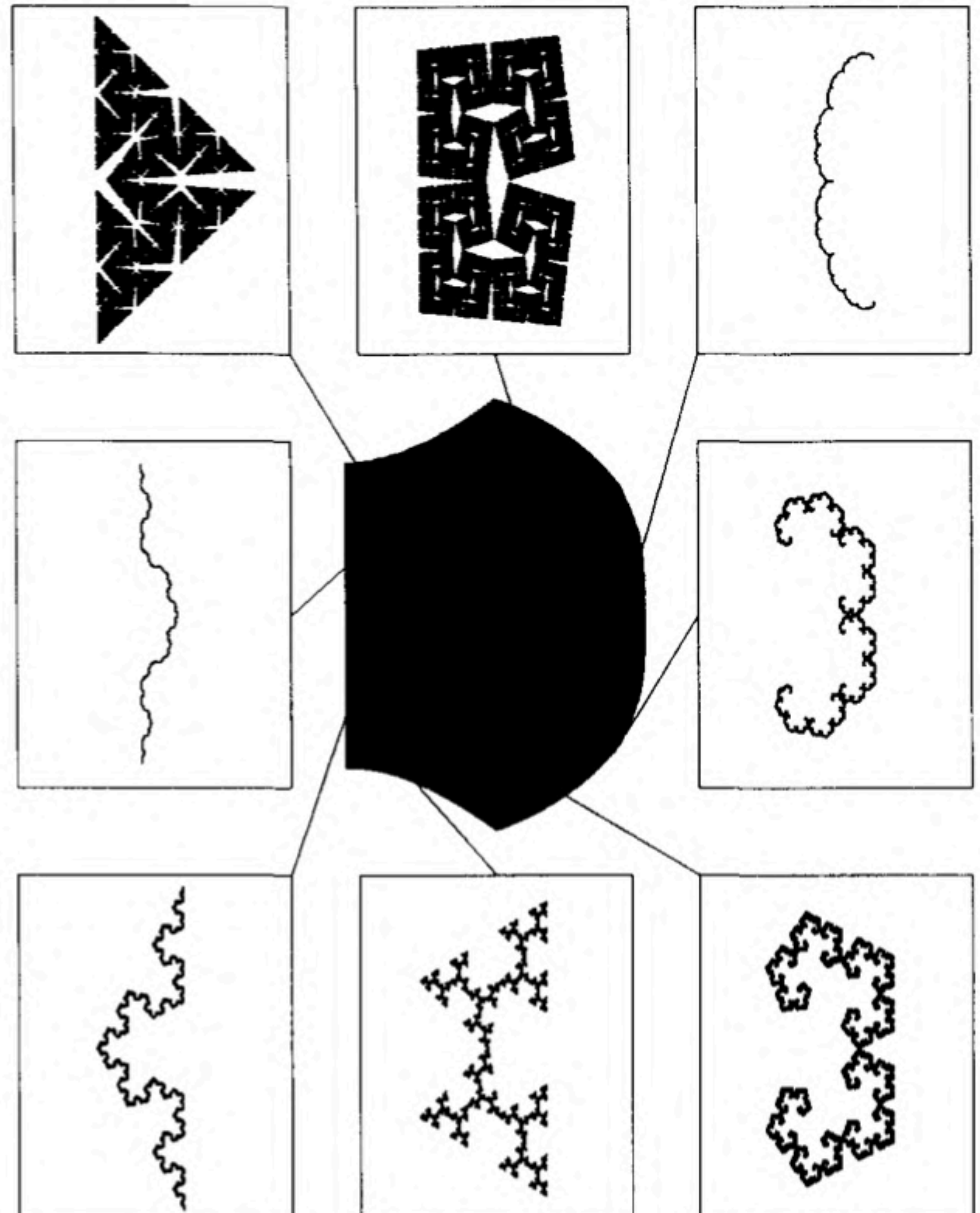


Figure 8.4.1(a) *The complement of the Mandelbrot set $\mathcal{M}_1$ associated with the family of IFS $\{\mathbb{C};\ w_1(z) = \lambda z + 1,\ w_2(z) = \lambda^* z - 1\}$. Points in the complement of the Mandelbrot set are colored black. The boundary of $\mathcal{M}_1$ is smooth and does not reveal much information about the family of fractals which it represents. The figure also shows attractors of the IFS corresponding to various points on the boundary of $\mathcal{M}_1$. What a disappointing map this is!*



"What a disappointing map this is! There are no secret bays, jutting peninsulas, nor ragged rocks in the coastline." -Michael F. Barnsley

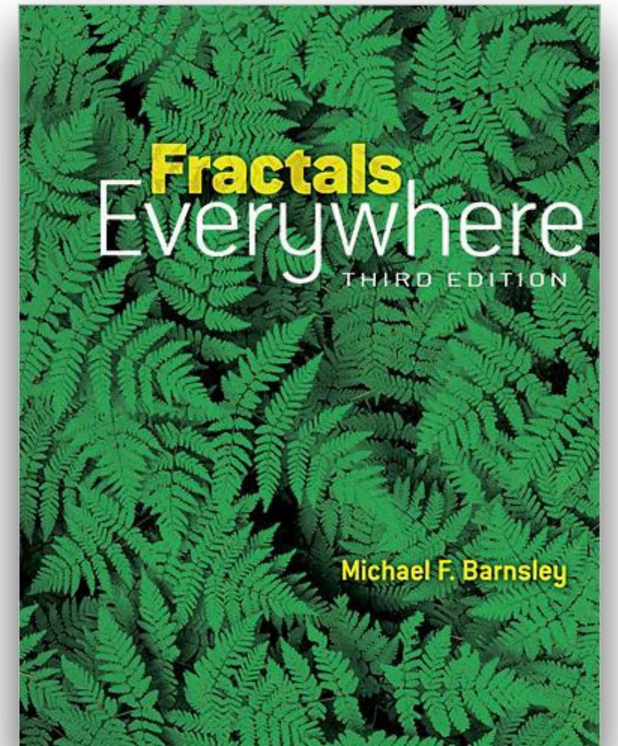
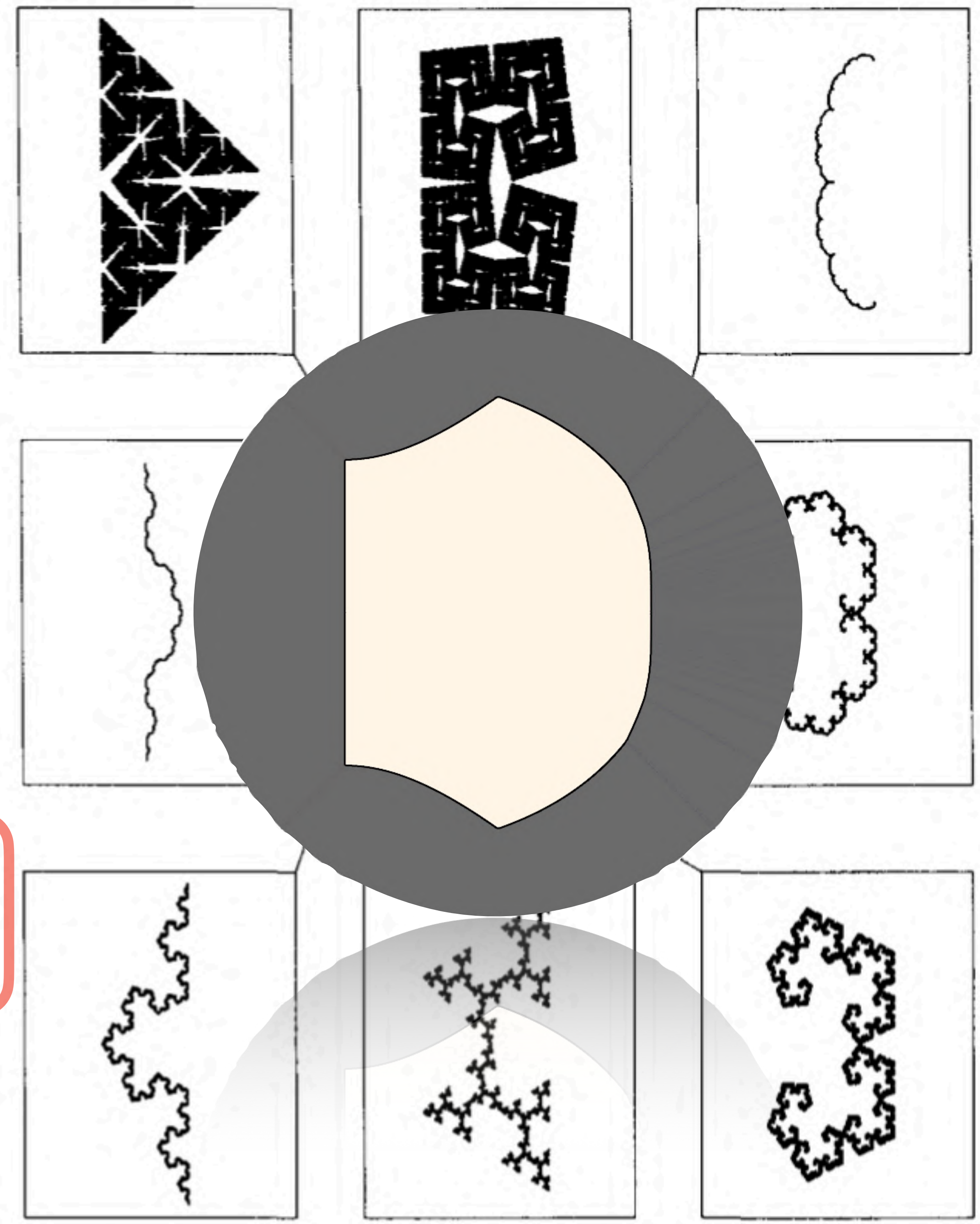
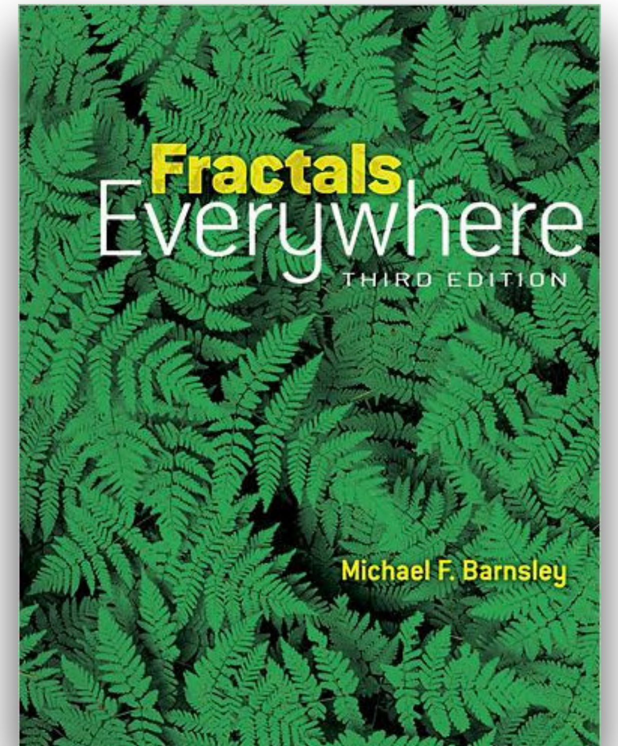
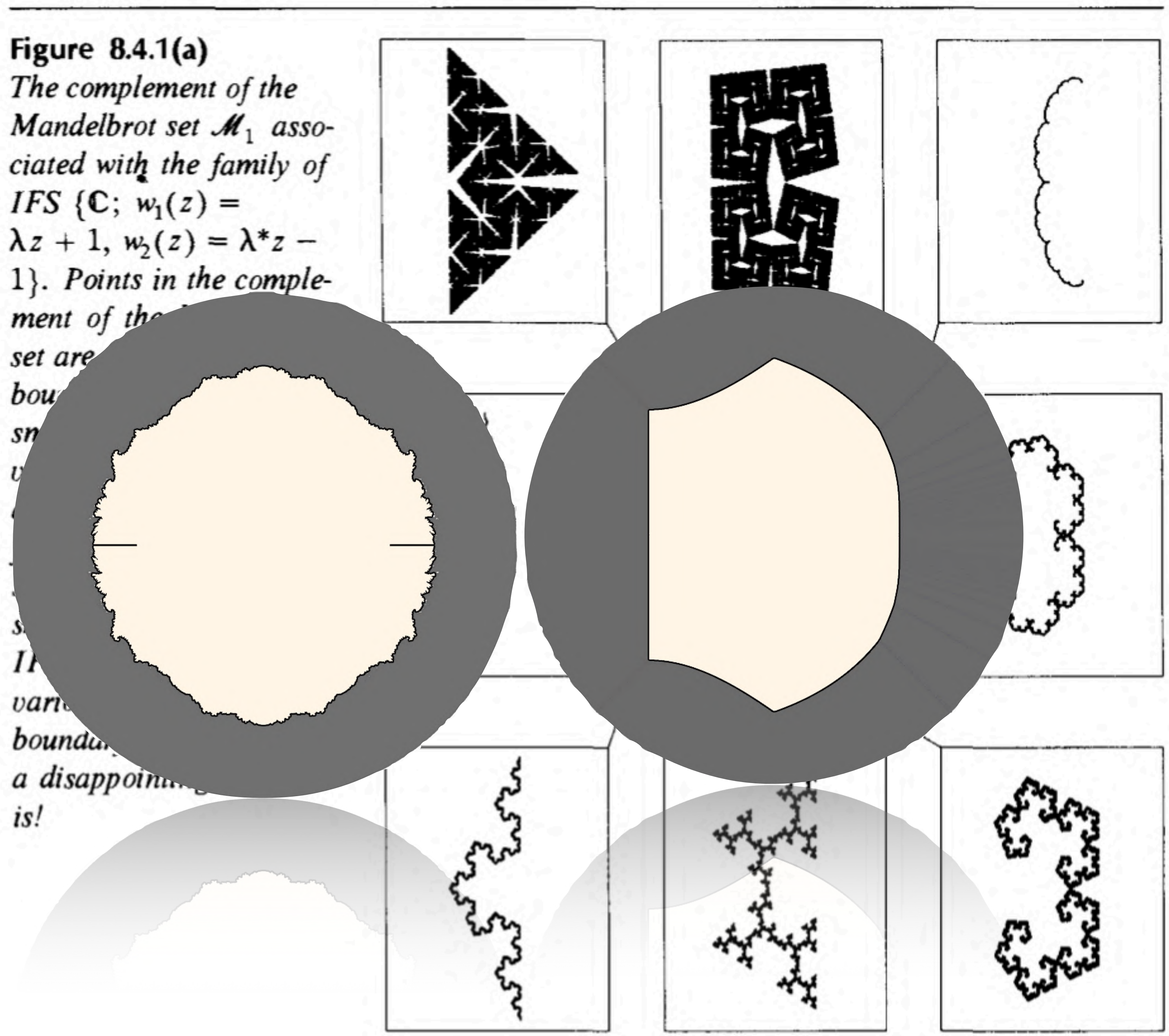


Figure 8.4.1(a) *The complement of the Mandelbrot set $\mathcal{M}_1$ associated with the family of IFS $\{\mathbb{C}; w_1(z) = \lambda z + 1, w_2(z) = \lambda^* z - 1\}$. Points in the complement of the Mandelbrot set are colored black. The boundary of $\mathcal{M}_1$ is smooth and does not reveal much information about the family of fractals which it represents. The figure also shows attractors of the IFS corresponding to various points on the boundary of $\mathcal{M}_1$. What a disappointing map this is!*





"What a disappointing map this is! There are no secret bays, jutting peninsulas, nor ragged rocks in the coastline." -Michael F. Barnsley

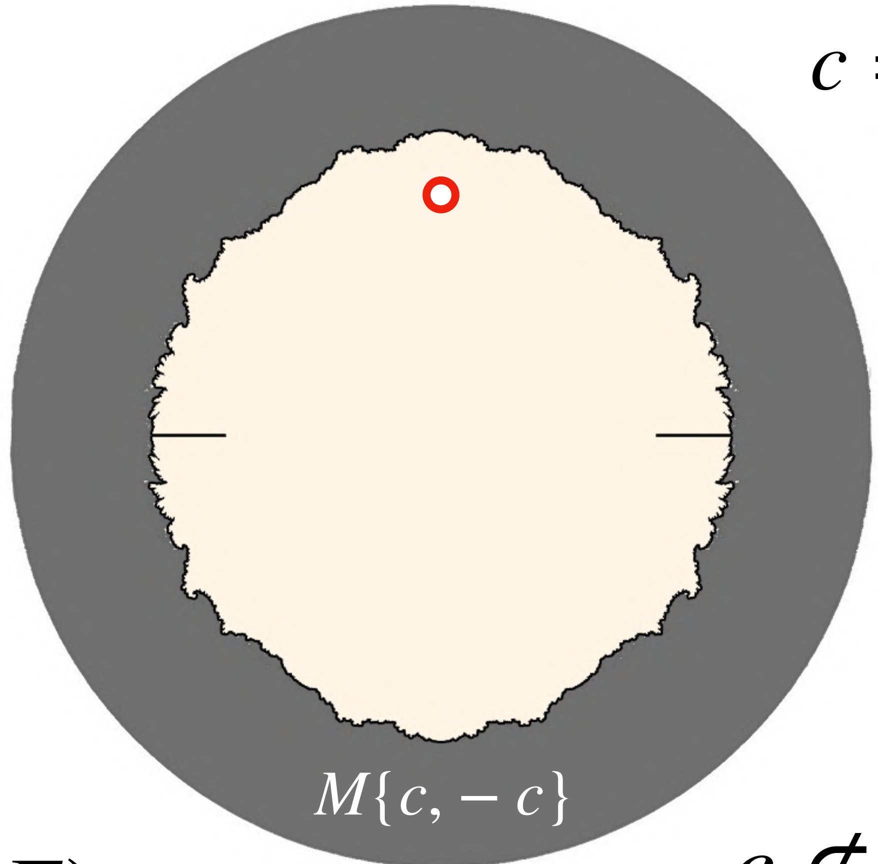


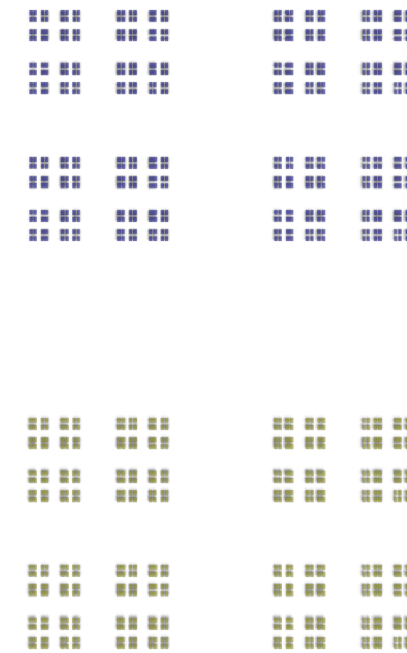
Complex-parametric families of self-similar sets

$c = 3i/5$

$f_1(z) := 1 + c \cdot z$

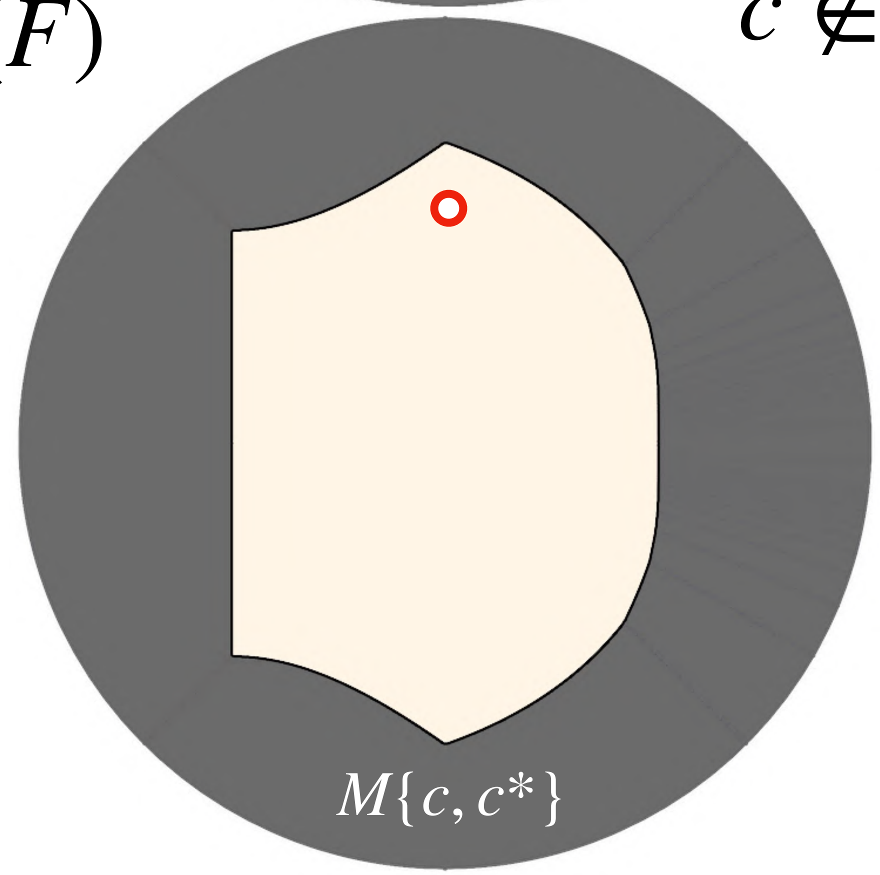
$f_2(z) := 1 - c \cdot z$





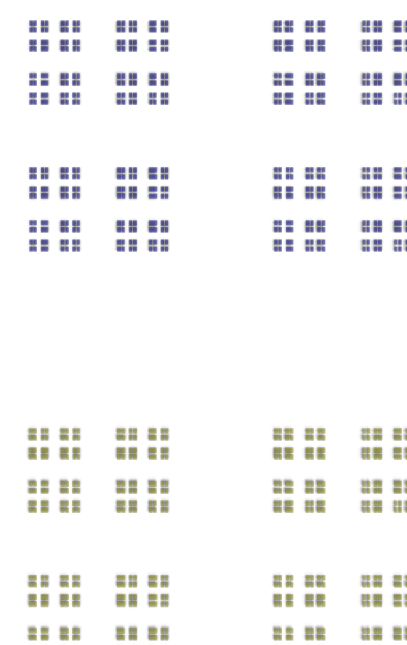
$$F := f_1(F) \cup f_2(F)$$

$c \notin M$ **if** $f_1(F) \cap f_2(F) = \emptyset$



$f_1(z) := 1 + c \cdot z$

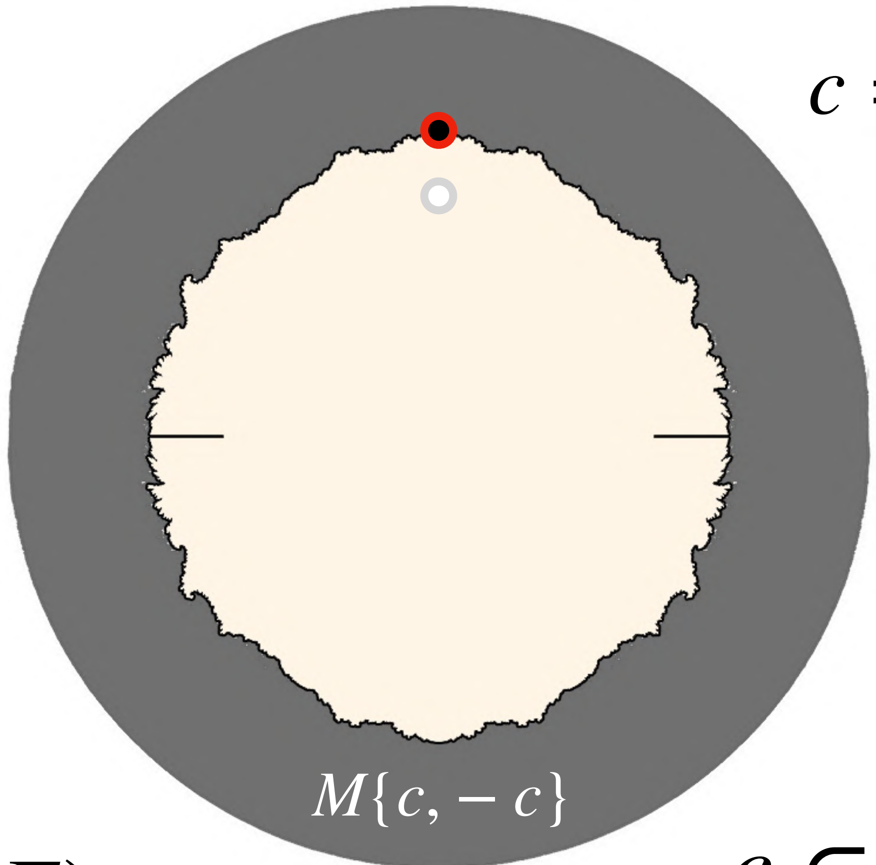
$f_2(z) := 1 + c^* \cdot z$

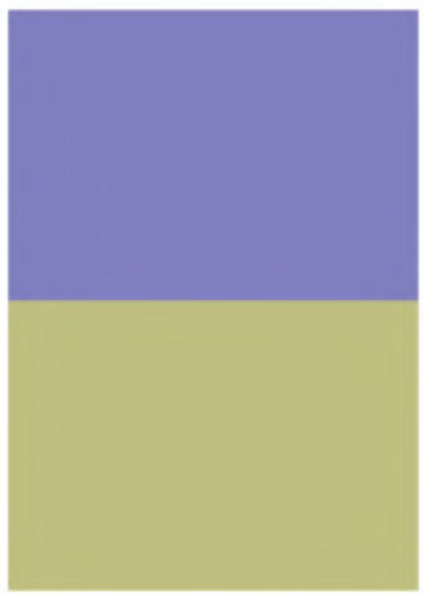


Complex-parametric families of self-similar sets

$c = i/\sqrt{2}$

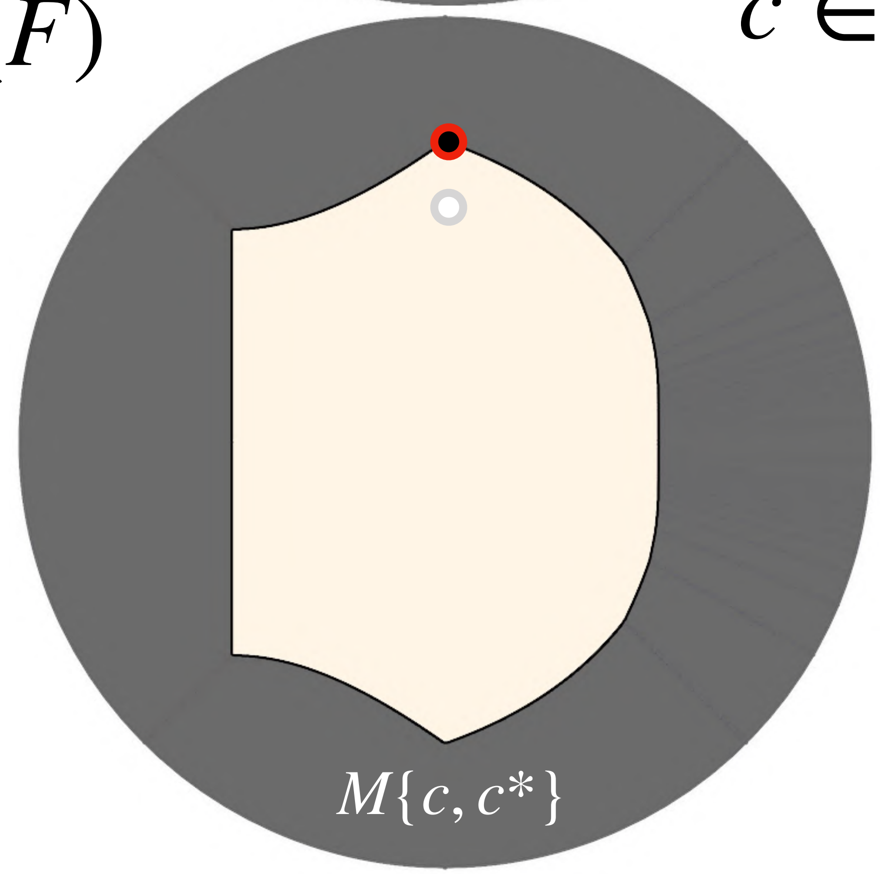
$f_1(z) := 1 + c \cdot z$
$f_2(z) := 1 - c \cdot z$

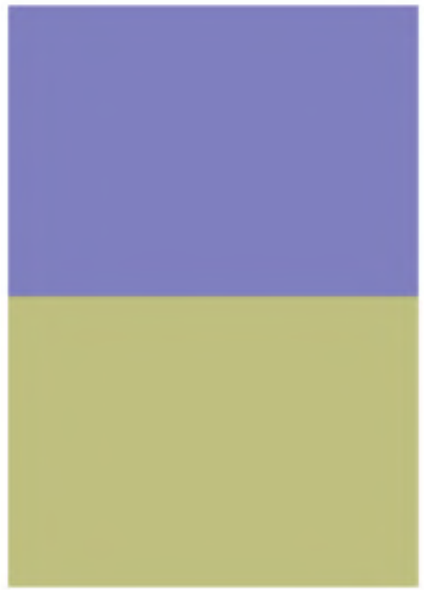




$F := f_1(F) \cup f_2(F)$

$c \in M$ **if** $f_1(F) \cap f_2(F) \neq \varnothing$





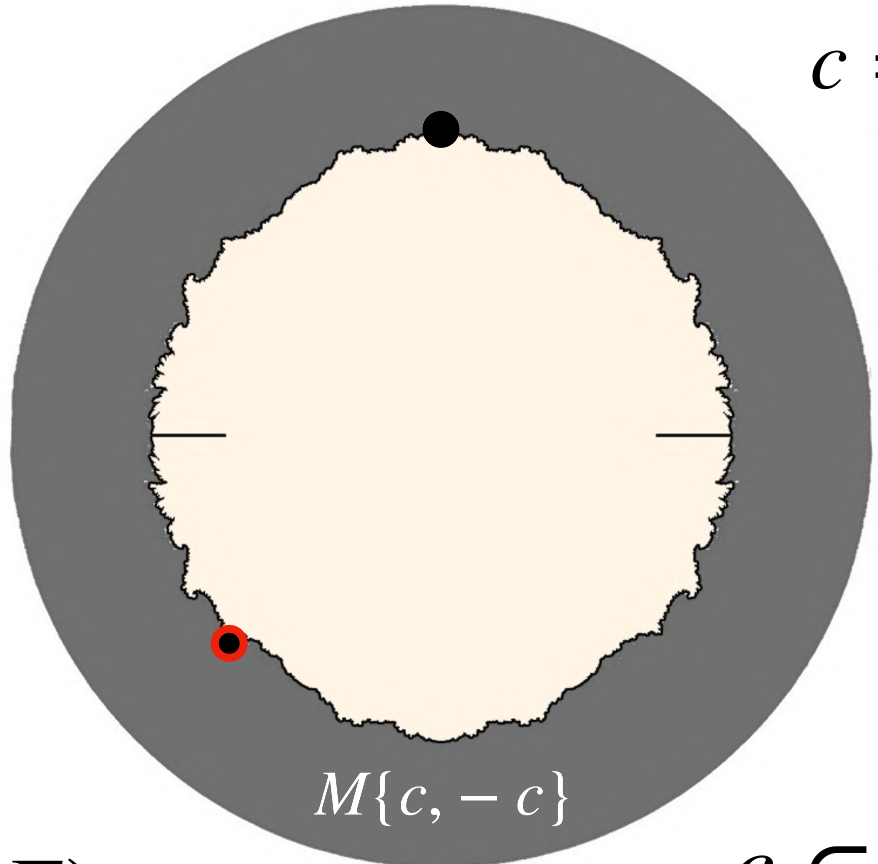
$f_1(z) := 1 + c \cdot z$
$f_2(z) := 1 + c^* \cdot z$

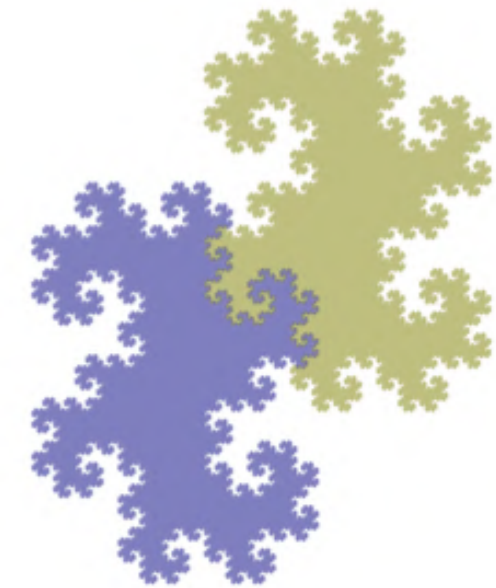
Complex-parametric families of self-similar sets

$c = -1/2 - i/2$

$f_1(z) := 1 + c \cdot z$

$f_2(z) := 1 - c \cdot z$



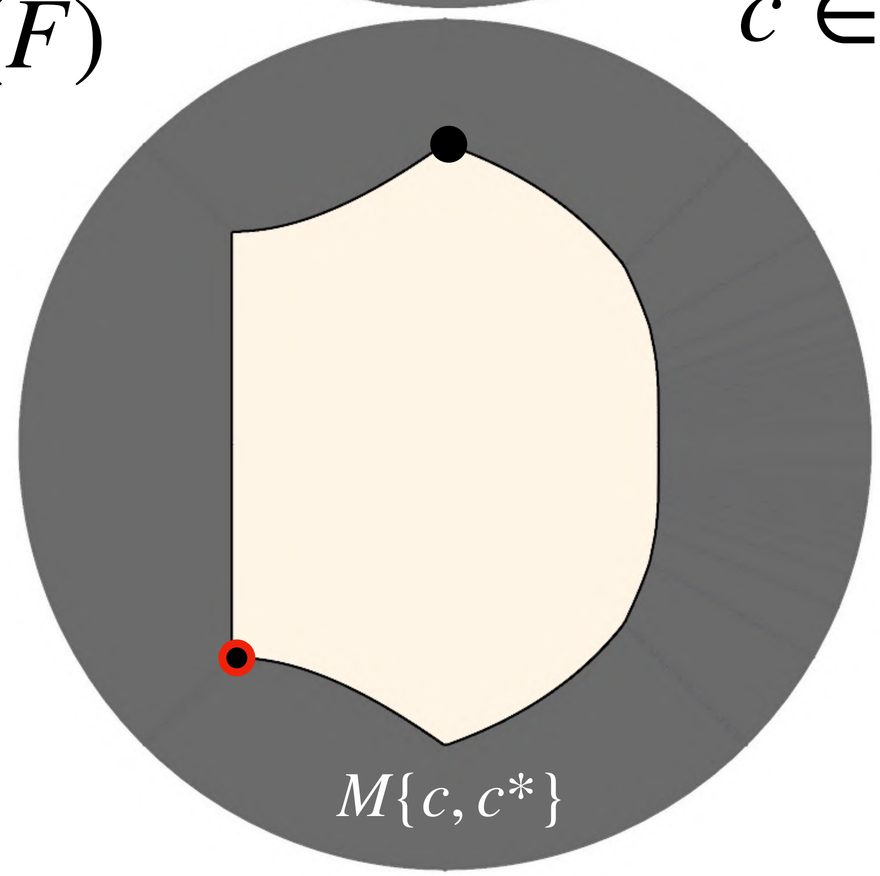


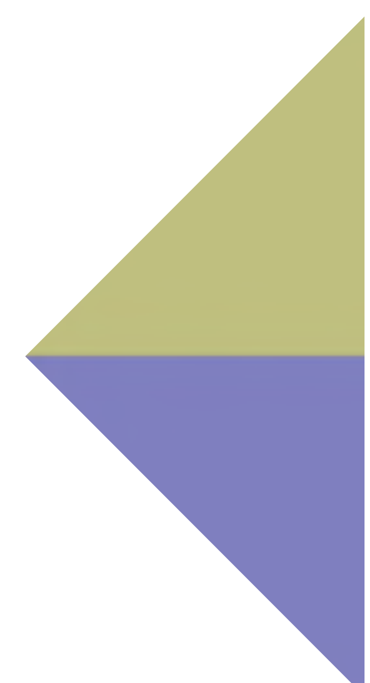
$$F := f_1(F) \cup f_2(F)$$

$c \in M$ **if** $f_1(F) \cap f_2(F) \neq \emptyset$

$f_1(z) := 1 + c \cdot z$

$f_2(z) := 1 + c^* \cdot z$



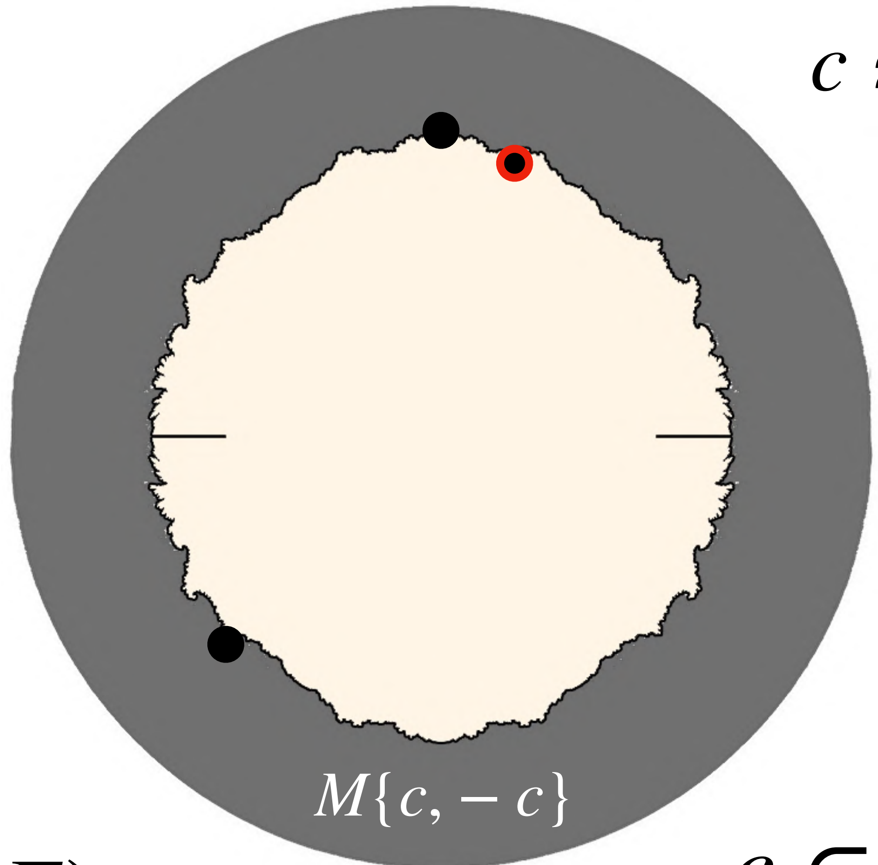


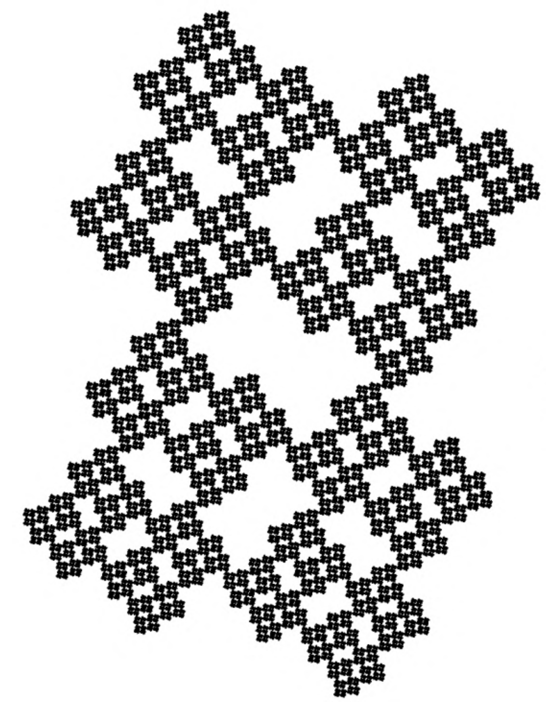
Complex-parametric families of self-similar sets

$c \approx 0.1028 + i0.6655$

$f_1(z) := 1 + c \cdot z$

$f_2(z) := 1 - c \cdot z$



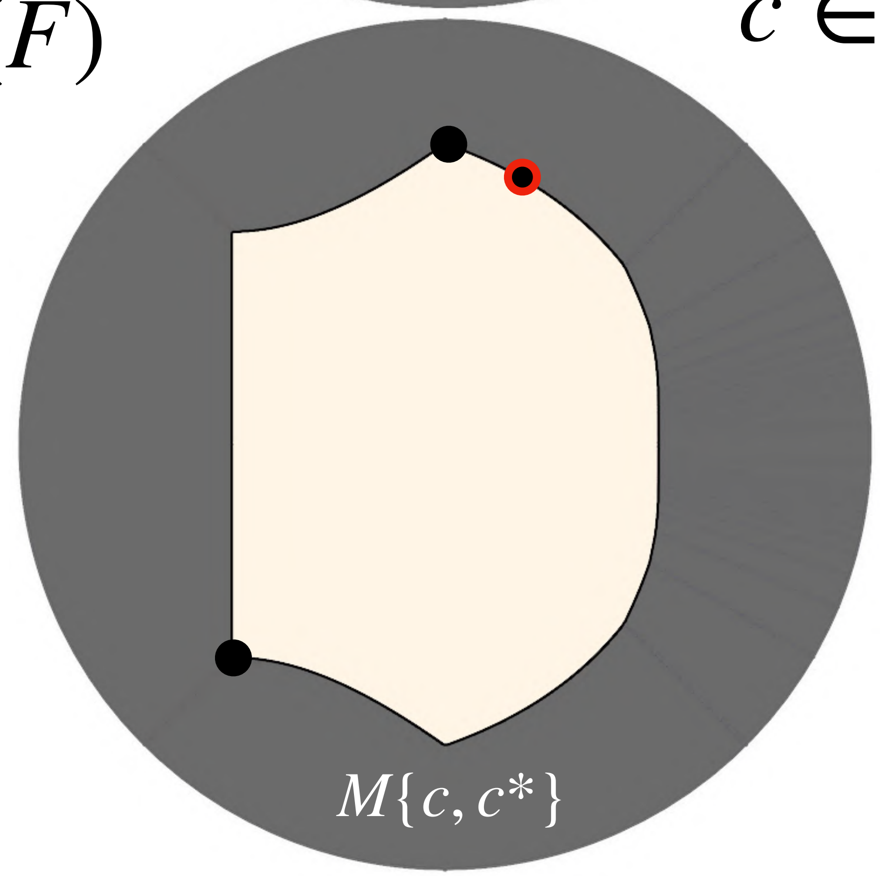


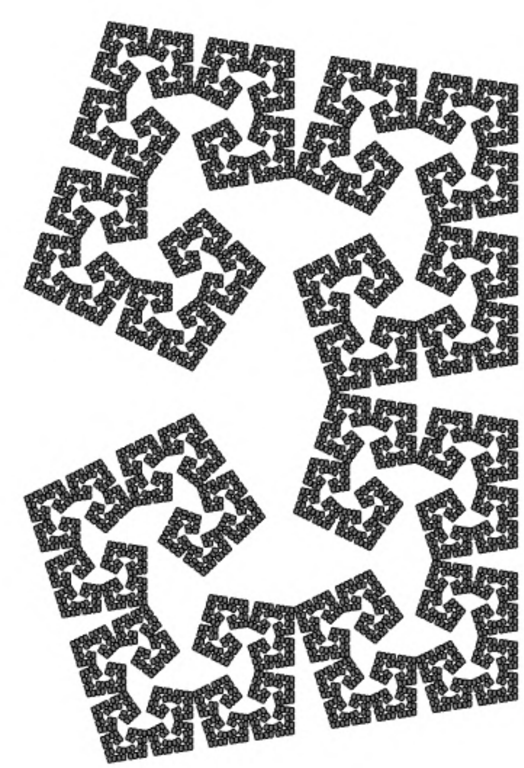
$F := f_1(F) \cup f_2(F)$

$c \in M$ **if** $f_1(F) \cap f_2(F) \neq \emptyset$

$f_1(z) := 1 + c \cdot z$

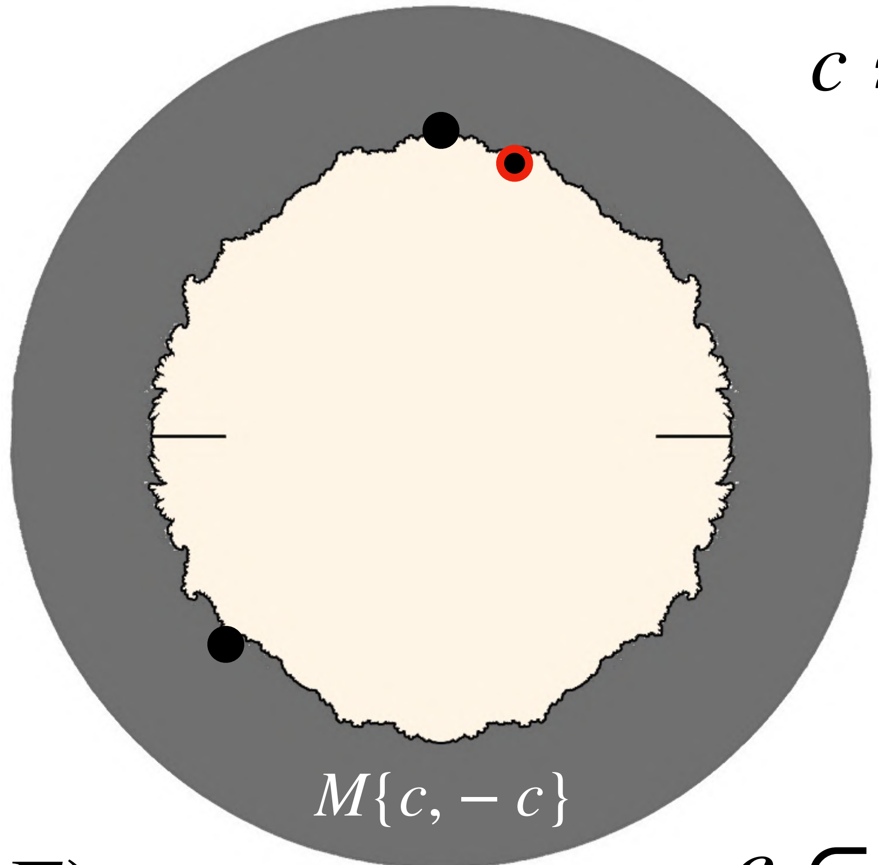
$f_2(z) := 1 + c^* \cdot z$

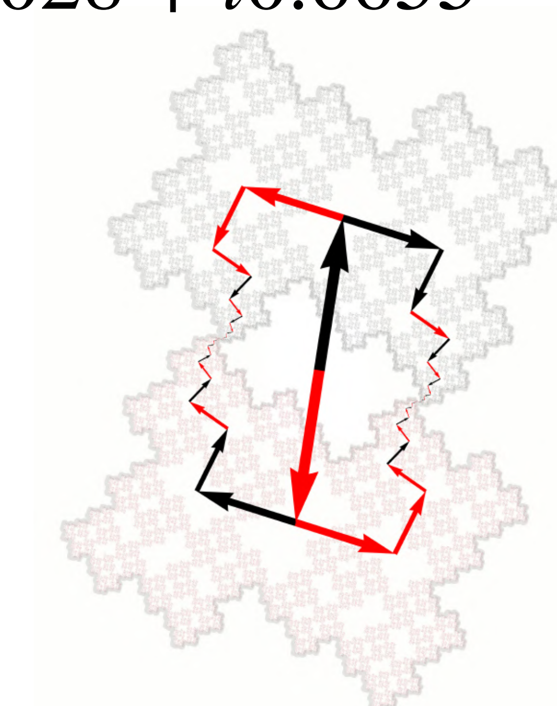




Complex-parametric families of self-similar sets

$c \approx 0.1028 + i0.6655$

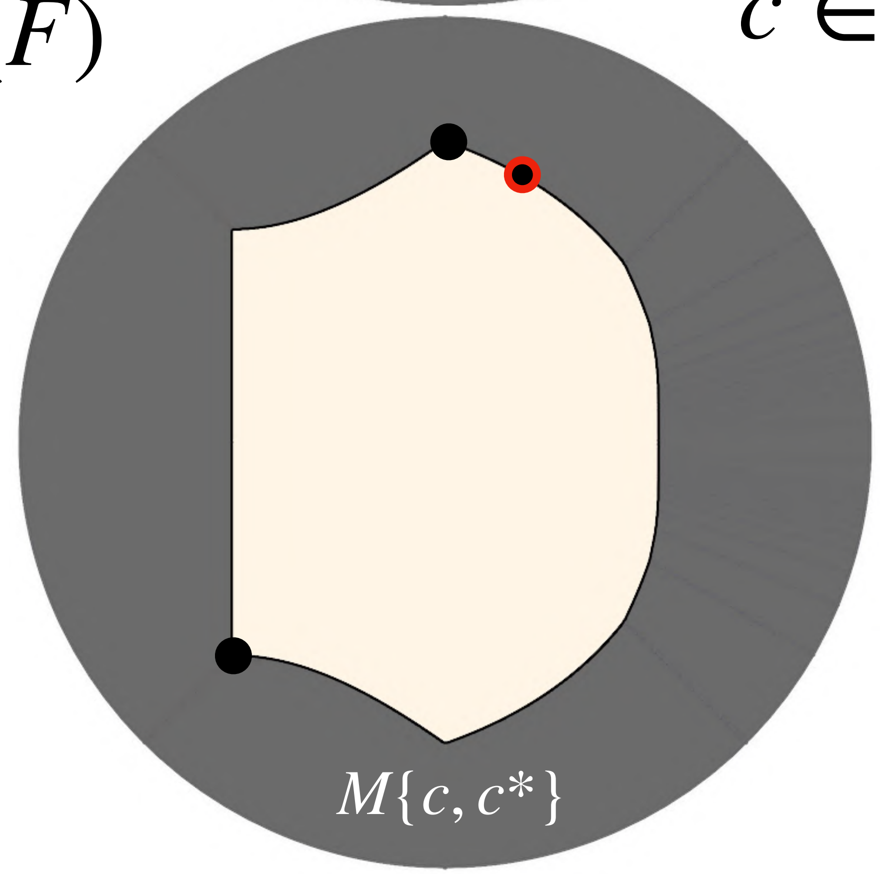


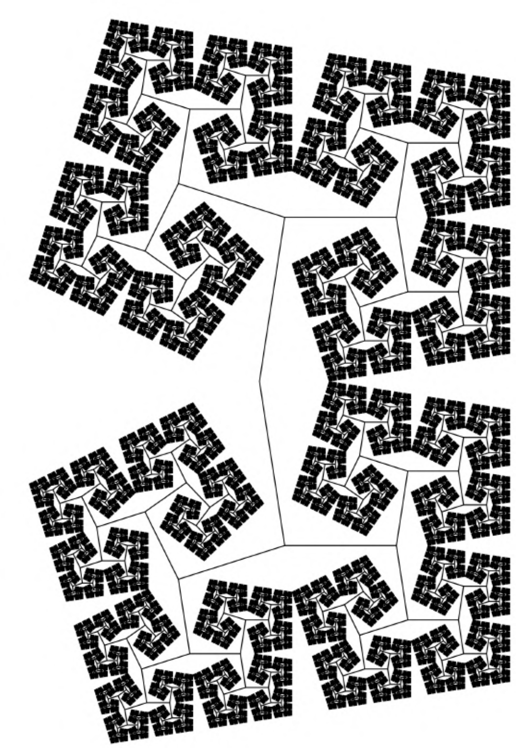


$$f_1(z) := 1 + c \cdot z$$
$$f_2(z) := 1 - c \cdot z$$

$$F := f_1(F) \cup f_2(F)$$

$c \in M$ **if** $f_1(F) \cap f_2(F) \neq \varnothing$



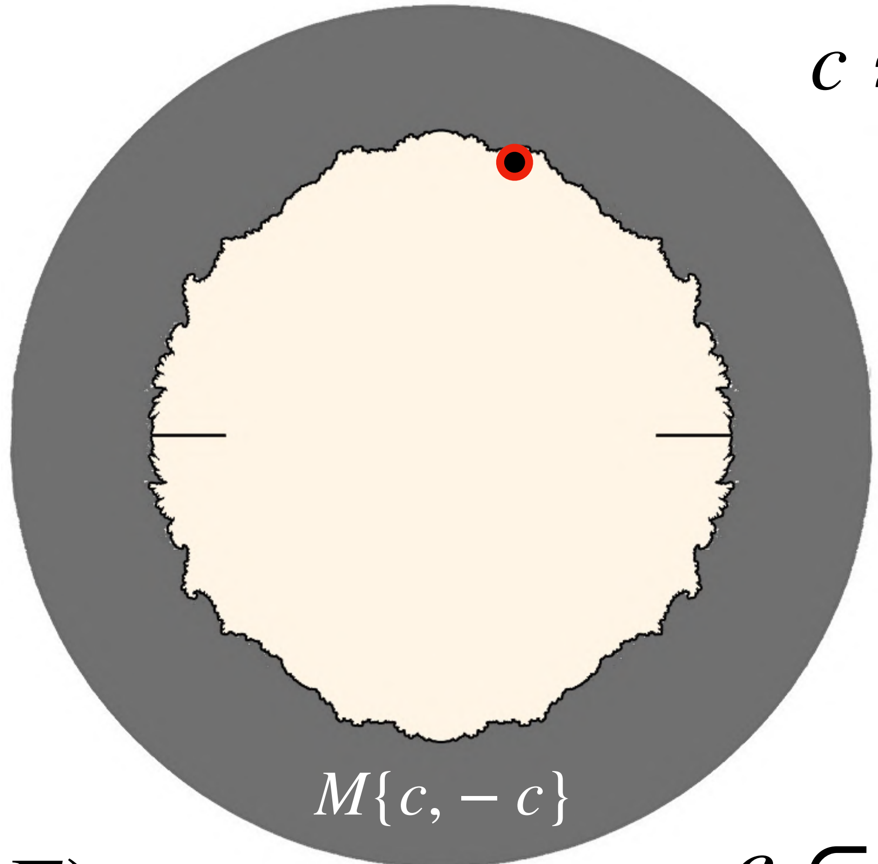


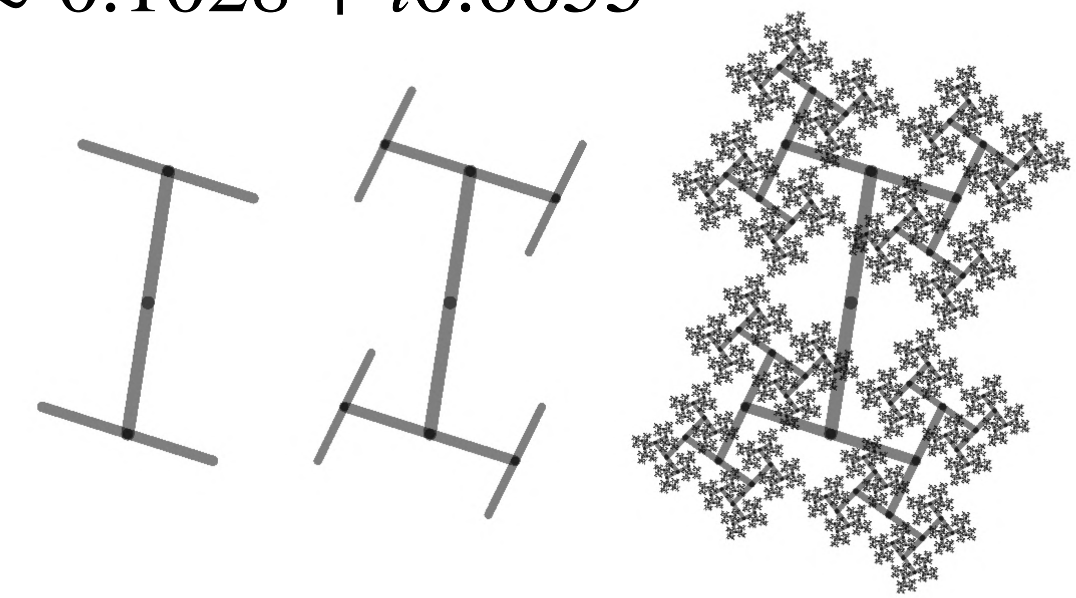
$$f_1(z) := 1 + c \cdot z$$
$$f_2(z) := 1 + c^* \cdot z$$

Complex-parametric families of self-similar sets

$$c \approx 0.1028 + i0.6655$$

$$f_1(z) := 1 + c \cdot z$$
$$f_2(z) := 1 - c \cdot z$$

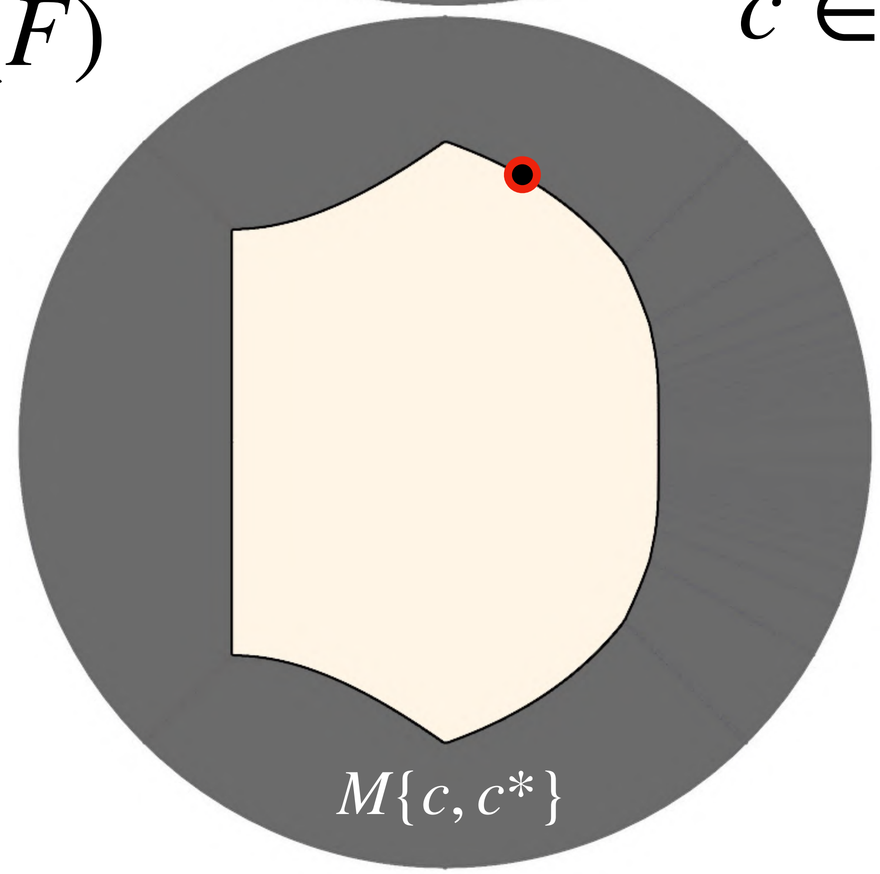


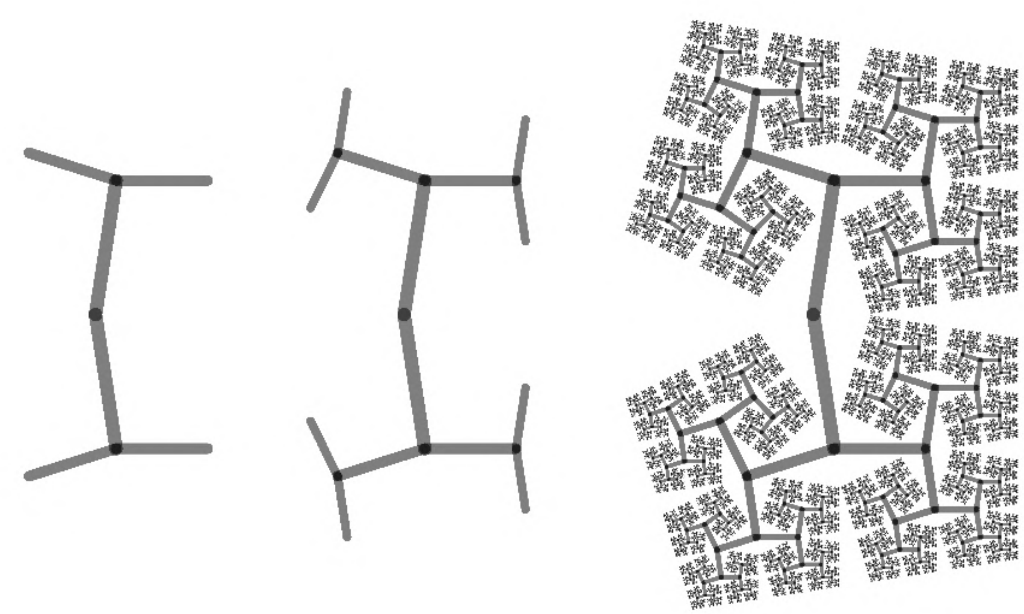


$$F := f_1(F) \cup f_2(F)$$

$c \in M$ **if** $f_1(F) \cap f_2(F) \neq \emptyset$

$$f_1(z) := 1 + c \cdot z$$
$$f_2(z) := 1 + c^* \cdot z$$





complex-valued alphabet $A = \{c_1, c_2\}$ with $0 < |c_1|, |c_2| < 1$

letters $w_1, w_2, w_3, \ldots \in \{c_1, c_2\}$

word $w = w_1 w_2 w_3 \ldots$

$$\varphi(w) = 1 + w_1 + w_1 \cdot w_2 + w_1 \cdot w_2 \cdot w_3 + \ldots$$

$$\varphi(2) = 1 + c_2$$

$$\varphi(1) = 1 + c_1$$

$$\varphi(11) = 1 + c_1 + c_1^2$$

$$\varphi(112) = 1 + c_1 + c_1^2 + c_1^2 c_2$$

nodes

empty string e_0 **root** $\varphi(e_0) = 1$

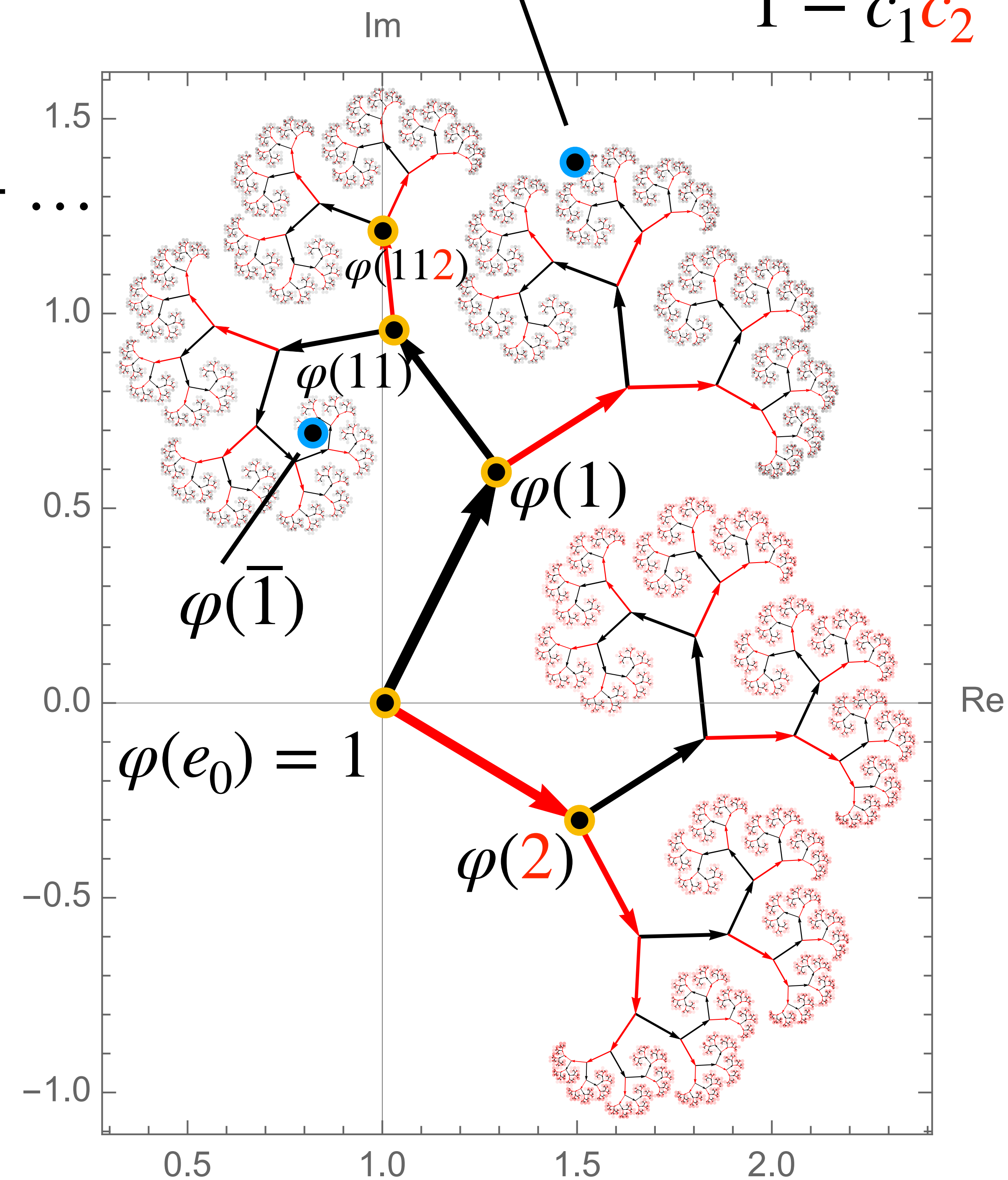
$$w = w_1 w_2 w_3 \ldots w_k \ldots \in \{c_1, c_2\}^\infty = A^\infty$$

tip points $\varphi(w)$

$$\varphi(111\ldots) = 1 + c_1 + c_1^2 + \ldots = \frac{1}{1 - c_1}$$

$$\varphi(1212\ldots) = 1 + c_1 + c_1 c_2 + \ldots = \frac{1 + c_1}{1 - c_1 c_2}$$

$$\varphi(\overline{12}) = \frac{1 + c_1}{1 - c_1 c_2}$$

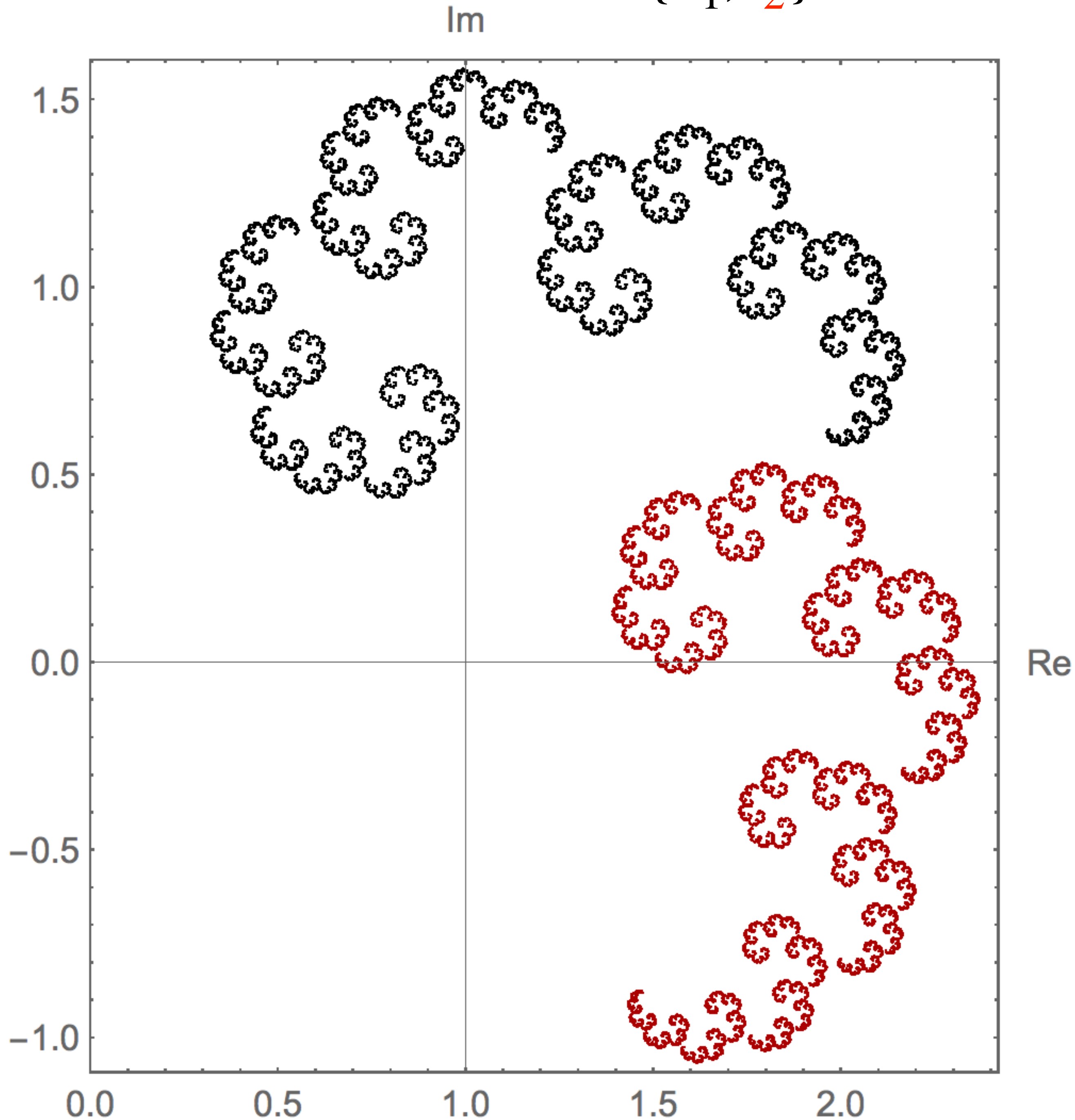


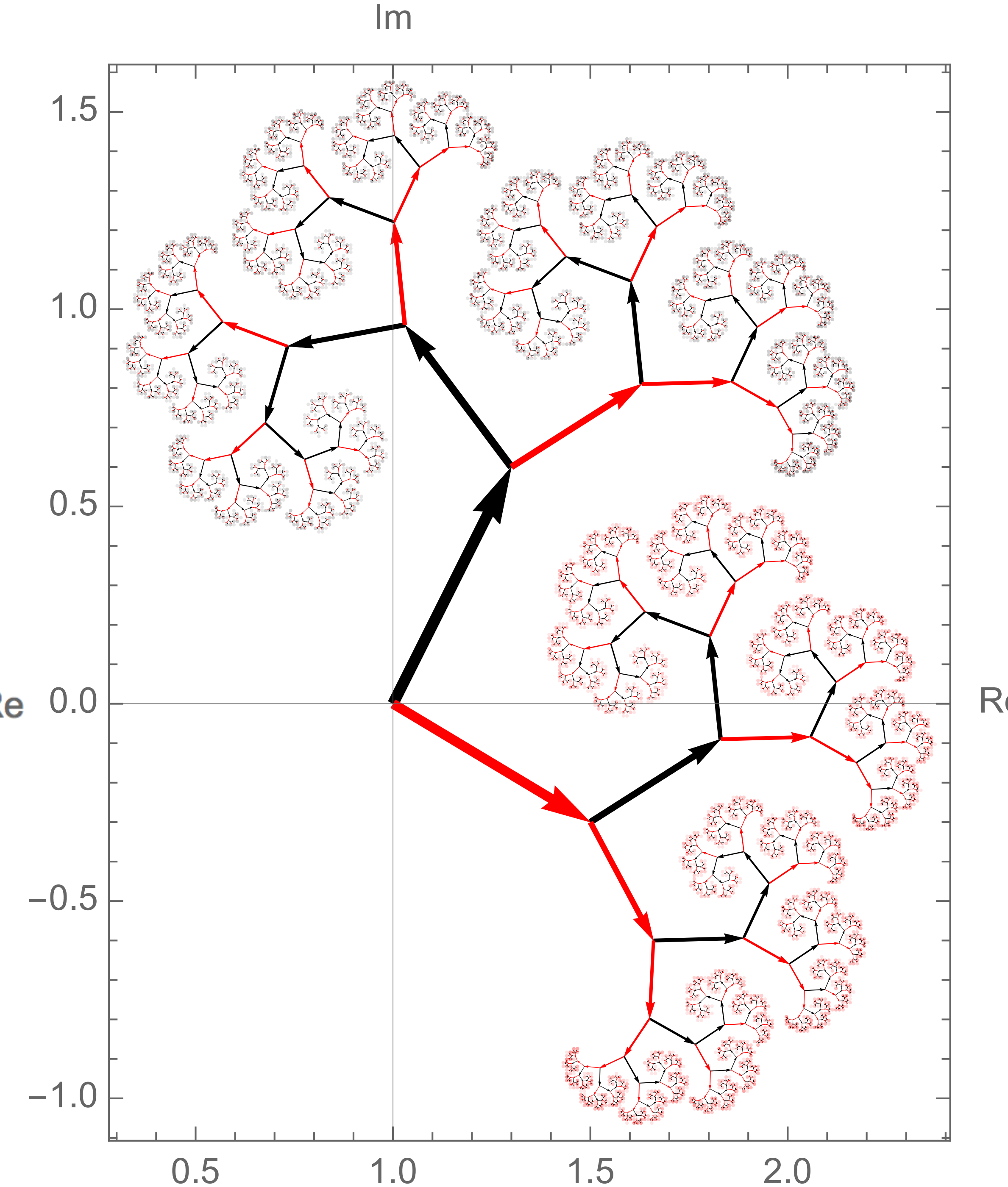
complex tree

$$T_A = T\{c_1, c_2\} = T\{.3 + i.6,\ .5 - i.3\}$$

$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \varphi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$





tipset

$F\{.3 + i.6,\ .5 - i.3\}$

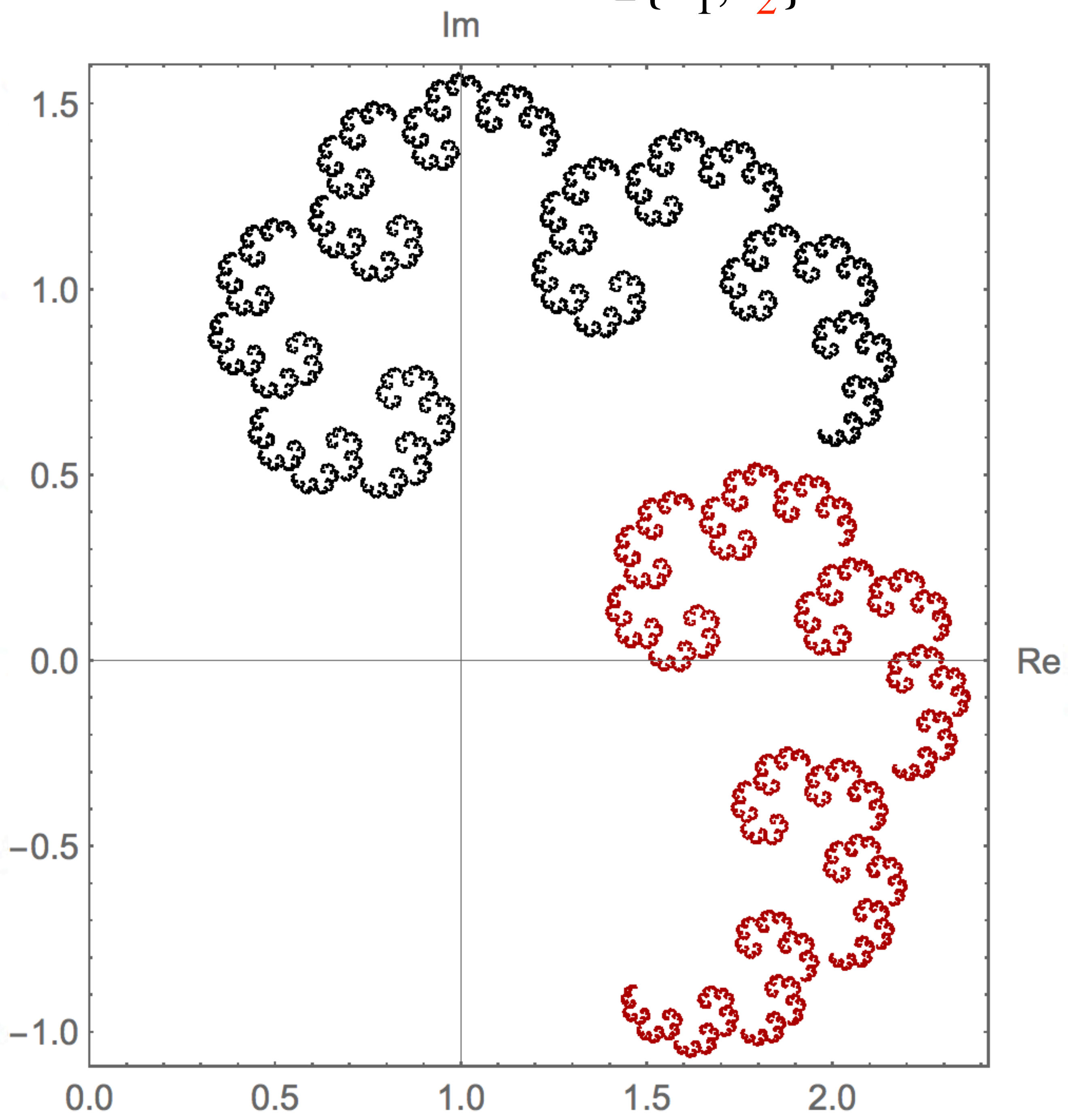
complex tree

$T\{.3 + i.6,\ .5 - i.3\}$

$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \varphi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$



$$f_1(F_A) = 1 + c_1 \cdot F_A$$

$$f_2(F_A) = 1 + c_2 \cdot F_A$$

The tipset is a self-similar set

Definition (punctured open unit disk)

$\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Definition (collinear digit set)

Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

Definition (collinear fractal)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,
$\boldsymbol{E}(c, n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \mathcal{A}_n \right\}$.

Definition (Mandelbrot set for collinear fractals)

$\mathcal{M}_n := \left\{ c^{-1} \in \mathbb{D}^* : \boldsymbol{E}(c, n) \text{ is connected} \right\}$.

Definition (*punctured open unit disk*)

$\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Definition (*collinear digit set*)

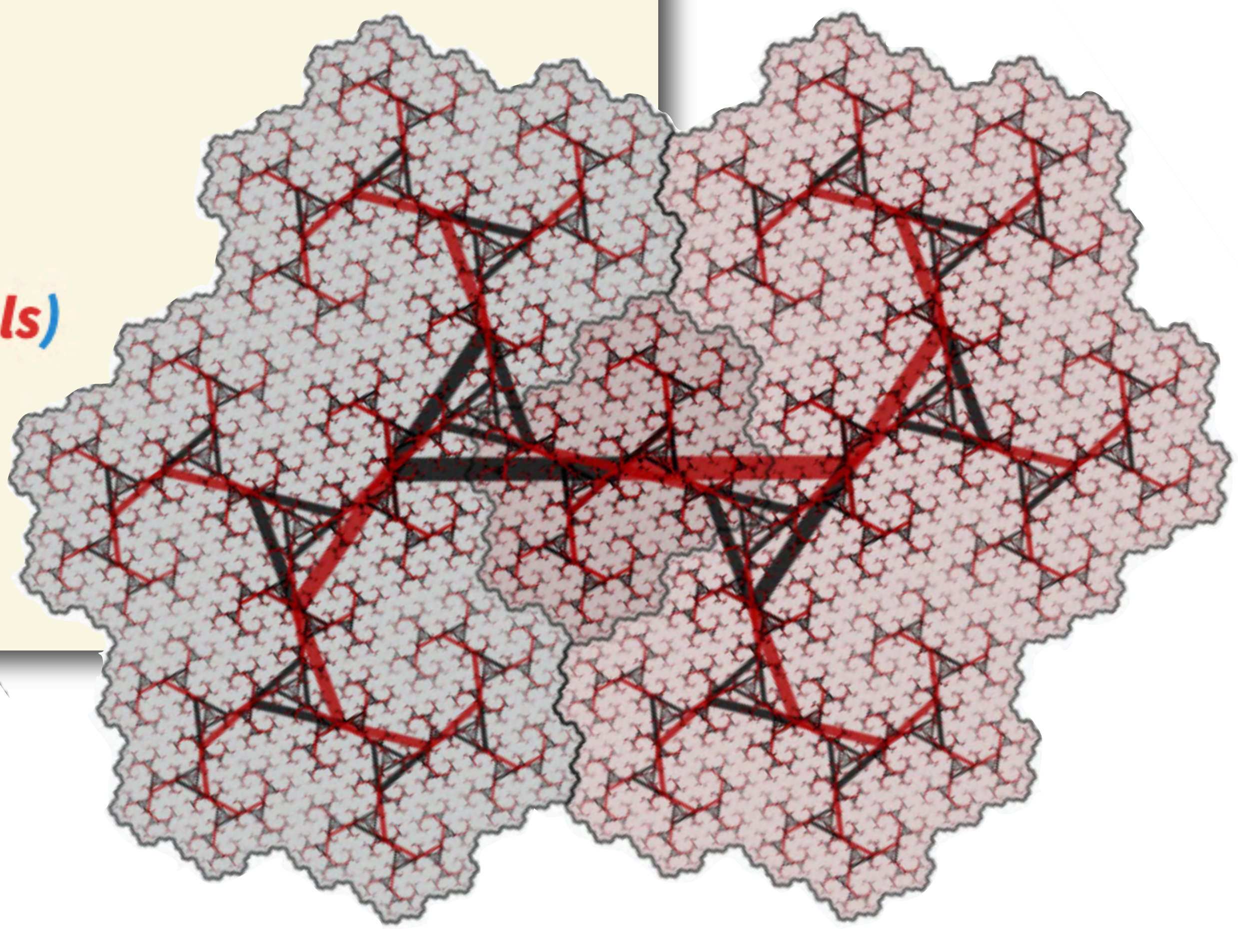
Set of $n \geqslant 2$ integers evenly spaced from $-n+1$ to $n-1$,
$\mathcal{A}_n := \{-n+1, -n+3, \ldots, n-3, n-1\}$.

Definition (*collinear fractal*)

Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,
$\boldsymbol{E}(c, n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} \,:\, a_k \in \mathcal{A}_n \right\}$.

Definition (*Mandelbrot set for collinear fractals*)

$\mathcal{M}_n := \left\{ c^{-1} \in \mathbb{D}^* \,:\, \boldsymbol{E}(c, n) \text{ is connected} \right\}$.



Definition (First-level piece)

$\boldsymbol{E}_t(c,n) := t + \frac{\boldsymbol{E}(c,n)}{c}$ s.t. $t \in \mathcal{A}_n$.

Proposition (IFS)

The set $\boldsymbol{E}(c,n)$ is the attractor of the **iterated function system** consisting of n maps $z \mapsto t + z/c$ with $t \in \mathcal{A}_n$, i.e.

$$\boldsymbol{E}(c,n) = \bigcup_{t \in \mathcal{A}_n} \boldsymbol{E}_t(c,n)$$

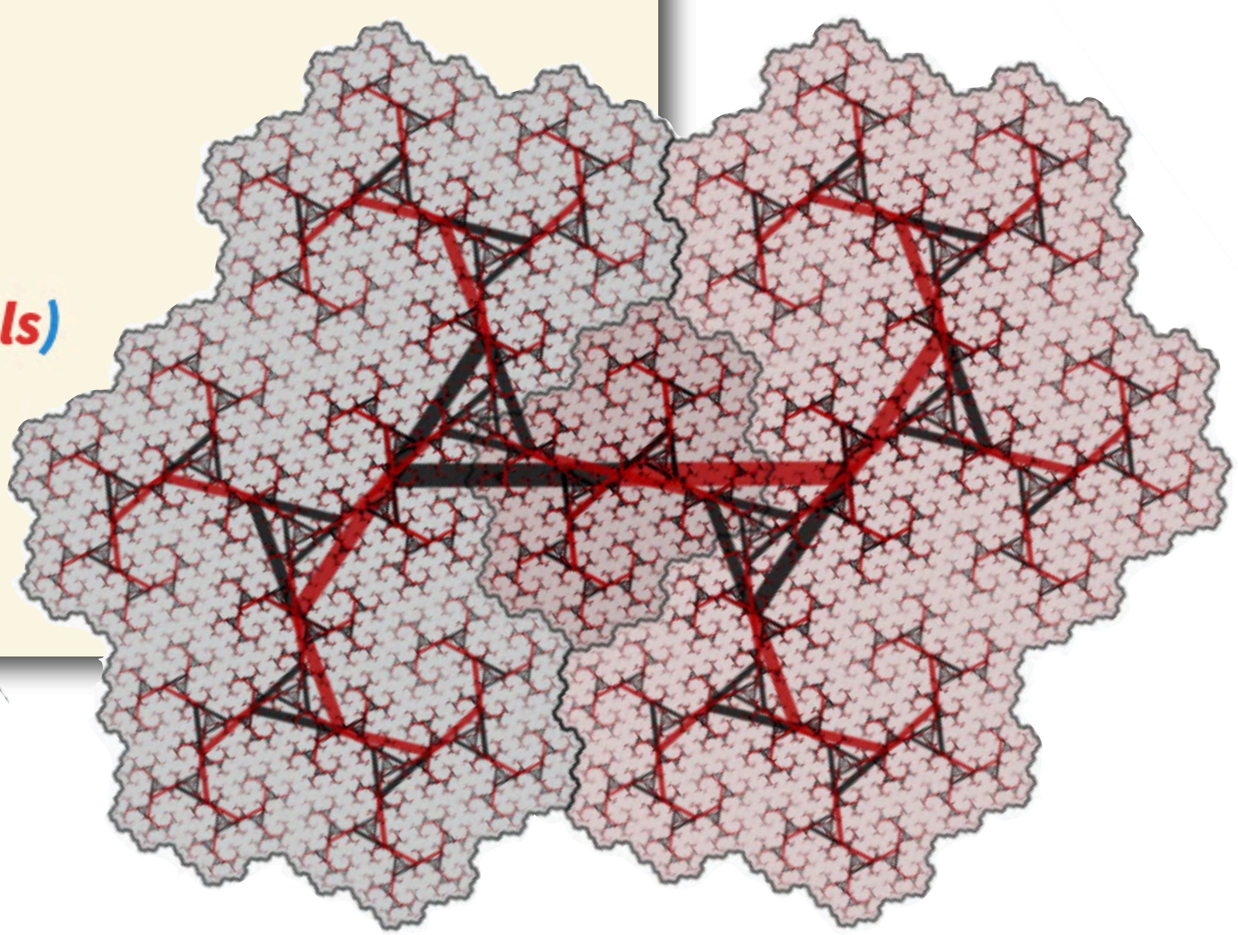
Definition (collinear fractal)

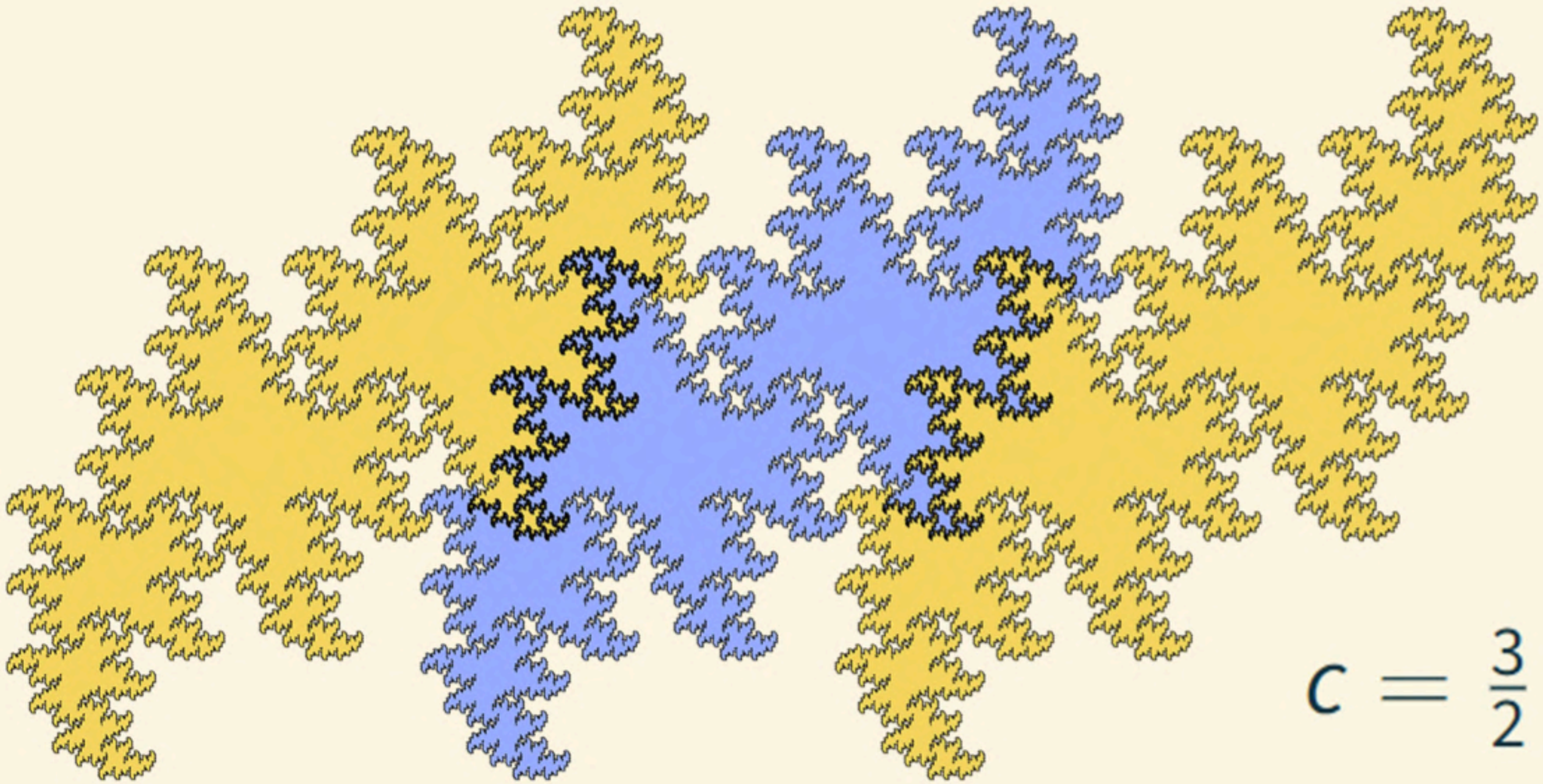
Self-similar set parameterized by $c^{-1} \in \mathbb{D}^*$,

$$\boldsymbol{E}(c,n) := \left\{ \sum_{k=0}^{\infty} a_k c^{-k} \, : \, a_k \in \mathcal{A}_n \right\}.$$

Definition (Mandelbrot set for collinear fractals)

$$\mathcal{M}_n := \left\{ c^{-1} \in \mathbb{D}^* \, : \, \boldsymbol{E}(c,n) \text{ is connected} \right\}.$$





$$c = \frac{3}{2} + \frac{i\sqrt{3}}{2}$$

Example (n = 3)

$$\begin{aligned} \boldsymbol{E}(c,3) &:= \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : \; a_k \in \{-2,0,2\} \right\} \\ &= \left(\frac{\boldsymbol{E}(c,3)}{c} - 2 \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} \right) \cup \left(\frac{\boldsymbol{E}(c,3)}{c} + 2 \right). \end{aligned}$$

M.F. Barnsley and A.N. Harrington / A Mandelbrot set for pairs of linear maps

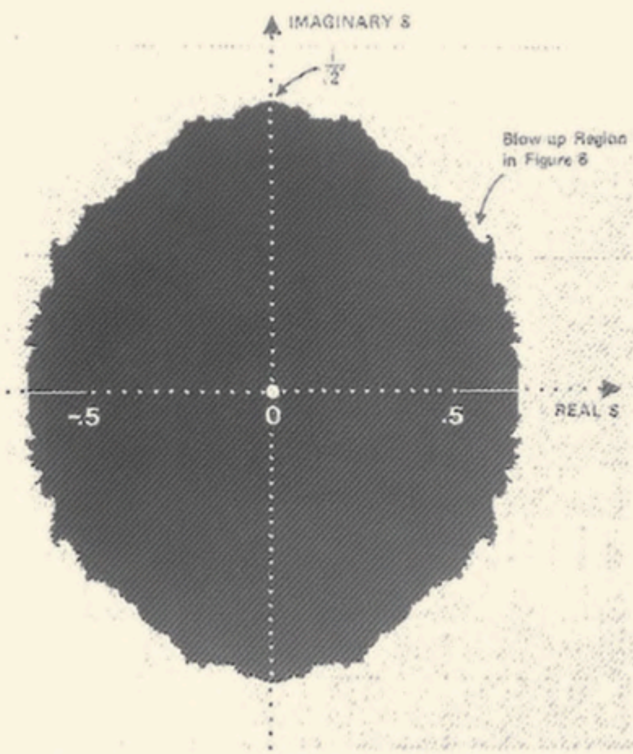


Fig. 5. The Mandelbrot set D is approximated in black. See text.

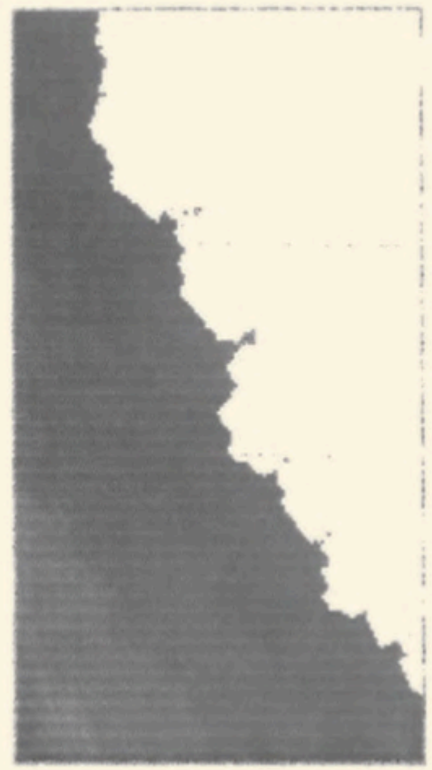


Fig. 6. Blow up of part of D, where $0.49 \le \mathrm{Re}\ s \le 0.55$ and $0.35 \le \mathrm{Im}\ s \le 0.45$.

In 1985, Michael Barnsley and Andrew Harrington introduced the Mandelbrot set $\mathcal{M}_2$ for 2-gon fractals $\boldsymbol{E}(c,2)$ as an analog of the Mandelbrot set for quadratic polynomials. Thirty years later, Danny Calegari, Sarah Koch, and Alden Walker (Section 6 in *Roots, Schottky semigroups, and a proof of Bandt's conjecture*, Ergod. Th. Dynam. Sys.37, no.8 (2017), 2487-2555.) briefly investigated the set of differences of $\boldsymbol{E}(c,2), \boldsymbol{E}(c,3), \boldsymbol{E}(c,5), \boldsymbol{E}(c,9), \ldots, \boldsymbol{E}(c,2^k+1)$.

Lemma

The set of differences between points in $\boldsymbol{E}(c,2^k+1)$ is

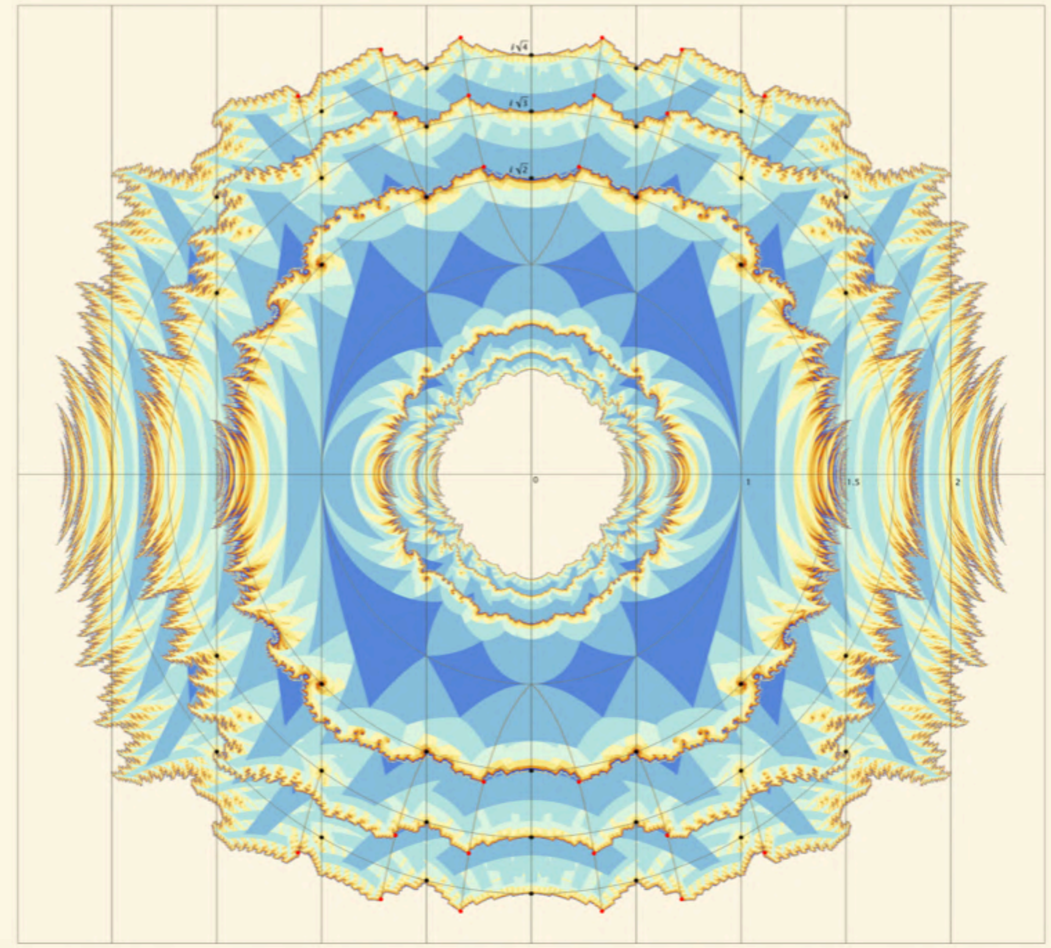
$$\boldsymbol{E}(c,2^{k+1}+1) = \{q_1 - q_2 \,:\, q_1, q_2 \in \boldsymbol{E}(c,2^k+1)\}.$$

Proposition

$$\mathcal{M}_{2^k+1} = \left\{ c^{-1} \in \mathbb{D}^* \,:\, 2c \in \boldsymbol{E}(c,2^{k+1}+1) \right\}.$$

What sequences of sets arise as iterated differences? What properties do these iterated IFS have?

-Calegari-Koch-Walker

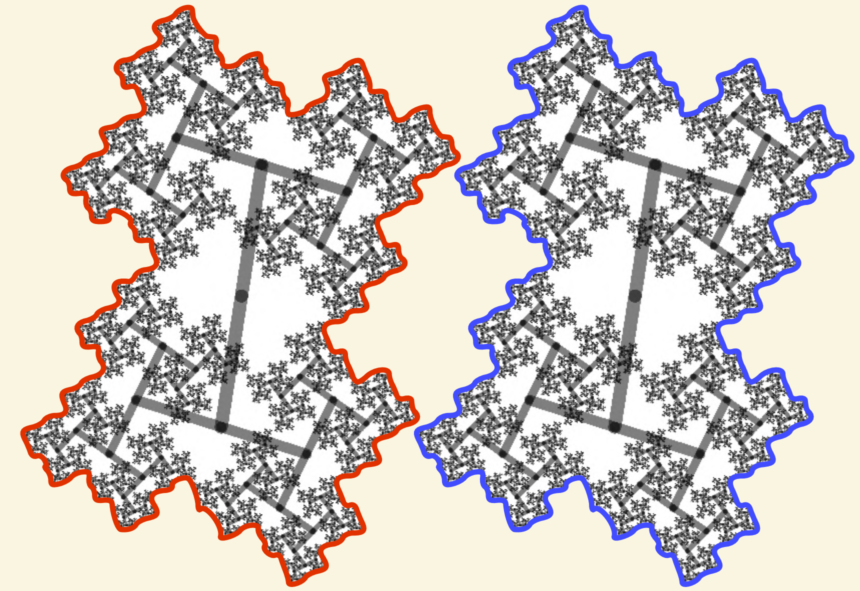


Our main contributions: combinatorial code structure, inner stability, components for any n>1, collinear tiles, structure theorem for the boundary of $\mathcal{M}_n$

- M.F.Barnsley and A.N. Harrington, *A Mandelbrot set for pairs of linear maps*, Physica 150 (1985) 421-432.
- T. Bousch, *Paires de similitudes $z \to sz+1, z \to sz-1$*, Preprint (1988).
- T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, thèse de l'Université Paris-Sud, Orsay (1992).
- C. Bandt, *On the Mandelbrot set for pairs of linear maps*, Nonlinearity **15** (2002), no. 4, 1127–1147
- D. Calegari, S. Koch and A. Walker, *Roots, Schottky semigroups, and a proof of Bandt's conjecture*, Ergod. Th & Dynam. Sys.37, no.8 (2017), 2487-2555.
- Y. Himeki and Y. Ishii, *$\mathcal{M}_4$ is regular closed*, Ergod. Th & Dynam. Sys. 40, no.1 (2020), 213-220.
- J. Hutchinson, *Fractals and Self-Similarity*, Indiana Univ. Math. J. 30, no. 5 (1981), 713-747.
- A. M. Odlyzko and B. Poonen, *Zeros of polynomials with 0, 1 coefficients*, L'Enseignement Math.39 (1993), 317-348.
- B. Solomyak, *Conjugates of Beta-Numbers and the Zero-Free Domain for a Class of Analytic Functions*, Proceedings of the London Mathematical Society, s3-68, 3 (1994), 477–498.
- B. Solomyak, *Mandelbrot set for a pair of line maps : the local geometry*, Anal. Theory Appl. 20 (2004), 149-157.
- B. Solomyak, *On the 'Mandelbrot set' for pairs of linear maps : asymptotic self-similarity*, Nonlinearity 18 (2005), 1927-1943.
- Silvestri, S. & Pérez, R. Accessibility of the Boundary of the Thurston Set. *Experimental Mathematics*. pp. 1-18 (2021)
- M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), 381–414.
- J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
- K.-H. Indlekofer, A. Járai, and I. Kátai, *On some properties of attractors generated by iterated function systems*, Acta Sci. Math. (Szeged) **60** (1995), 411–427.
- K. G. Hare, N. Sidorov, *On a family of self-affine sets: topology, uniqueness, simultaneous expansions*, Ergodic Theory Dynam. Systems 37 (2017), 193–227.
- B. Solomyak *Connectedness locus for pairs of affine maps and zeros of power series*, J. Fractal Geom. (2) (2015) no.3 281-308.
- P. Shmerkin and B. Solomyak, *Zeros of $\{-1,0,1\}$ power series and connectedness loci for self-affine sets*, Exp. Math. 15 (2006), 499-511.
- V. F. Sirvent and J. M. Thuswaldner, *On the Fibonacci-Mandelbrot set*, Indagationes Mathematicae 26.1 (2015), 174-190.
- B. Solomyak and H. Xu, *On the 'Mandelbrot set' for a pair of linear maps and complex Bernoulli convolutions*, Nonlinearity 16 (2003), 1733-1749.
- Y. Nakajima, *Mandelbrot set for fractal n-gons and zeros of power series* `arXiv:2008.12915 [math.DS]`, 2023.
- F. Beaucoup, P. Borwein, D. W. Boyd, and C. Pinner. Power series with restricted coefficients and a root on a given ray. *Mathematics of Computation*, 67(222):715–736, 1998.

Proposition

$$\mathcal{M}_n = \left\{ c^{-1} \in \mathbb{D}^* : 2c \in \boldsymbol{E}(c, 2n-1) \right\}$$



Proof.

$\boldsymbol{E}(c, n)$ is connected if and only if its n first-level pieces intersect pairwise, $\boldsymbol{E}_{t+2}(c, n) \cap \boldsymbol{E}_t(c, n) \neq \emptyset$ with $t+2, t \in \mathcal{A}_n$. By self-similarity, $c\big(\boldsymbol{E}_{t+2}(c, n) \cap \boldsymbol{E}_t(c, n)\big) - t = \big(2c + \boldsymbol{E}(c, n)\big) \cap \boldsymbol{E}(c, n)$, so we have

$$\mathcal{M}_n = \left\{ c^{-1} \in \mathbb{D}^* : \exists p_1, p_2 \in \boldsymbol{E}(c, n) \text{ s.t. } 2c + p_1 = p_2 \right\}.$$

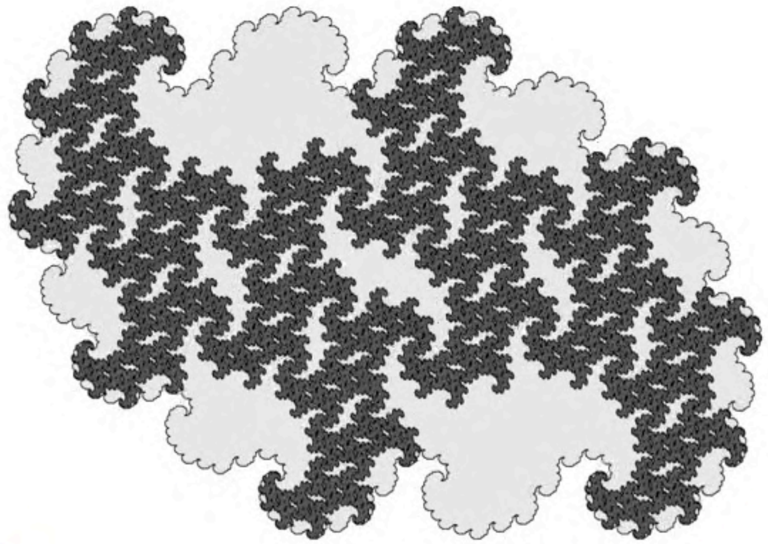
Now, since $(p_2 - p_1) \in \boldsymbol{E}(c, 2n-1)$, it follows that $\mathcal{M}_n$ is the set of parameters $c^{-1} \in \mathbb{D}^*$ such that $2c \in \boldsymbol{E}(c, 2n-1)$. ■

Animation: youtu.be/11NZDHNahJs

$c = 1.2345 + i\,0.7935$

$\mathrm{Arg}(c) = 0.5713 = 32.73°$

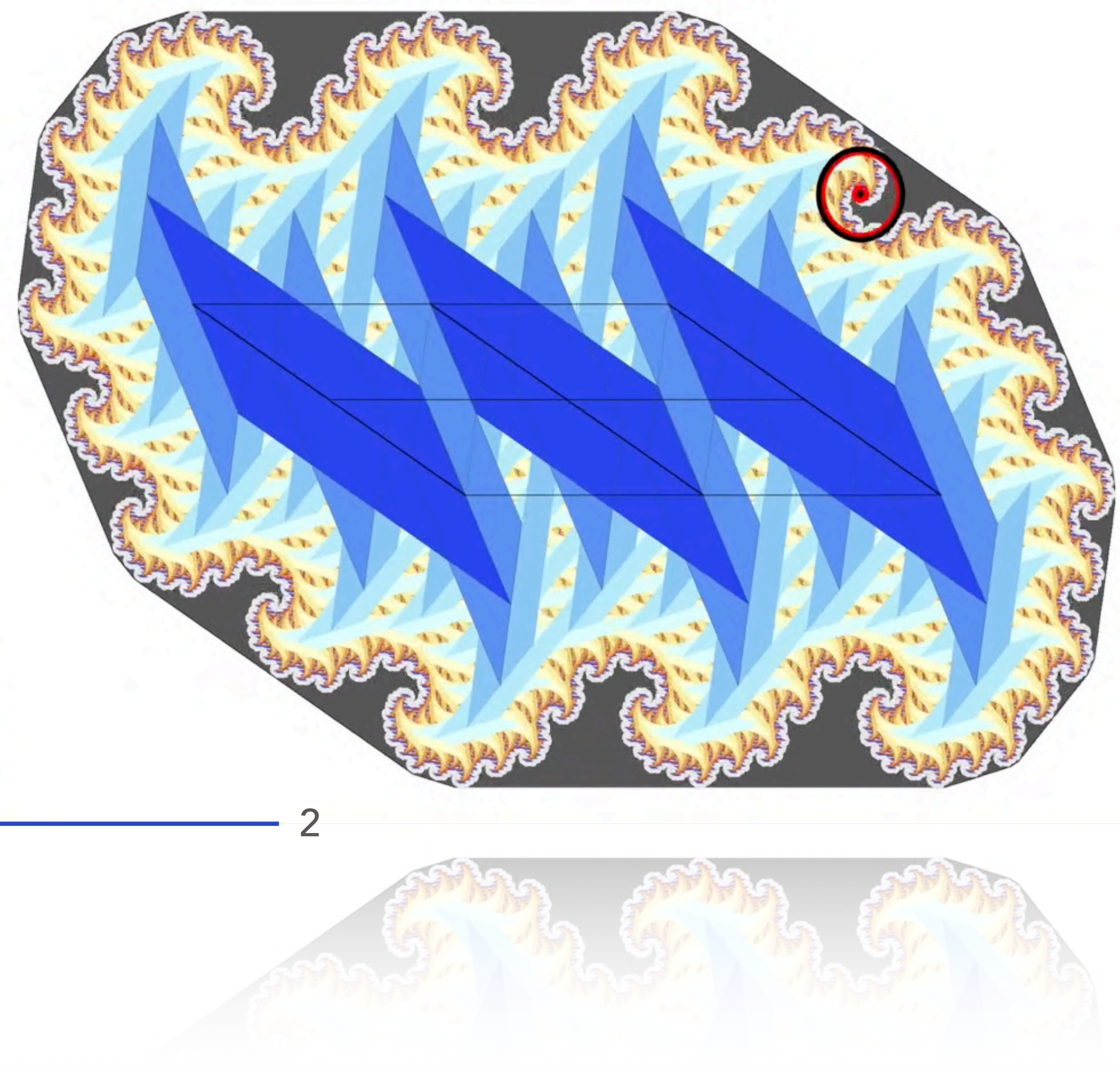
$r = |c^{-1}| = 0.6814$

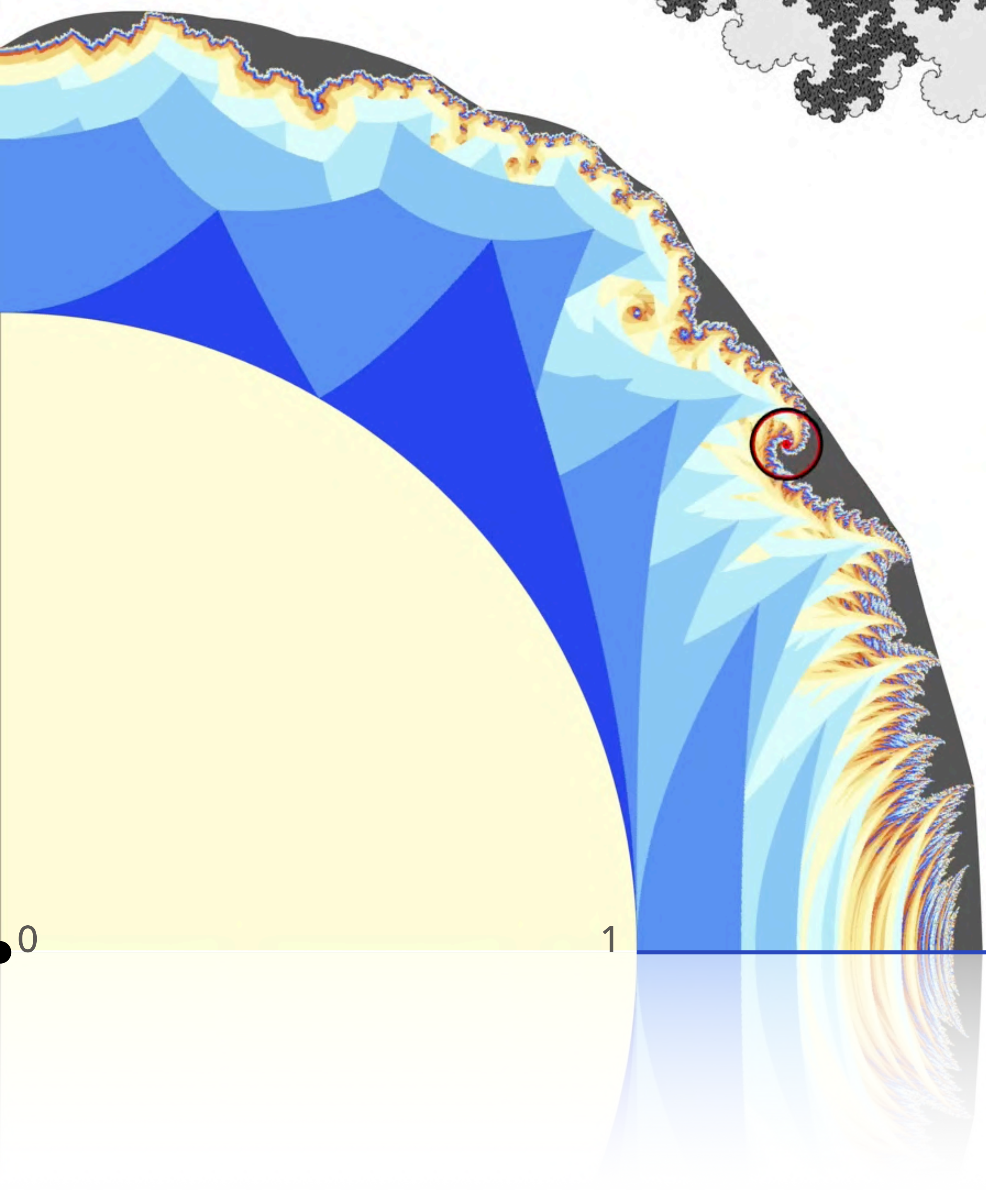


$$\dim_H E(c,2) = \frac{\log 2}{\log |c|} \approx 1.8071$$

"Asymptotic self-similarity"

$2c \in E(c,3)$



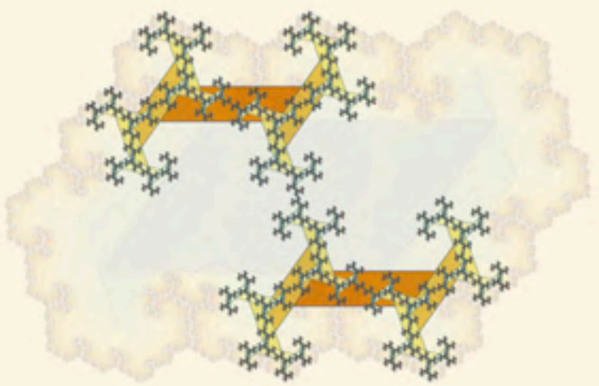


Lemma (Minkowski sum)

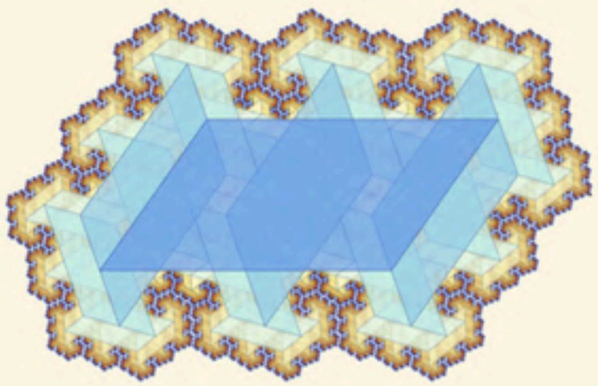
The Minkowski sum and geometric difference

$$\mathbf{E}(c, 2n-1) = \mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$$

$$= \mathbf{E}(c,n) \ominus \mathbf{E}(c,n)$$



$\mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$



$\mathbf{E}(c, 2n-1)$

Our main result, the combinatorial code structure of $\mathcal{M}_n$, is based on the following discovery that applies to all $c \in \mathcal{M}_n$.

Lemma (inner parallelogram)

Let $c = a + ib$, if $\mathbf{E}(c,n)$ is connected, then its Minkowski sum, $\mathbf{E}(c, 2n-1)$, contains $\mathbf{P}(c, 2n-1) = \mathbf{P}(c,n) \oplus \mathbf{P}(c,n) = 2\mathbf{P}(c,n)$ where $\mathbf{P}(c,n)$ is the parametric region centered at 0 given either by the parallelogram

$$\mathbf{P}(c,n) := \left\{ x+iy \in \mathbb{C} : |y| < \frac{|b|}{|c|^2}, -n+1-\frac{y \cdot a}{b} < x < n-1-\frac{y \cdot a}{b} \right\}$$

or, if $c \in \mathbb{R}$, by the interval

$$\mathbf{P}(c,n) := (1-n-|c^{-1}|, n-1+|c^{-1}|).$$

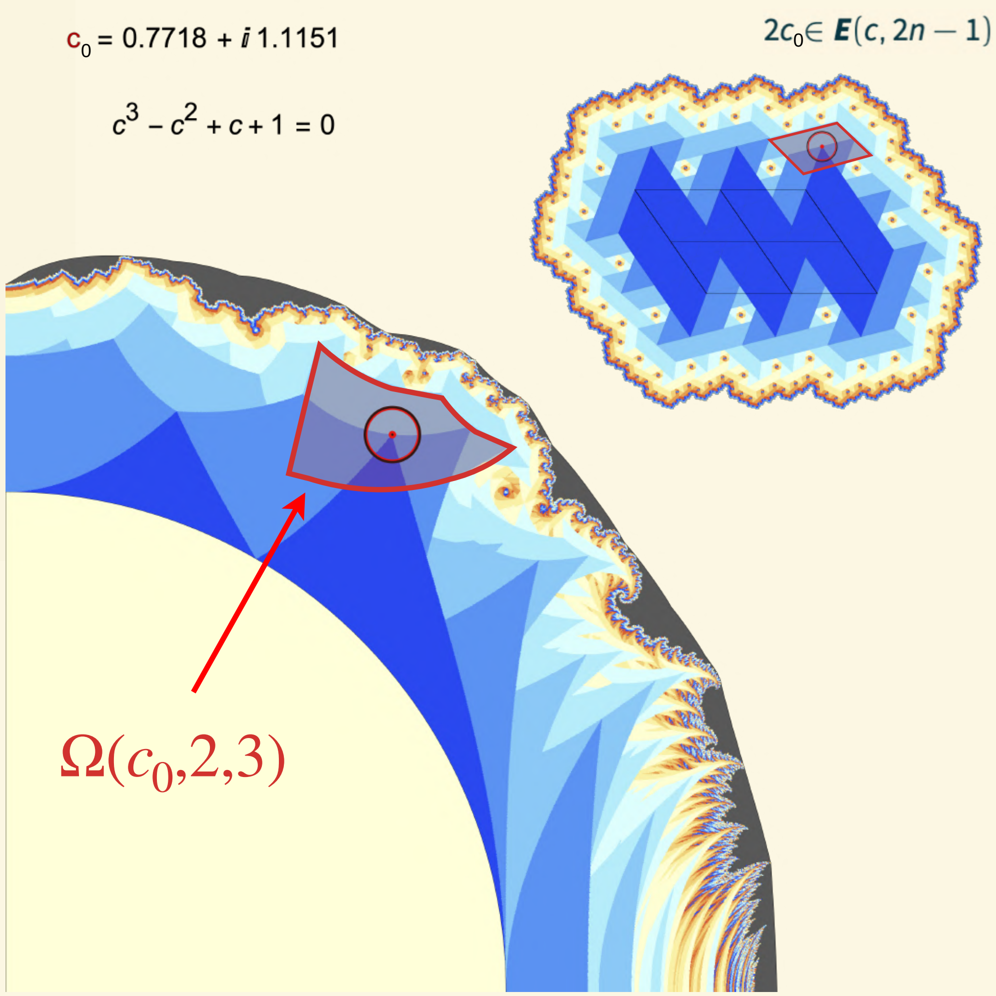
Definition (component $\Omega(c_0, n, m)$)

Let c_0 be a root of a polynomial $q(z)$ of degree m with coefficients restricted to the integers from $-n+1$ to $n-1$, i.e. $\mathcal{A}_n \cup \mathcal{A}_{n-1}$. The *component* $\Omega(c_0, n, m)$ is defined as the maximal connected open set containing c_0, of parameter values c close to c_0 for which the following condition holds as $c_0 \to c$

$$q(c) \in \frac{\mathbf{P}(c,n)}{c}.$$

Theorem (inner stability)

$\Omega(c_0, n, m) \subset \mathrm{int}(\mathcal{M}_n)$.

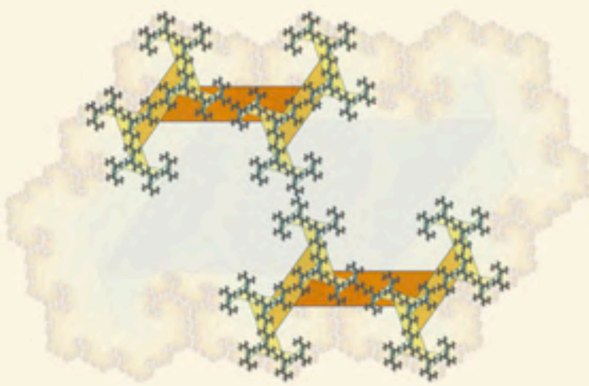


Lemma (Minkowski sum)

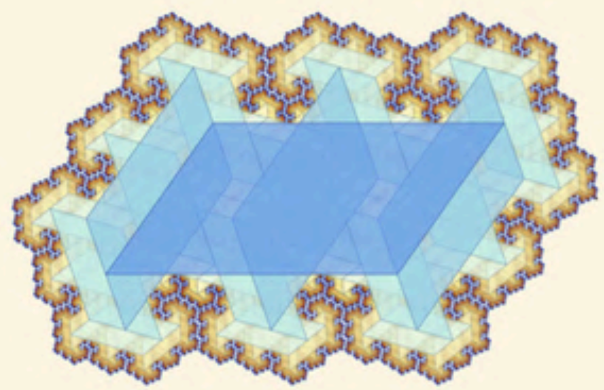
The Minkowski sum and geometric difference

$$\boldsymbol{E}(c, 2n-1) = \boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$$

$$= \boldsymbol{E}(c,n) \ominus \boldsymbol{E}(c,n)$$



$\boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$



$\boldsymbol{E}(c, 2n-1)$

Our main result, the combinatorial code structure of $\mathcal{M}_n$, is based on the following discovery that applies to all $c \in \mathcal{M}_n$.

Lemma (inner parallelogram)

Let $c = a + ib$, if $\boldsymbol{E}(c,n)$ is connected, then its Minkowski sum, $\boldsymbol{E}(c, 2n-1)$, contains $\boldsymbol{P}(c, 2n-1) = \boldsymbol{P}(c,n) \oplus \boldsymbol{P}(c,n) = 2\boldsymbol{P}(c,n)$ where $\boldsymbol{P}(c,n)$ is the parametric region centered at 0 given either by the parallelogram

$$\boldsymbol{P}(c,n) := \left\{ x+iy \in \mathbb{C} : |y| < \frac{|b|}{|c|^2}, -n+1-\frac{y \cdot a}{b} < x < n-1-\frac{y \cdot a}{b} \right\}$$

or, if $c \in \mathbb{R}$, by the interval

$$\boldsymbol{P}(c,n) := (1-n-|c^{-1}|, n-1+|c^{-1}|).$$

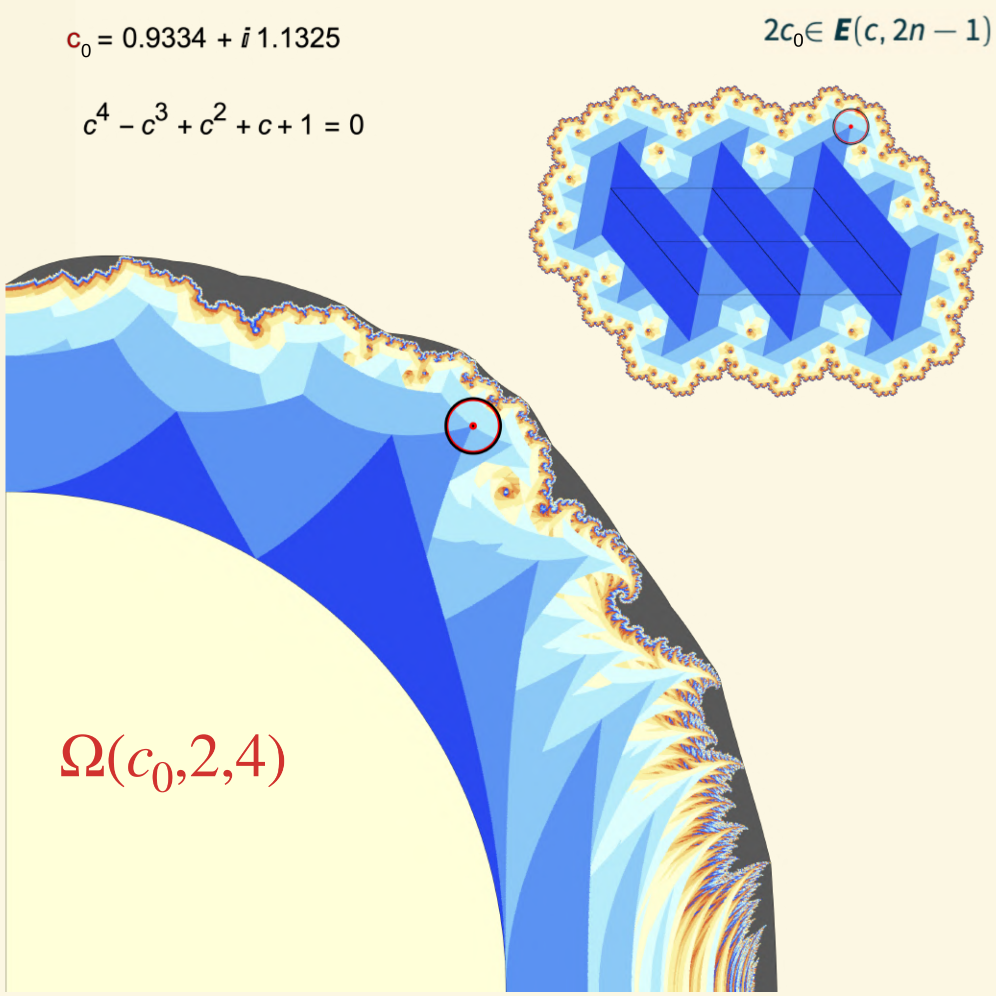
Definition (component $\boldsymbol{\Omega}(c_0, n, m)$)

Let c_0 be a root of a polynomial $q(z)$ of degree m with coefficients restricted to the integers from $-n+1$ to $n-1$, i.e. $\mathcal{A}_n \cup \mathcal{A}_{n-1}$. The *component* $\boldsymbol{\Omega}(c_0, n, m)$ is defined as the maximal connected open set containing c_0, of parameter values c close to c_0 for which the following condition holds as $c_0 \to c$

$$q(c) \in \frac{\boldsymbol{P}(c,n)}{c}.$$

Theorem (inner stability)

$\boldsymbol{\Omega}(c_0, n, m) \subset \operatorname{int}(\mathcal{M}_n)$.

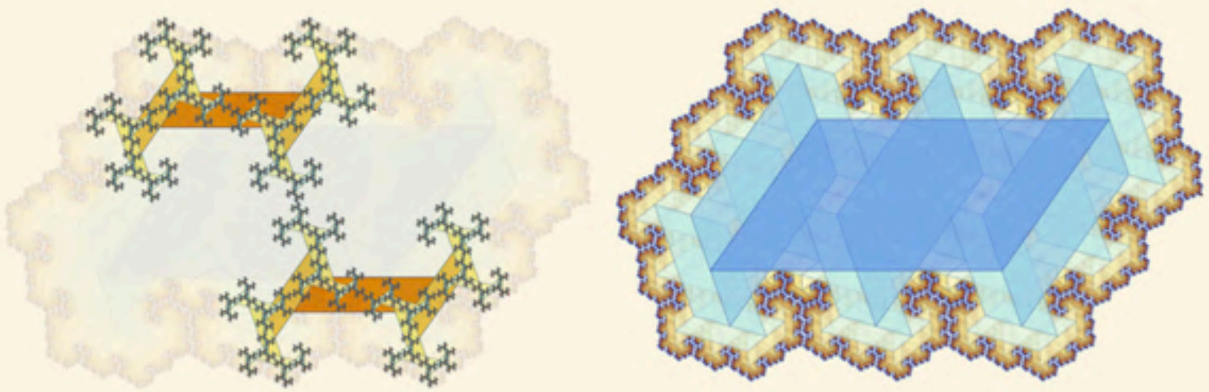


Lemma (Minkowski sum)

The Minkowski sum and geometric difference

$$\boldsymbol{E}(c, 2n-1) = \boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$$

$$= \boldsymbol{E}(c,n) \ominus \boldsymbol{E}(c,n)$$



$\boldsymbol{E}(c,n) \oplus \boldsymbol{E}(c,n)$ $\quad$ $\boldsymbol{E}(c, 2n-1)$

Our main result, the combinatorial code structure of $\mathcal{M}_n$, is based on the following discovery that applies to all $c \in \mathcal{M}_n$.

Lemma (inner parallelogram)

Let $c = a + ib$, if $\boldsymbol{E}(c,n)$ is connected, then its Minkowski sum, $\boldsymbol{E}(c, 2n-1)$, contains $\boldsymbol{P}(c, 2n-1) = \boldsymbol{P}(c,n) \oplus \boldsymbol{P}(c,n) = 2\boldsymbol{P}(c,n)$ where $\boldsymbol{P}(c,n)$ is the parametric region centered at 0 given either by the parallelogram

$$\boldsymbol{P}(c,n) := \left\{ x+iy \in \mathbb{C} : |y| < \frac{|b|}{|c|^2}, -n+1-\frac{y \cdot a}{b} < x < n-1-\frac{y \cdot a}{b} \right\}$$

or, if $c \in \mathbb{R}$, by the interval

$$\boldsymbol{P}(c,n) := (1-n-|c^{-1}|, n-1+|c^{-1}|).$$

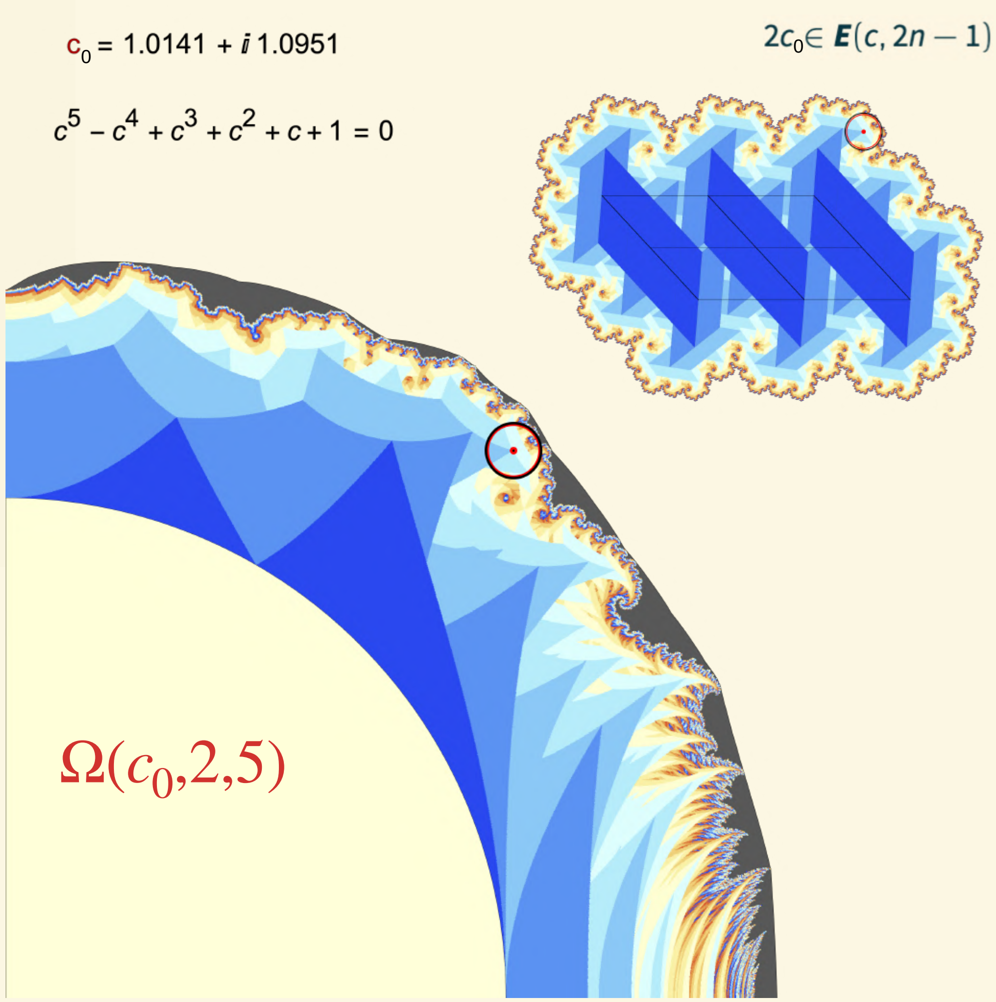
Definition (component $\boldsymbol{\Omega}(c_0, n, m)$)

Let c_0 be a root of a polynomial $q(z)$ of degree m with coefficients restricted to the integers from $-n+1$ to $n-1$, i.e. $\mathcal{A}_n \cup \mathcal{A}_{n-1}$. The *component* $\boldsymbol{\Omega}(c_0, n, m)$ is defined as the maximal connected open set containing c_0, of parameter values c close to c_0 for which the following condition holds as $c_0 \to c$

$$q(c) \in \frac{\boldsymbol{P}(c,n)}{c}.$$

Theorem (inner stability)

$\boldsymbol{\Omega}(c_0, n, m) \subset \operatorname{int}(\mathcal{M}_n)$.

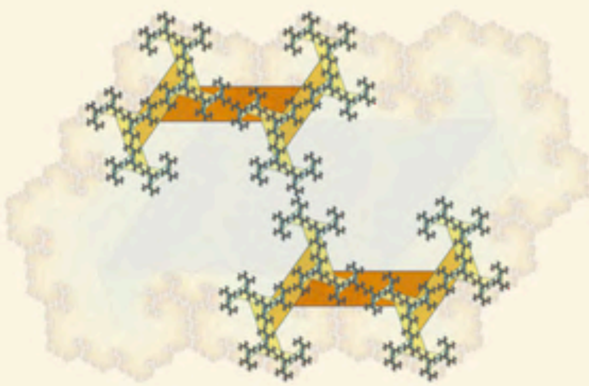


Lemma (Minkowski sum)

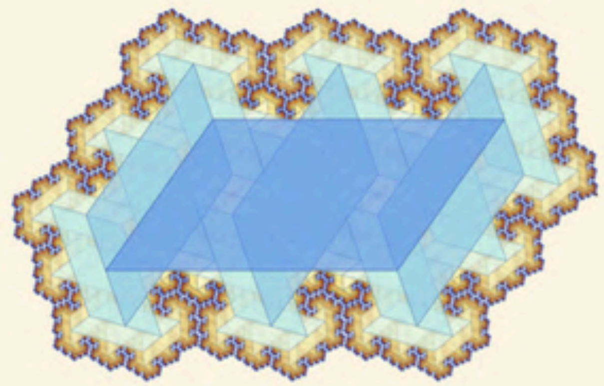
The Minkowski sum and geometric difference

$$\mathbf{E}(c, 2n-1) = \mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$$

$$= \mathbf{E}(c,n) \ominus \mathbf{E}(c,n)$$



$\mathbf{E}(c,n) \oplus \mathbf{E}(c,n)$



$\mathbf{E}(c, 2n-1)$

Our main result, the combinatorial code structure of $\mathcal{M}_n$, is based on the following discovery that applies to all $c \in \mathcal{M}_n$.

Lemma (inner parallelogram)

Let $c = a + ib$, if $\mathbf{E}(c,n)$ is connected, then its Minkowski sum, $\mathbf{E}(c, 2n-1)$, contains $\mathbf{P}(c, 2n-1) = \mathbf{P}(c,n) \oplus \mathbf{P}(c,n) = 2\mathbf{P}(c,n)$ where $\mathbf{P}(c,n)$ is the parametric region centered at 0 given either by the parallelogram

$$\mathbf{P}(c,n) := \left\{ x+iy \in \mathbb{C} : |y| < \frac{|b|}{|c|^2}, -n+1-\frac{y\cdot a}{b} < x < n-1-\frac{y\cdot a}{b} \right\}$$

or, if $c \in \mathbb{R}$, by the interval

$$\mathbf{P}(c,n) := (1-n-|c^{-1}|, n-1+|c^{-1}|).$$

Definition (component $\mathbf{\Omega}(c_0, n, m)$)

Let c_0 be a root of a polynomial $q(z)$ of degree m with coefficients restricted to the integers from $-n+1$ to $n-1$, i.e. $\mathcal{A}_n \cup \mathcal{A}_{n-1}$. The *component* $\mathbf{\Omega}(c_0, n, m)$ is defined as the maximal connected open set containing c_0, of parameter values c close to c_0 for which the following condition holds as $c_0 \to c$

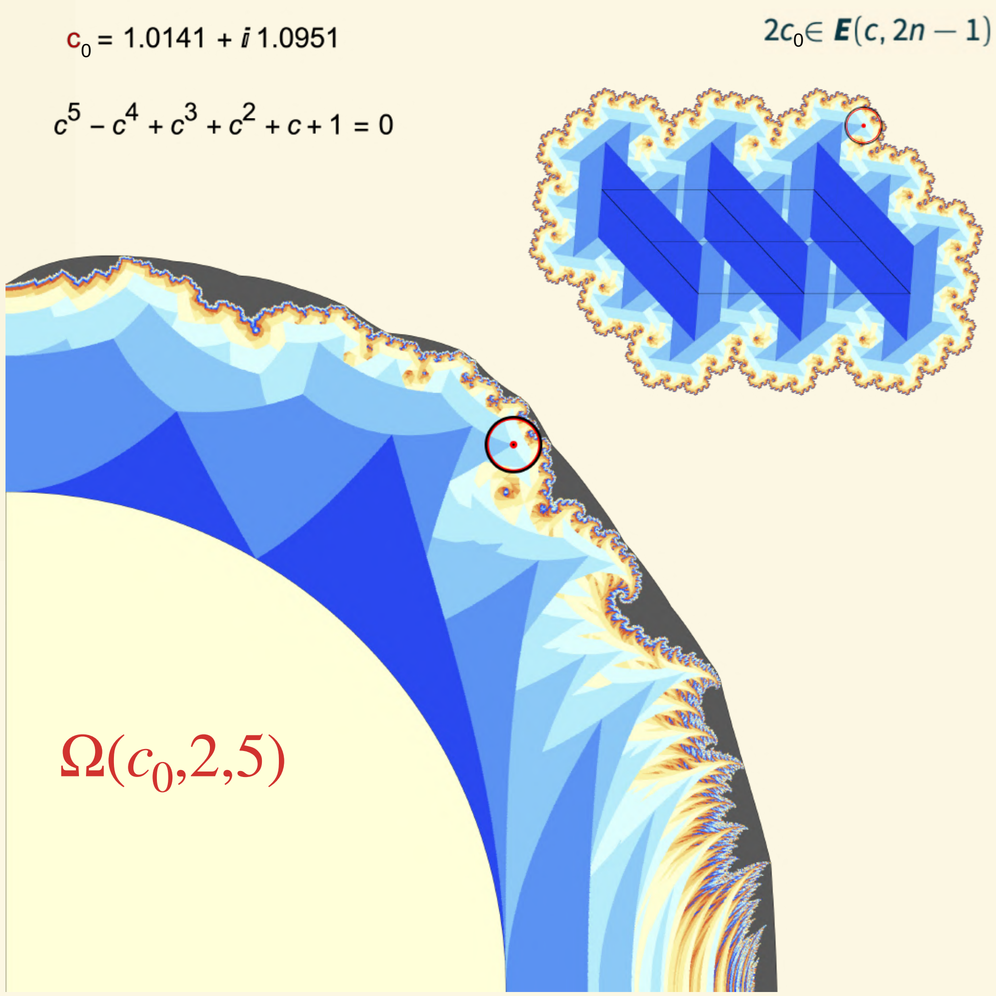
$$q(c) \in \frac{\mathbf{P}(c,n)}{c}.$$

Theorem (inner stability)

$\mathbf{\Omega}(c_0, n, m) \subset \operatorname{int}(\mathcal{M}_n)$.

Theorem (parameters in $\partial\mathcal{M}_n$)

For each $c \in \partial\mathcal{M}_n$ there exists a sequence $\{\mathbf{\Omega}(c_0, n, m)\}_m$ of components of $\mathcal{M}_n$ such that $c_0 \to c$ as $m \to \infty$.



Proposition

Let $\boldsymbol{E}(c, 2n-1, *)$ denote the set of all polynomials with coefficients in $\mathcal{A}_{2n-1}$. We have $\boldsymbol{E}(c, 2n-1) = \operatorname{clos}(\boldsymbol{E}(c, 2n-1, *))$.

Lemma

$\mathcal{M}_n = \operatorname{clos}(\mathcal{M}_{n,0})$,
where

$$\mathcal{M}_{n,0} := \left\{ c^{-1} \in \mathbb{D}^* : \quad 2c \in \boldsymbol{E}(c, 2n-1, *) \right\}.$$

Corollary ($\mathcal{M}_n \setminus \mathbb{R}$ is regular-closed)

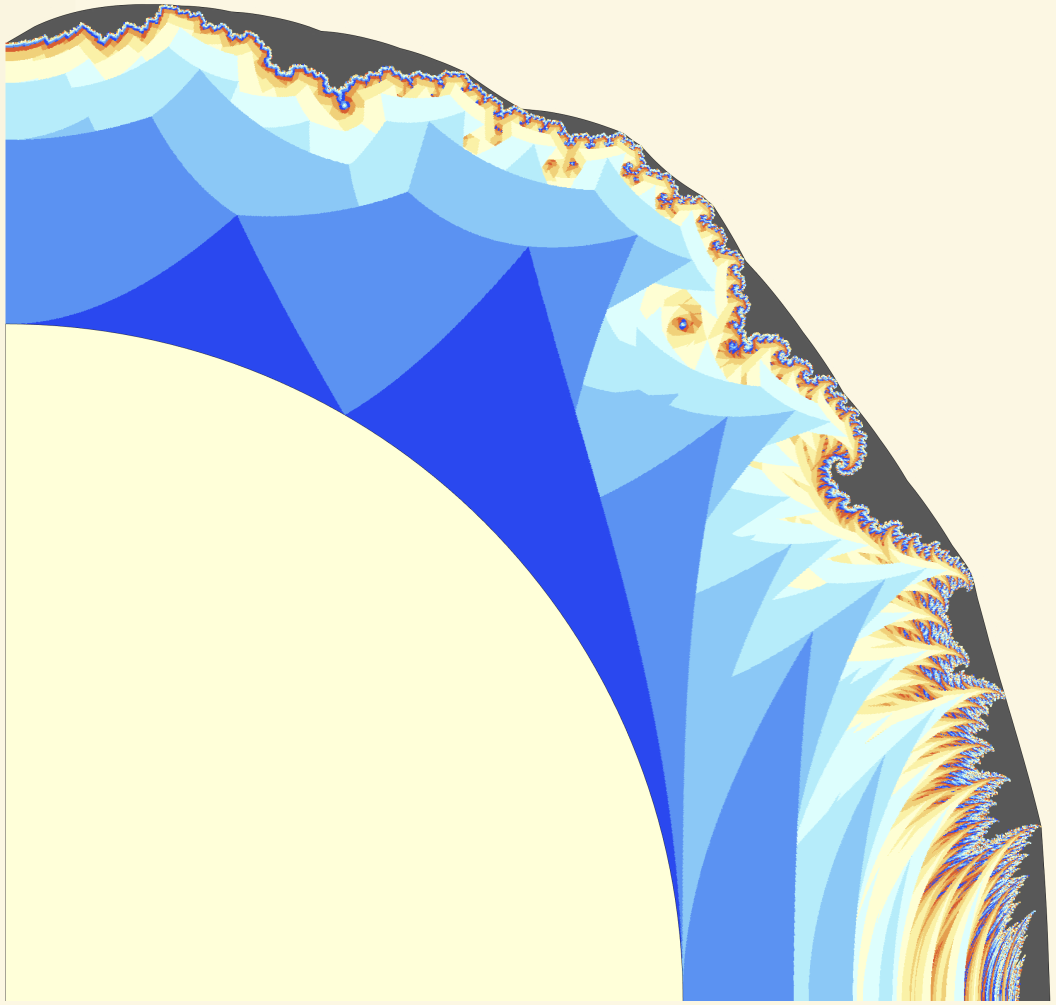
The interior of $\mathcal{M}_n$ is dense away from $\mathcal{M}_n \cap \mathbb{R}$, that is,

$$\operatorname{clos}\big(\operatorname{int}(\mathcal{M}_n)\big) \cup (\mathcal{M}_n \cap \mathbb{R}) = \mathcal{M}_n.$$

Definition (component $\boldsymbol{\Omega}(c_0, n, m)$)

Let $c_0 \in \mathcal{M}_{n,0}$ be a root of a polynomial $q(z)$ of degree m with coefficients restricted to the integers from $-n+1$ to $n-1$, i.e. $\mathcal{A}_n \cup \mathcal{A}_{n-1}$. The *component* $\boldsymbol{\Omega}(c_0, n, m)$ is defined as the maximal connected open set containing c_0, of parameter values c close to c_0 for which the following condition holds as $c_0 \to c$

$$q(c) \in \frac{\boldsymbol{P}(c, n)}{c}.$$

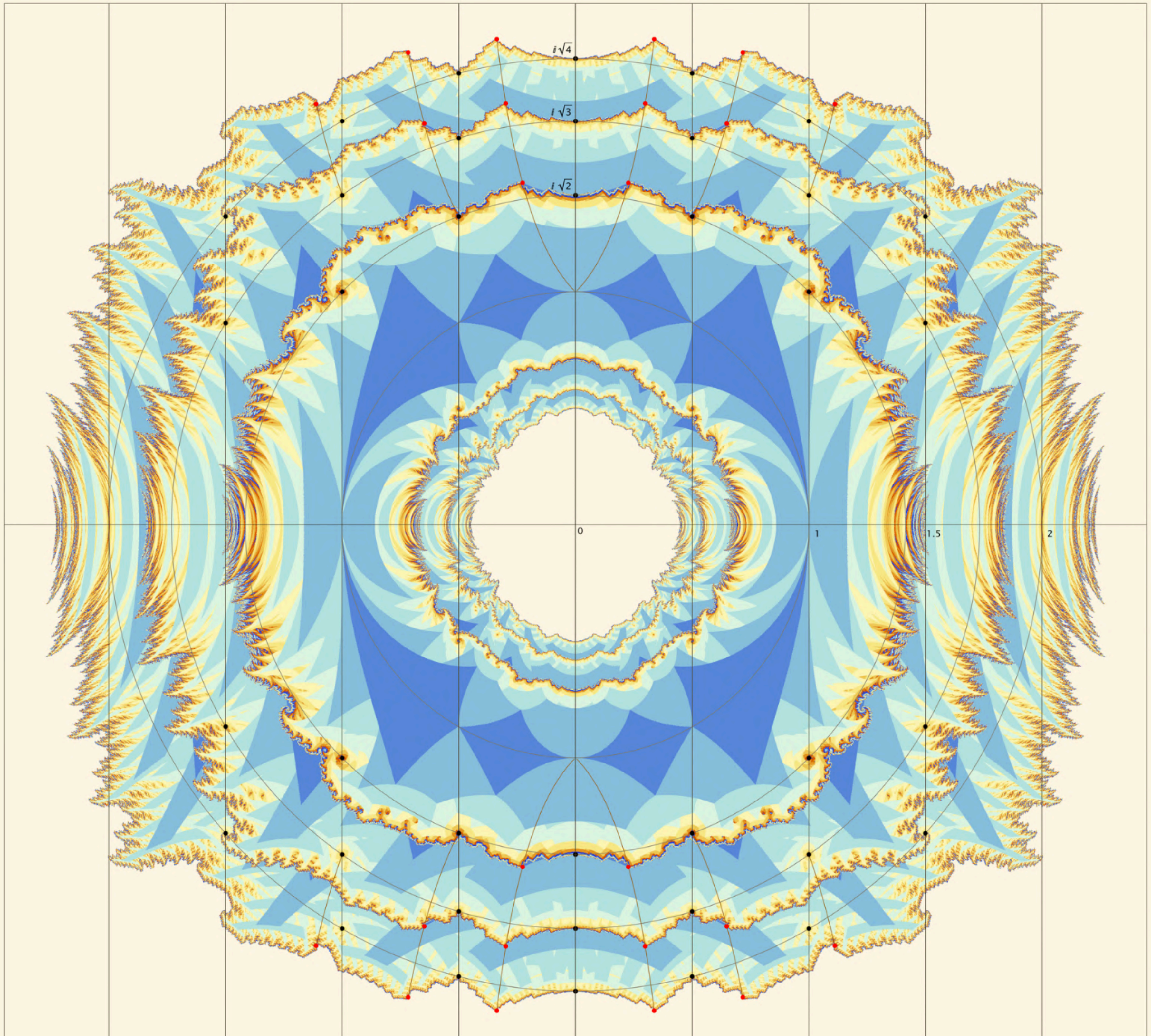


Theorem (inner stability)

$\boldsymbol{\Omega}(c_0, n, m) \subset \operatorname{int}(\mathcal{M}_n)$.

Theorem (parameters in $\partial\mathcal{M}_n$)

For each $c \in \partial\mathcal{M}_n$ there exists a sequence $\{\boldsymbol{\Omega}(c_0, n, m)\}_m$ of components of $\mathcal{M}_n$ such that $c_0 \to c$ as $m \to \infty$.



Lemma

$\mathcal{M}_n \subset \mathcal{M}_{n+1}$

Proposition (nested components)

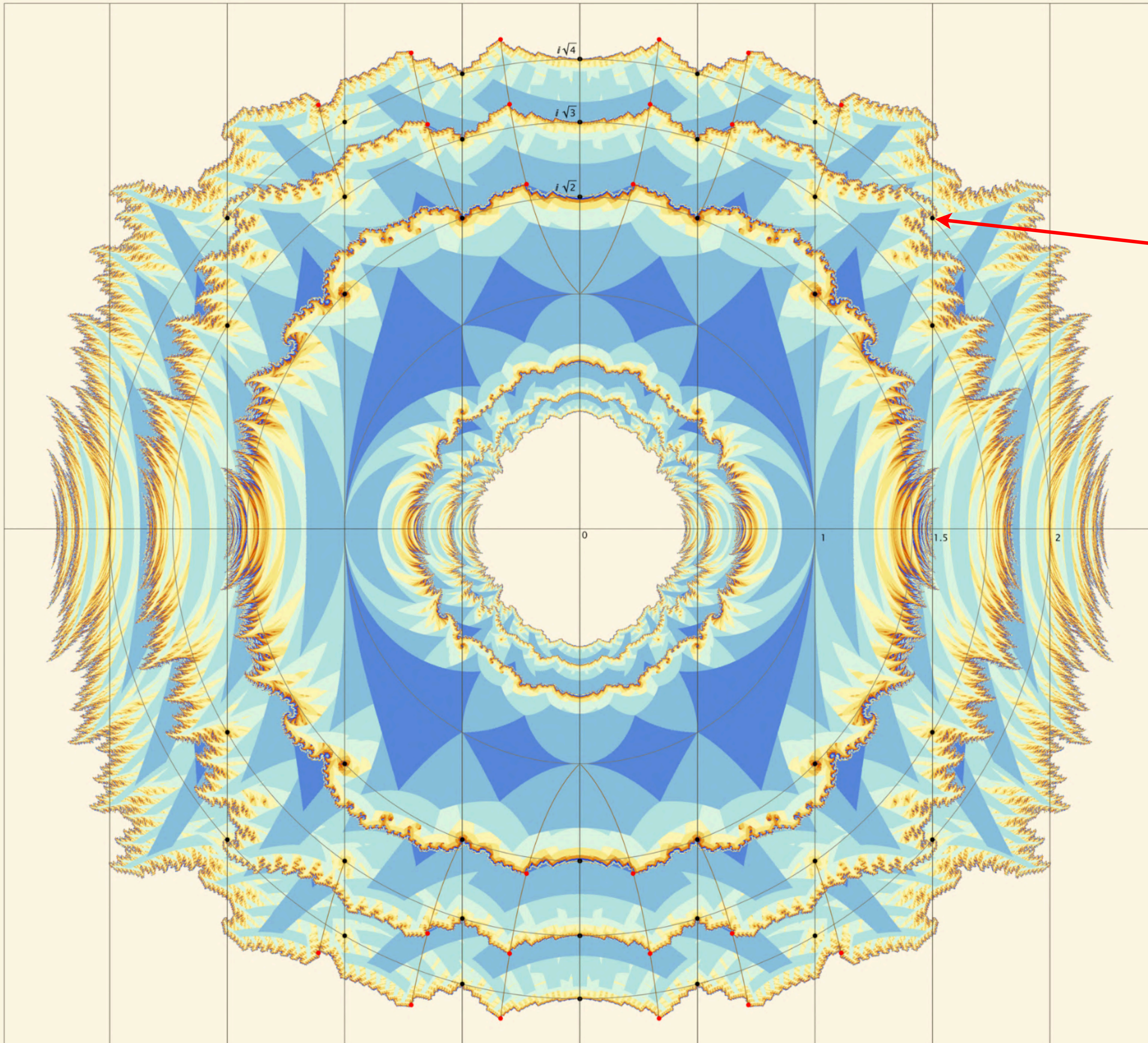
$\mathbf{\Omega}(c_0, n, m) \subset \mathbf{\Omega}(c_0, n+1, m)$

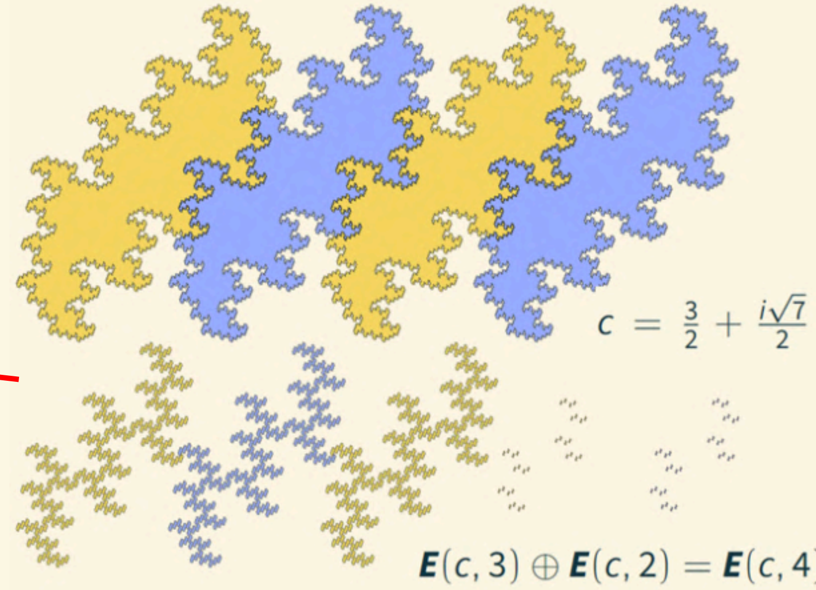
Lemma (*n*-fold iterated sumset)

$$\mathbf{E}(c,n) = \left\{ p_1 + p_2 + \ldots + p_{n-1} \,:\, p_1, p_2, \ldots, p_{n-1} \in \mathbf{E}(c,2) \right\}.$$

Corollary (dilated set)

$$\mathbf{E}(c, n+1) = \mathbf{E}(c,n) \oplus \mathbf{E}(c,2)$$





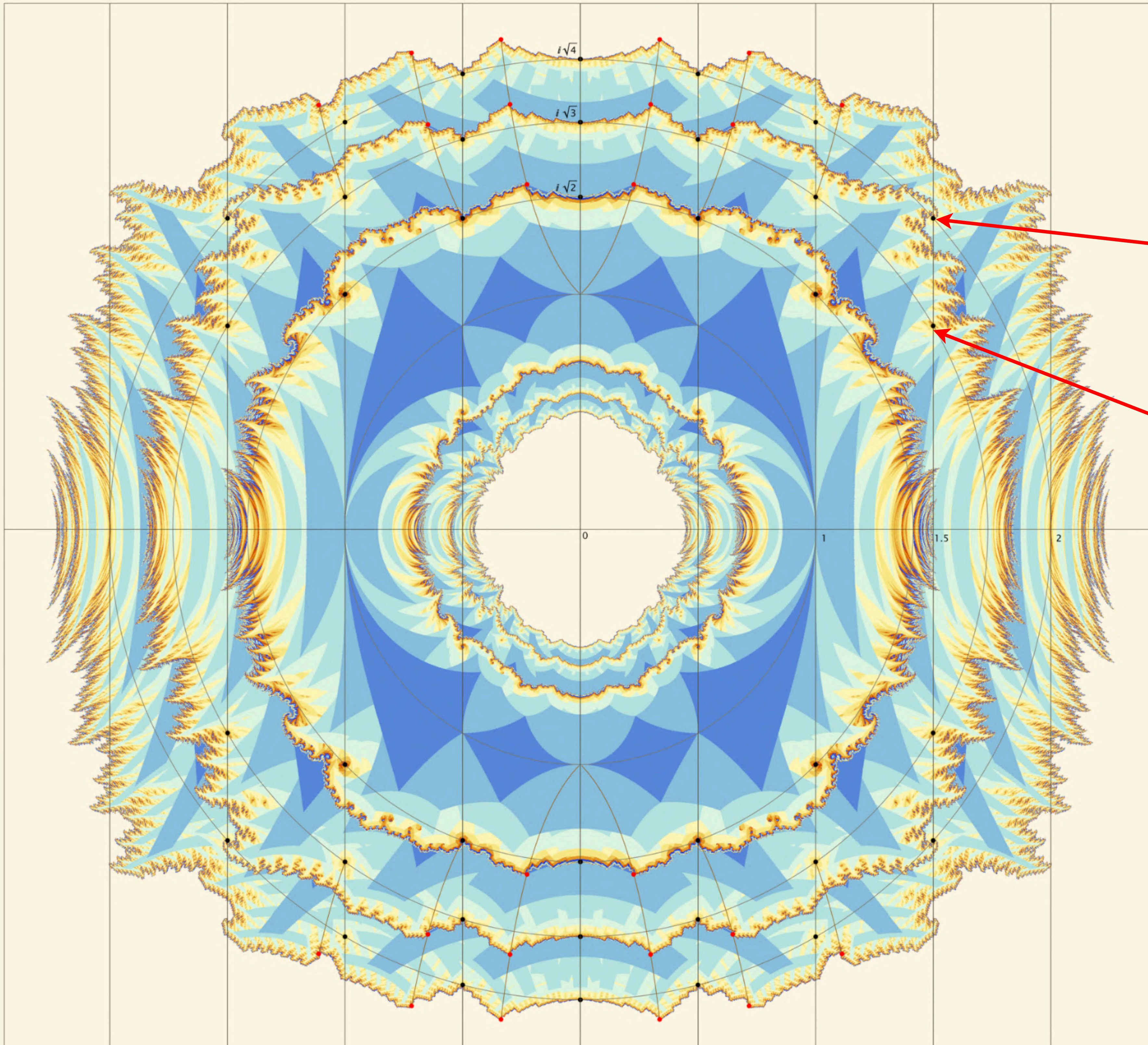
Lemma

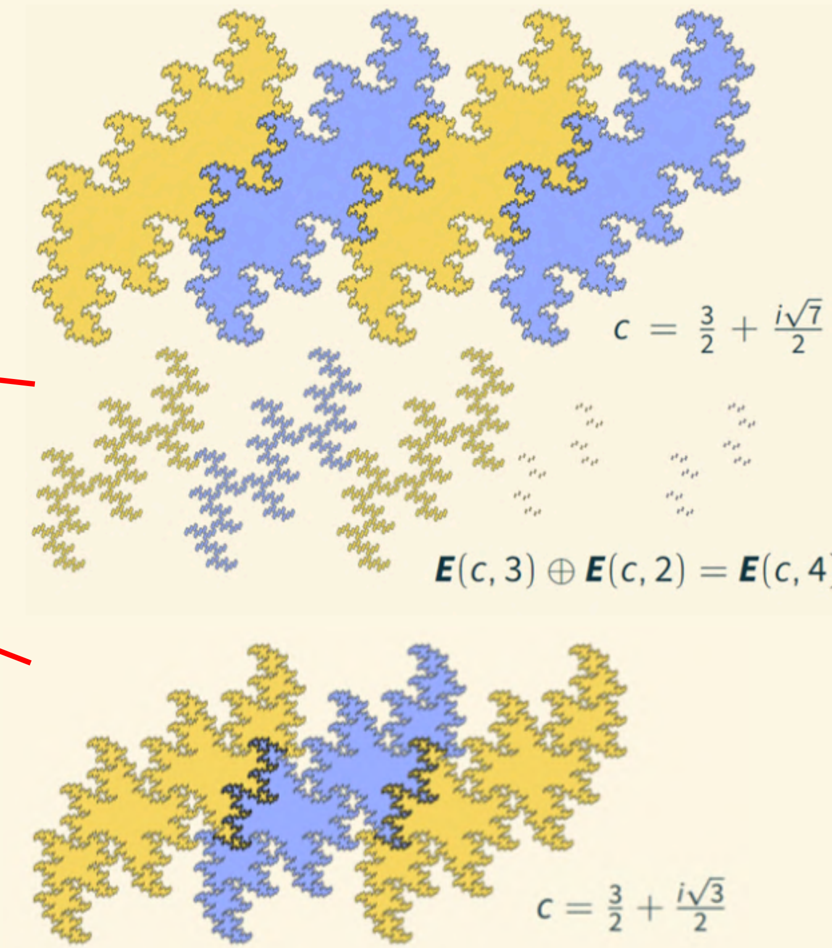
$$\mathcal{M}_n \subset \mathcal{M}_{n+1}$$

Proposition (nested components)

$$\boldsymbol{\Omega}(c_0, n, m) \subset \boldsymbol{\Omega}(c_0, n+1, m)$$

a=3



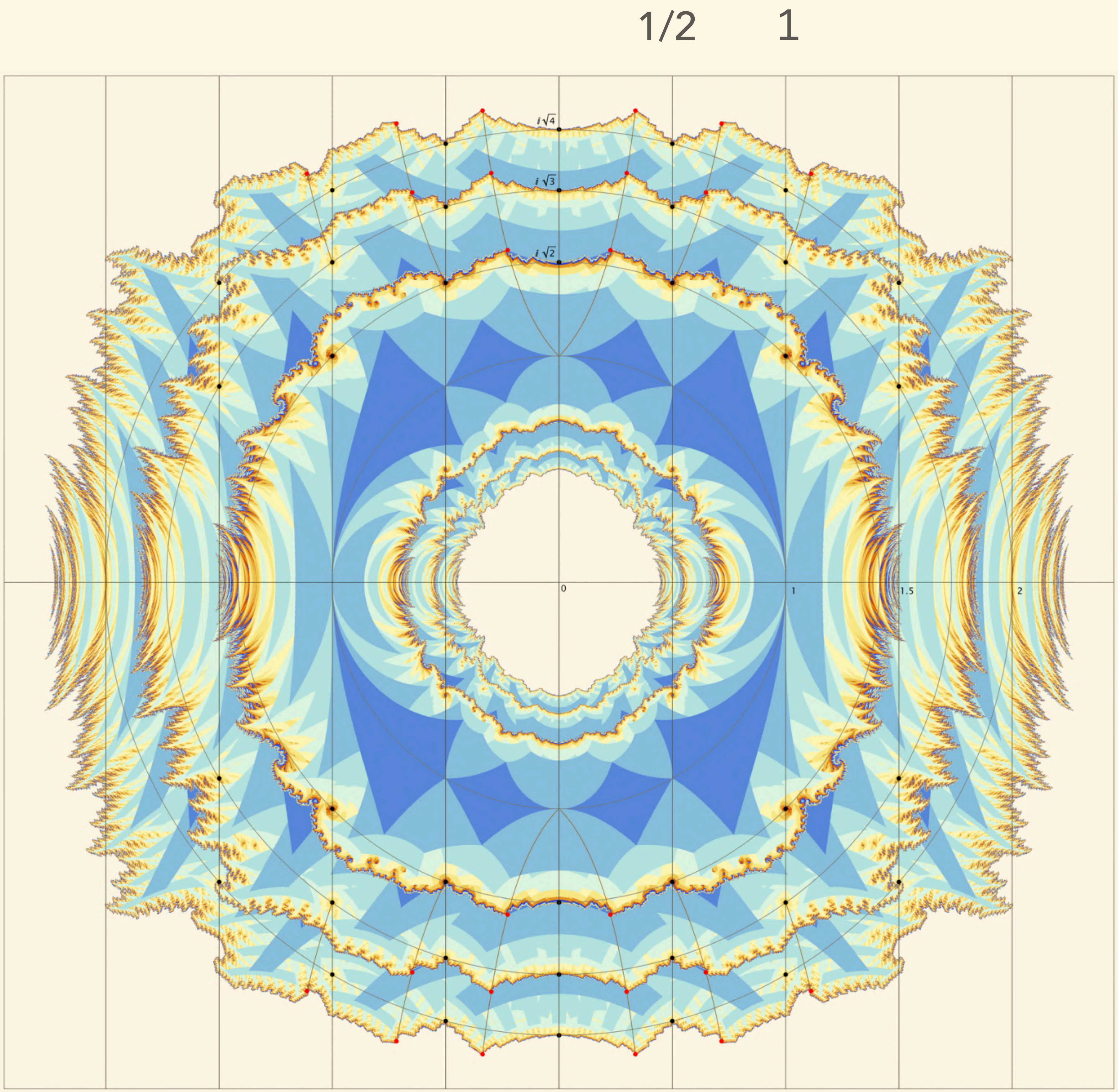


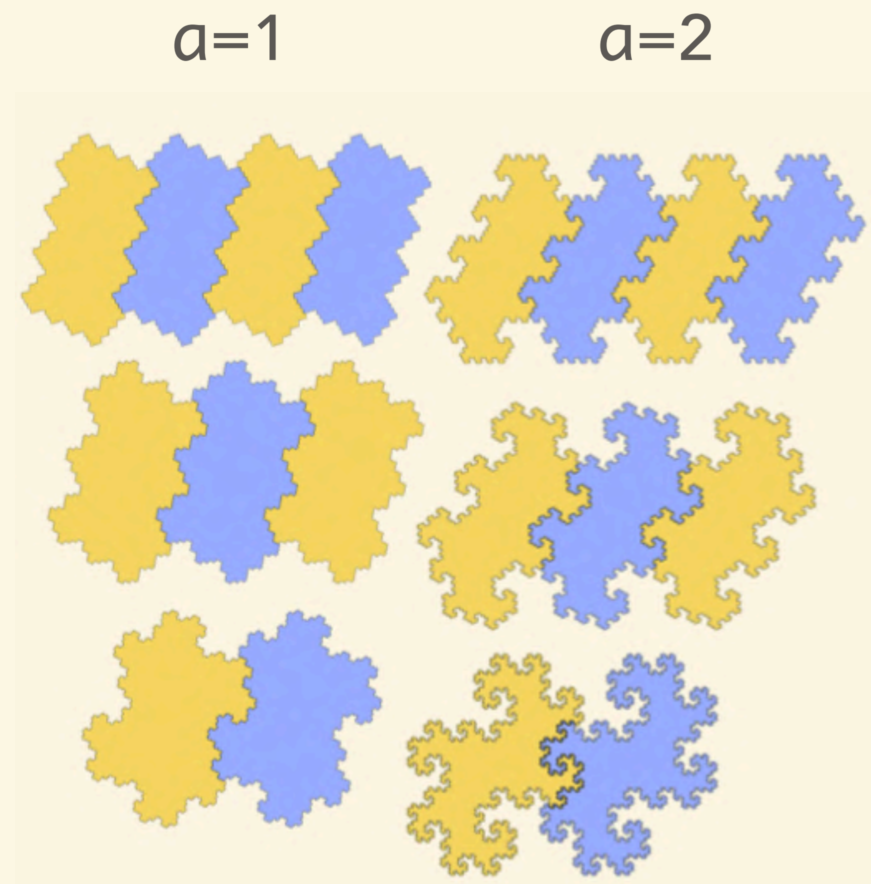
Theorem (self-affine tiles)

Let

$$c = \frac{a}{2} + \frac{i}{2}\sqrt{4b + a - 2\left\lfloor \frac{a+1}{2} \right\rfloor},$$

where $a \geqslant 0$ and $b \geqslant 1$ are a pair of integers. If $|c| = \sqrt{n}$ then $\boldsymbol{E}(c, n)$ is a planar self-affine tile with a collinear digit set.





Theorem (self-affine tiles)

Let

$$c = \frac{a}{2} + \frac{i}{2}\sqrt{4b + a - 2\left\lfloor \frac{a+1}{2} \right\rfloor},$$

where $a \geqslant 0$ and $b \geqslant 1$ are a pair of integers. If $|c| = \sqrt{n}$ then $\boldsymbol{E}(c, n)$ is a planar self-affine tile with a collinear digit set.

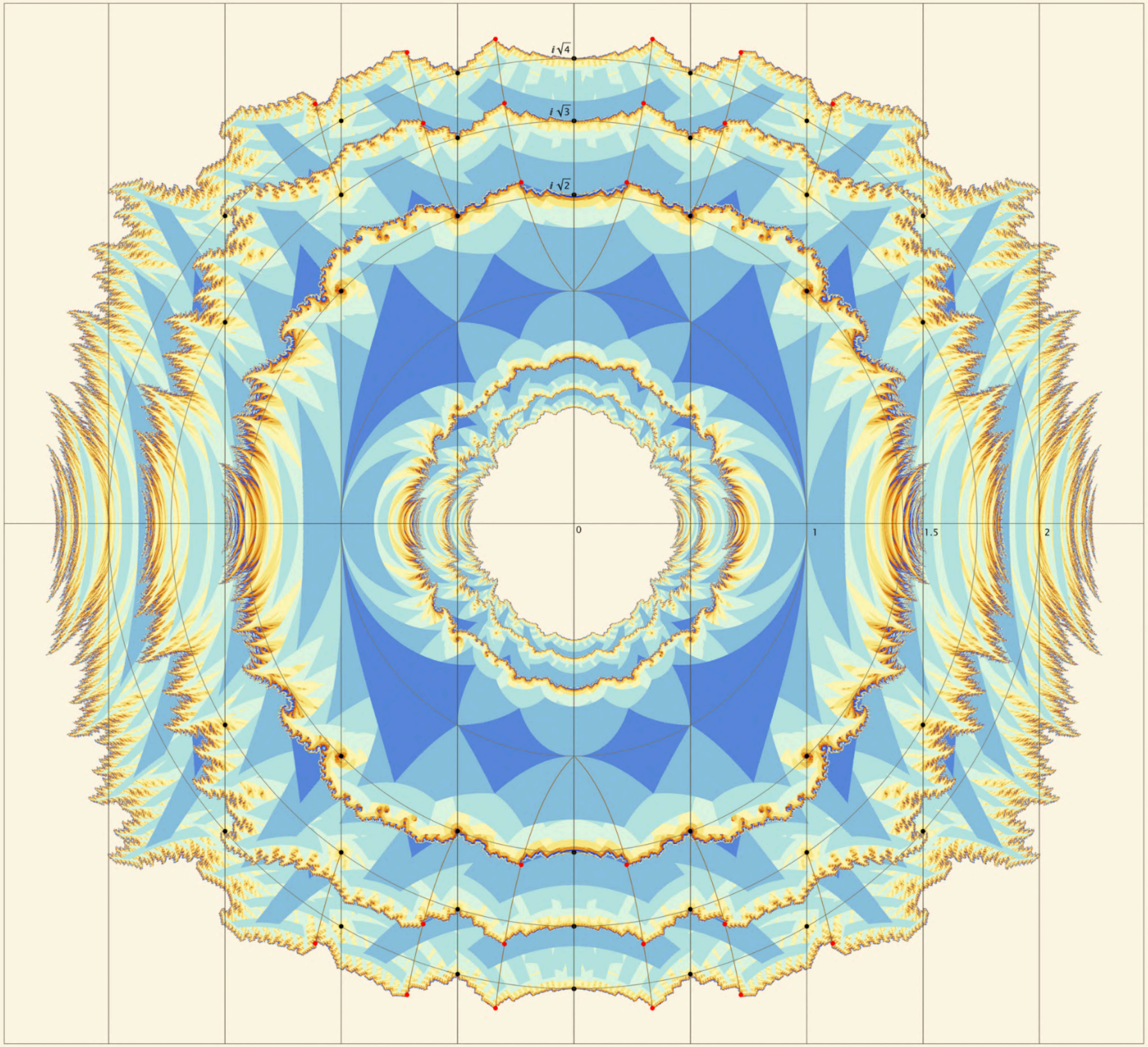
Shigeki Akiyama, Benoît Loridant, and Jörg Thuswaldner, *Topology of planar self-affine tiles with collinear digit set*, J. Fractal Geom., 8 (2021), no. 1, pp. 53–93.

Lemma

$\sqrt{n} \leqslant |c| \leqslant n$ for all $n \geqslant 2$ and $c \in \partial\mathcal{M}_n$, with equality achieved for $c = \pm i\sqrt{n}$ and $c = \pm n$ respectively.

Lemma

$\sqrt{n} \leqslant |c| < 1 + \sqrt{n-1}$ for all $n \geqslant 2$ and $c \in \partial\mathcal{M}_n \backslash \mathbb{R}$.



Lemma

$\sqrt{n} \leqslant |c| \leqslant n$ for all $n \geqslant 2$ and $c \in \partial\mathcal{M}_n$, with equality achieved for $c = \pm i\sqrt{n}$ and $c = \pm n$ respectively.

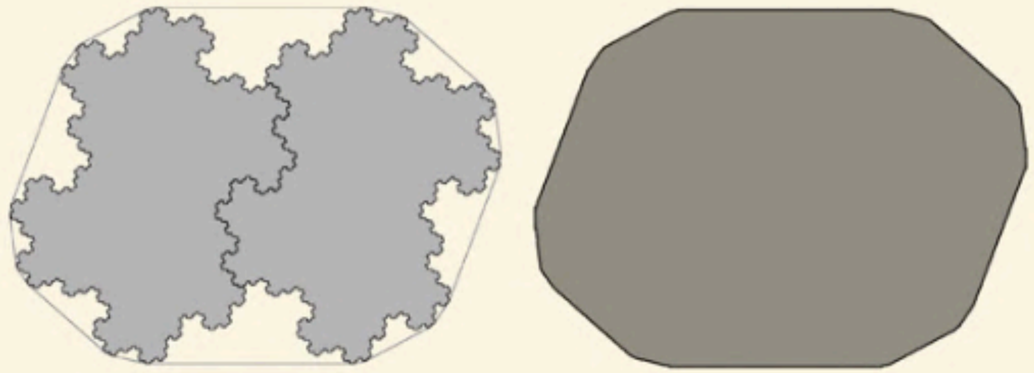
Lemma

$\sqrt{n} \leqslant |c| < 1 + \sqrt{n-1}$ for all $n \geqslant 2$ and $c \in \partial\mathcal{M}_n \backslash \mathbb{R}$.

Lemma (convex hull)

The convex hull of $\boldsymbol{E}(c,n)$, denoted by $\boldsymbol{H}(c,n) := \mathrm{Conv}(\boldsymbol{E}(c,n))$, is the set of power series with real coefficients in $\boldsymbol{I}_n := [-n+1, n-1]$,

$$\boldsymbol{H}(c,n) = \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in \boldsymbol{I}_n \right\}.$$



Definition

Let $\mathcal{M}_n^*$ be the complement of the zero-free domain of

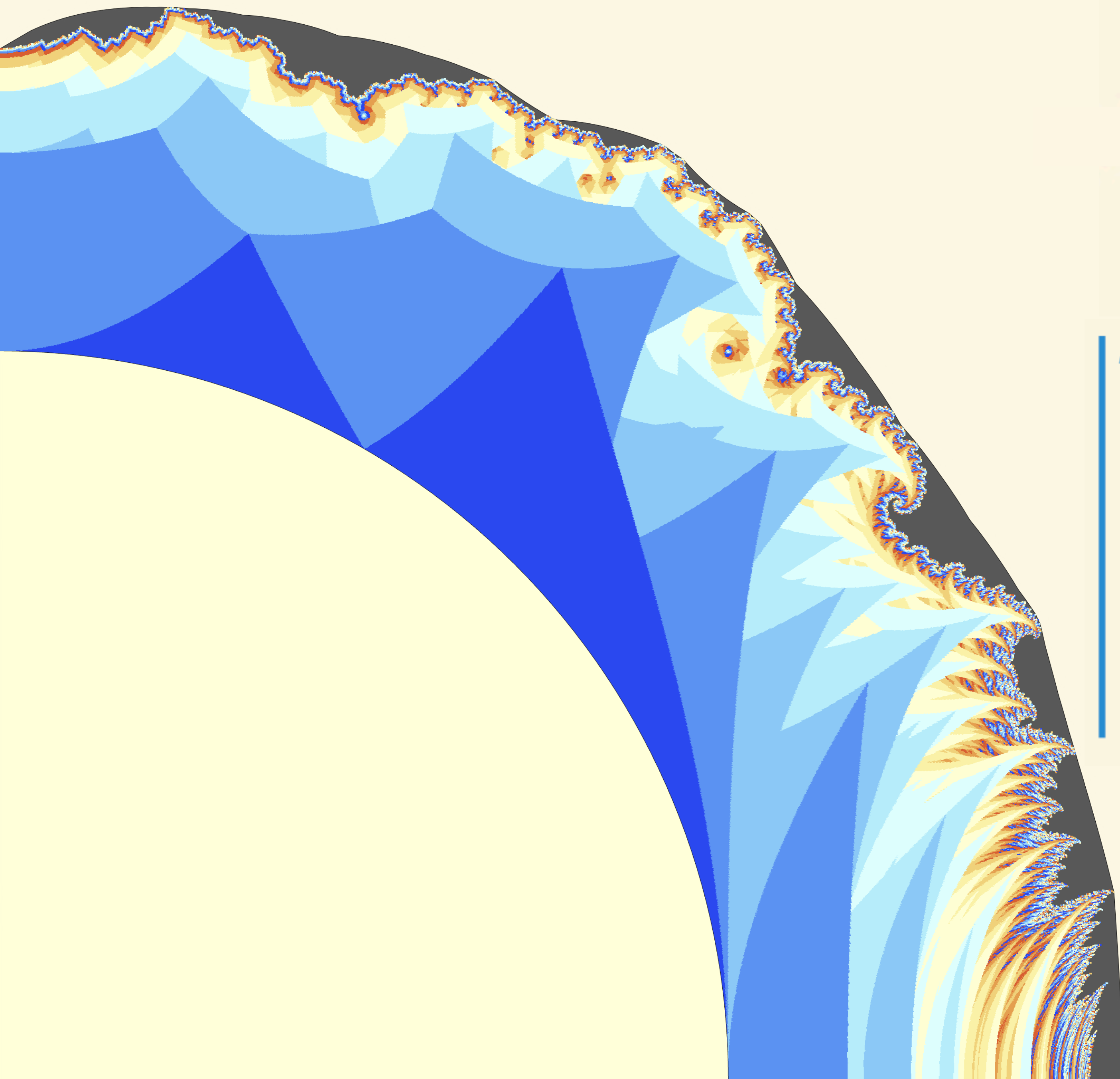
$$\mathcal{H}_n := \left\{ h(z) = 1 + \sum_{k=1}^{\infty} a_k z^k : a_k \in \boldsymbol{I}_n \right\}. \tag{1}$$

$$\mathcal{M}_n^* := \left\{ \lambda \in \mathbb{D}^* : \exists\, h(z) \in \mathcal{H}_n \text{ such that } h(\lambda) = 0 \right\}.$$

F. Beaucoup, P. Borwein, D. W. Boyd, and C. Pinner.
Power series with restricted coefficients and a root on a given ray.
Mathematics of Computation, 67(222):715–736, 1998.

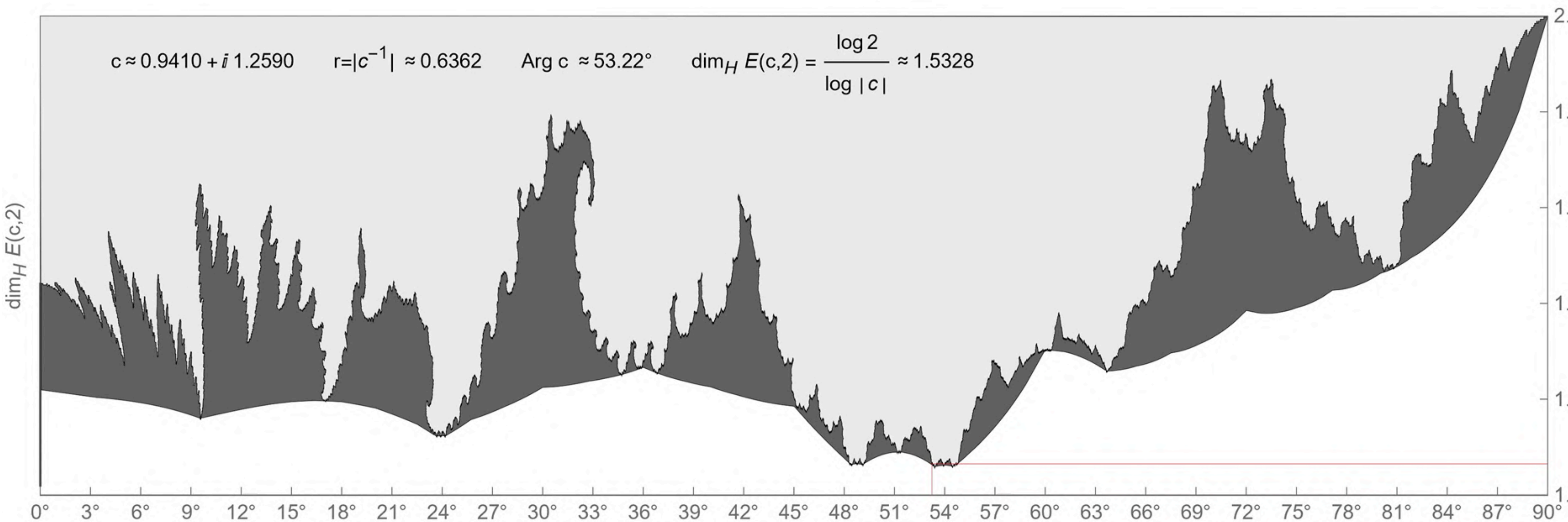
Lemma

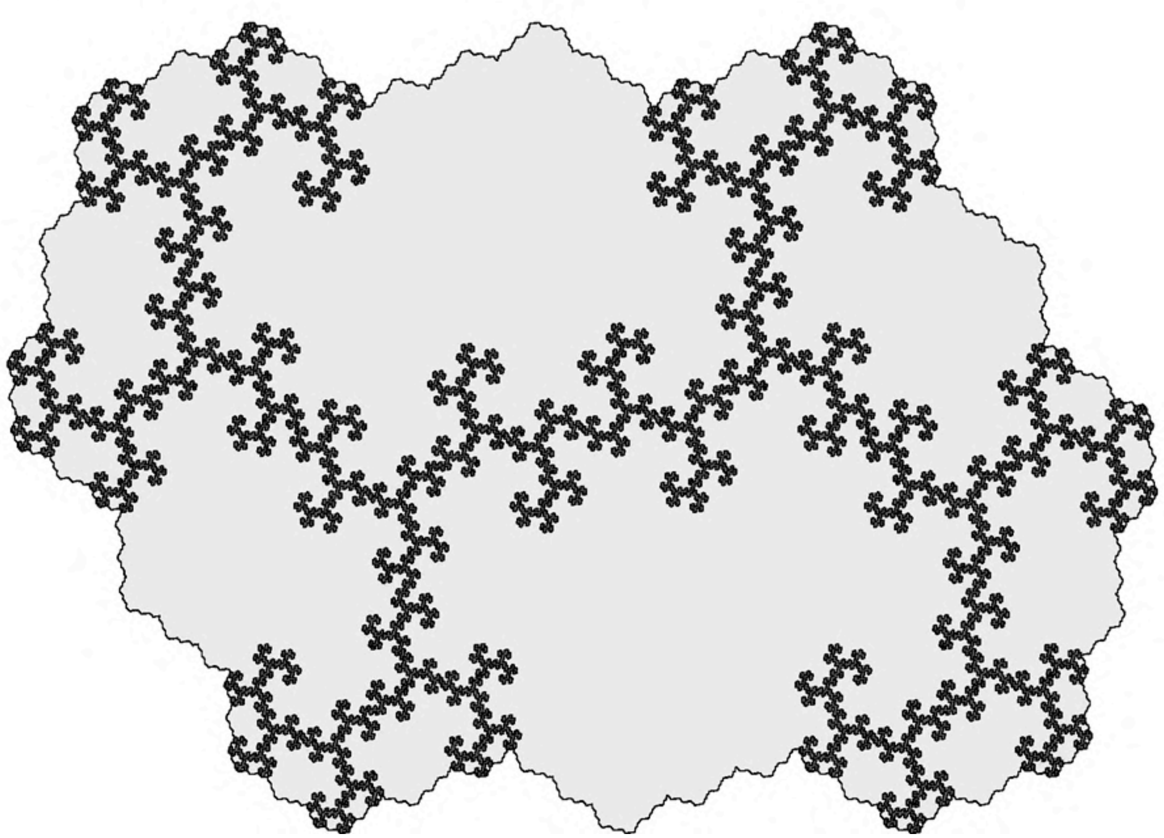
$$\mathcal{M}_n \subset \mathcal{M}_n^*.$$

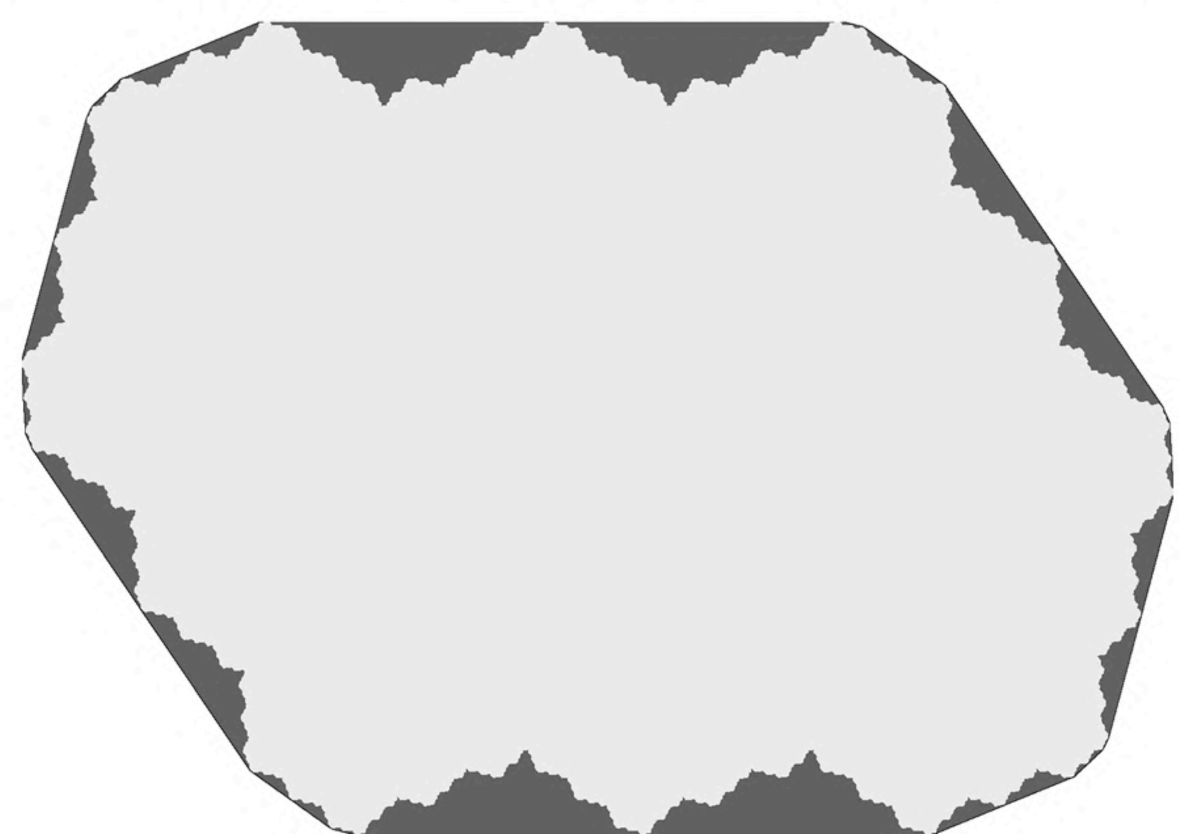


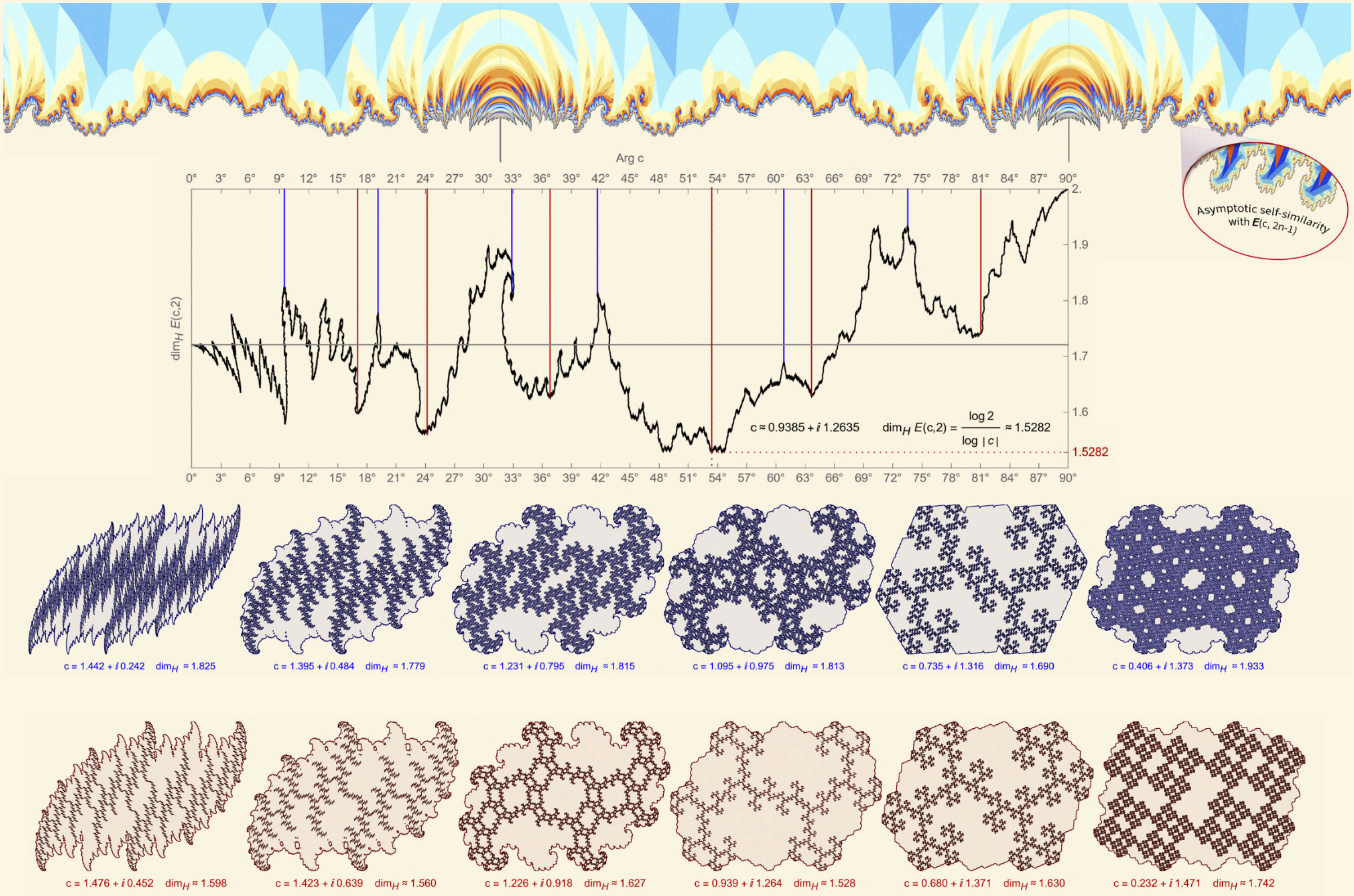
Hausdorff dimension of collinear fractals for the accessible boundary of the Mandelbrot set $\mathscr{M}_2$

youtu.be/dW4JOW-wj-Q

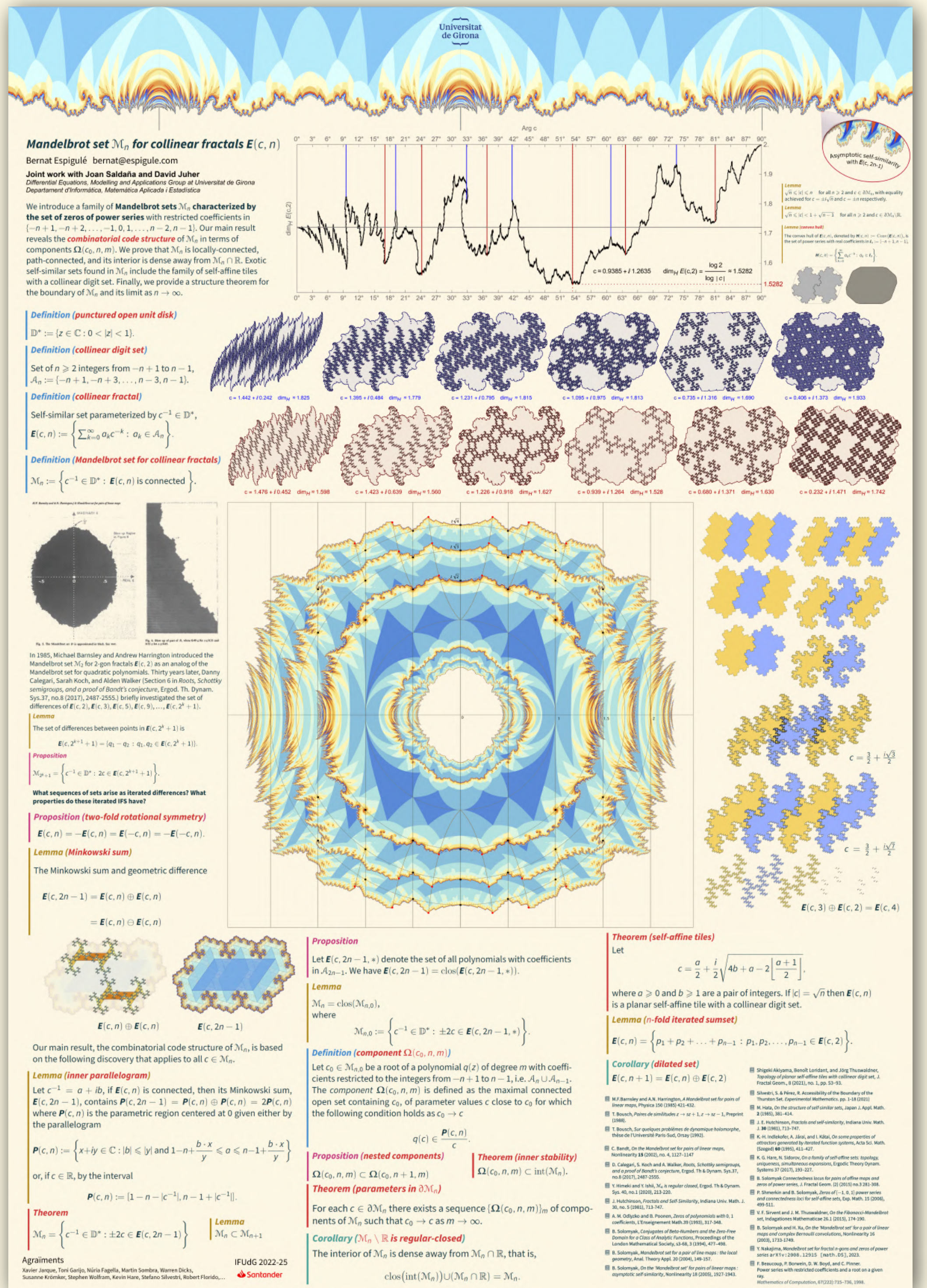


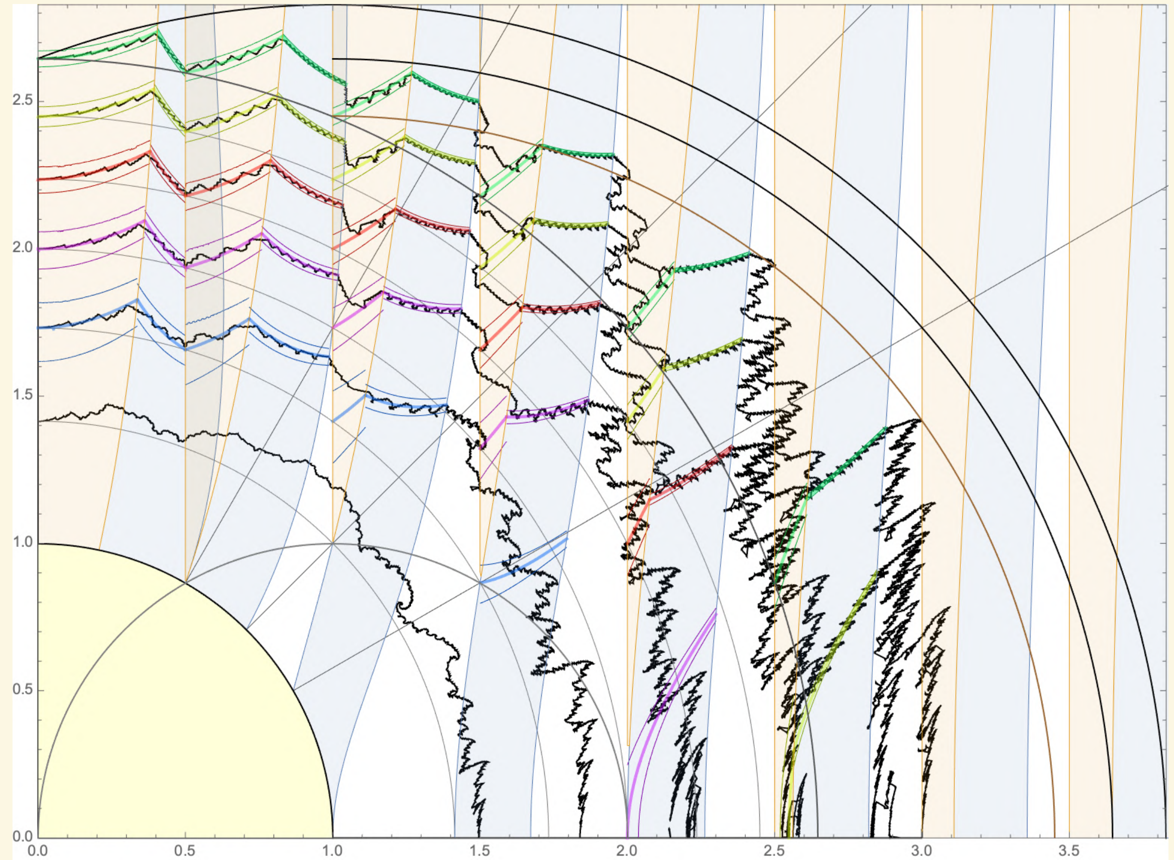






Arg c
Asymptotic self-similarity with E(c, 2n-1)
dim_H E(c,2)
c ≈ 0.9385 + i 1.2635
dim_H E(c,2) = log 2 / log |c| ≈ 1.5282
1.5282
c = 1.442 + i 0.242 dim_H ≈ 1.825
c = 1.395 + i 0.484 dim_H ≈ 1.779
c = 1.231 + i 0.795 dim_H ≈ 1.815
c = 1.095 + i 0.975 dim_H ≈ 1.813
c = 0.735 + i 1.316 dim_H ≈ 1.690
c = 0.406 + i 1.373 dim_H ≈ 1.933
c = 1.476 + i 0.452 dim_H ≈ 1.598
c = 1.423 + i 0.639 dim_H ≈ 1.560
c = 1.226 + i 0.918 dim_H ≈ 1.627
c = 0.939 + i 1.264 dim_H ≈ 1.528
c = 0.680 + i 1.371 dim_H ≈ 1.630
c = 0.232 + i 1.471 dim_H ≈ 1.742



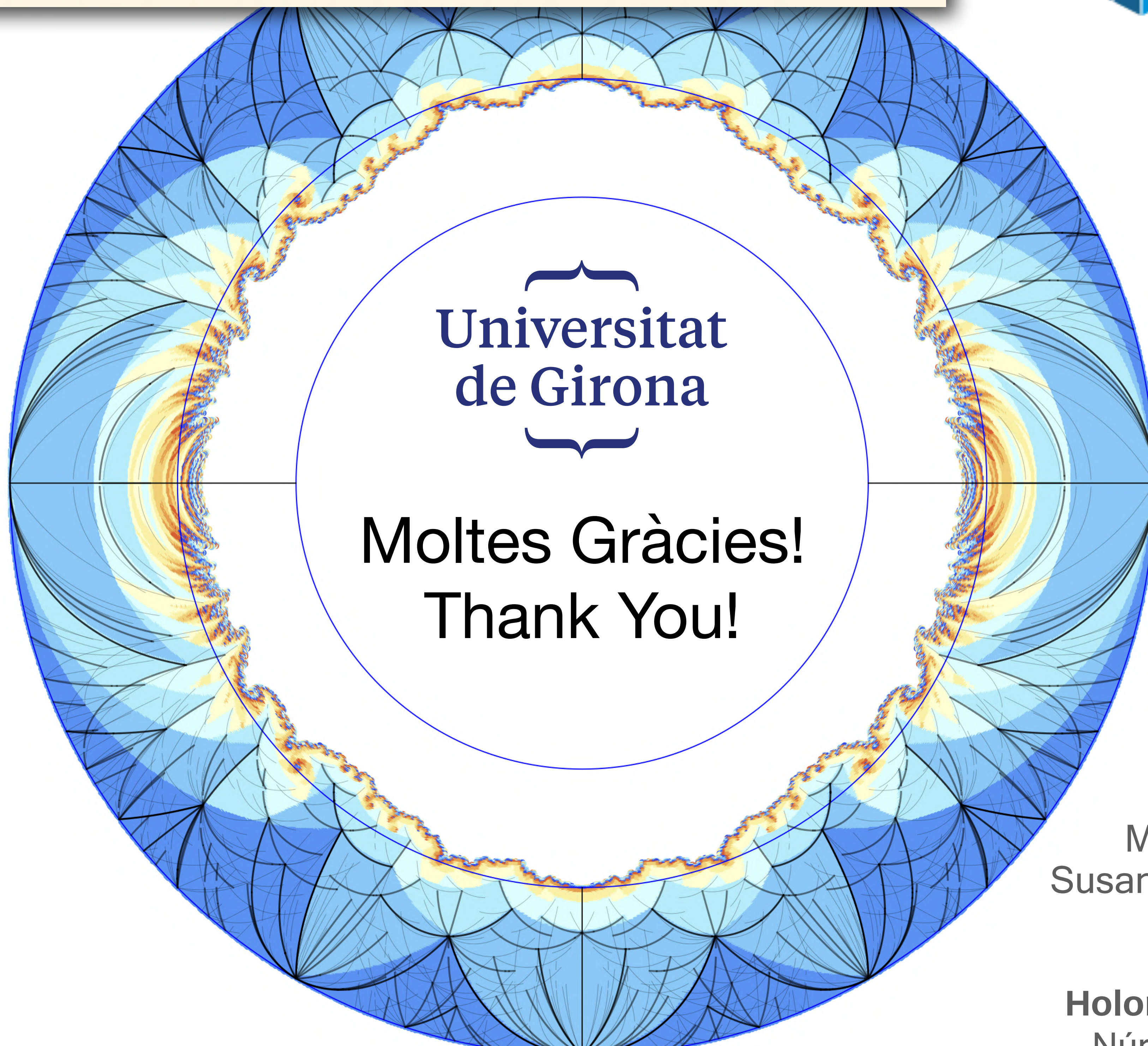


Our main contributions: combinatorial code structure, inner stability, components for any n>1, collinear tiles, ***structure theorem for the boundary of*** $\mathcal{M}_n$

- M.F.Barnsley and A.N. Harrington, *A Mandelbrot set for pairs of linear maps*, Physica 150 (1985) 421-432.
- T. Bousch, *Paires de similitudes* $z \to sz+1, z \to sz-1$, Preprint (1988).
- T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, thèse de l'Université Paris-Sud, Orsay (1992).
- C. Bandt, *On the Mandelbrot set for pairs of linear maps*, Nonlinearity **15** (2002), no. 4, 1127–1147
- D. Calegari, S. Koch and A. Walker, *Roots, Schottky semigroups, and a proof of Bandt's conjecture*, Ergod. Th & Dynam. Sys.37, no.8 (2017), 2487-2555.
- Y. Himeki and Y. Ishii, $\mathcal{M}_4$ *is regular closed*, Ergod. Th & Dynam. Sys. 40, no.1 (2020), 213-220.
- J. Hutchinson, *Fractals and Self-Similarity*, Indiana Univ. Math. J. 30, no. 5 (1981), 713-747.
- A. M. Odlyzko and B. Poonen, *Zeros of polynomials with* 0, 1 *coefficients*, L'Enseignement Math.39 (1993), 317-348.
- B. Solomyak, *Conjugates of Beta-Numbers and the Zero-Free Domain for a Class of Analytic Functions*, Proceedings of the London Mathematical Society, s3-68, 3 (1994), 477–498.
- B. Solomyak, *Mandelbrot set for a pair of line maps : the local geometry*, Anal. Theory Appl. 20 (2004), 149-157.
- B. Solomyak, *On the 'Mandelbrot set' for pairs of linear maps : asymptotic self-similarity*, Nonlinearity 18 (2005), 1927-1943.
- Silvestri, S. & Pérez, R. Accessibility of the Boundary of the Thurston Set. *Experimental Mathematics*. pp. 1-18 (2021)
- M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), 381–414.
- J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
- K.-H. Indlekofer, A. Járai, and I. Kátai, *On some properties of attractors generated by iterated function systems*, Acta Sci. Math. (Szeged) **60** (1995), 411–427.
- K. G. Hare, N. Sidorov, *On a family of self-affine sets: topology, uniqueness, simultaneous expansions*, Ergodic Theory Dynam. Systems 37 (2017), 193–227.
- B. Solomyak *Connectedness locus for pairs of affine maps and zeros of power series*, J. Fractal Geom. (2) (2015) no.3 281-308.
- P. Shmerkin and B. Solomyak, *Zeros of* $\{-1,0,1\}$ *power series and connectedness loci for self-affine sets*, Exp. Math. 15 (2006), 499-511.
- V. F. Sirvent and J. M. Thuswaldner, *On the Fibonacci-Mandelbrot set*, Indagationes Mathematicae 26.1 (2015), 174-190.
- B. Solomyak and H. Xu, *On the 'Mandelbrot set' for a pair of linear maps and complex Bernoulli convolutions*, Nonlinearity 16 (2003), 1733-1749.
- Y. Nakajima, *Mandelbrot set for fractal n-gons and zeros of power series* `arXiv:2008.12915 [math.DS]`, 2023.
- F. Beaucoup, P. Borwein, D. W. Boyd, and C. Pinner. Power series with restricted coefficients and a root on a given ray. *Mathematics of Computation*, 67(222):715–736, 1998.

Mandelbrot set $\mathcal{M}_n$ for collinear fractals $\mathbf{E}(c, n)$

International Online GSDUAB Seminar 2024



ICERM
Fall 2019, Illustrating Mathematics

2017-2018
IMUB

WOLFRAM

Stefano Silvestri,
Martin Sombra, Warren Dicks,
Susanne Krömker, Stephen Wolfram, …

Holomorphic Dynamics Group
Núria Fagella, Xavier Jarque,
Toni Garijo, Robert Florido, …

Universitat de Girona *Departament d'Informàtica, Matemàtica Aplicada i Estadística*
PhD supervisors: *Joan Saldaña and David Juher,* Differential Equations, Modelling and Applications Group