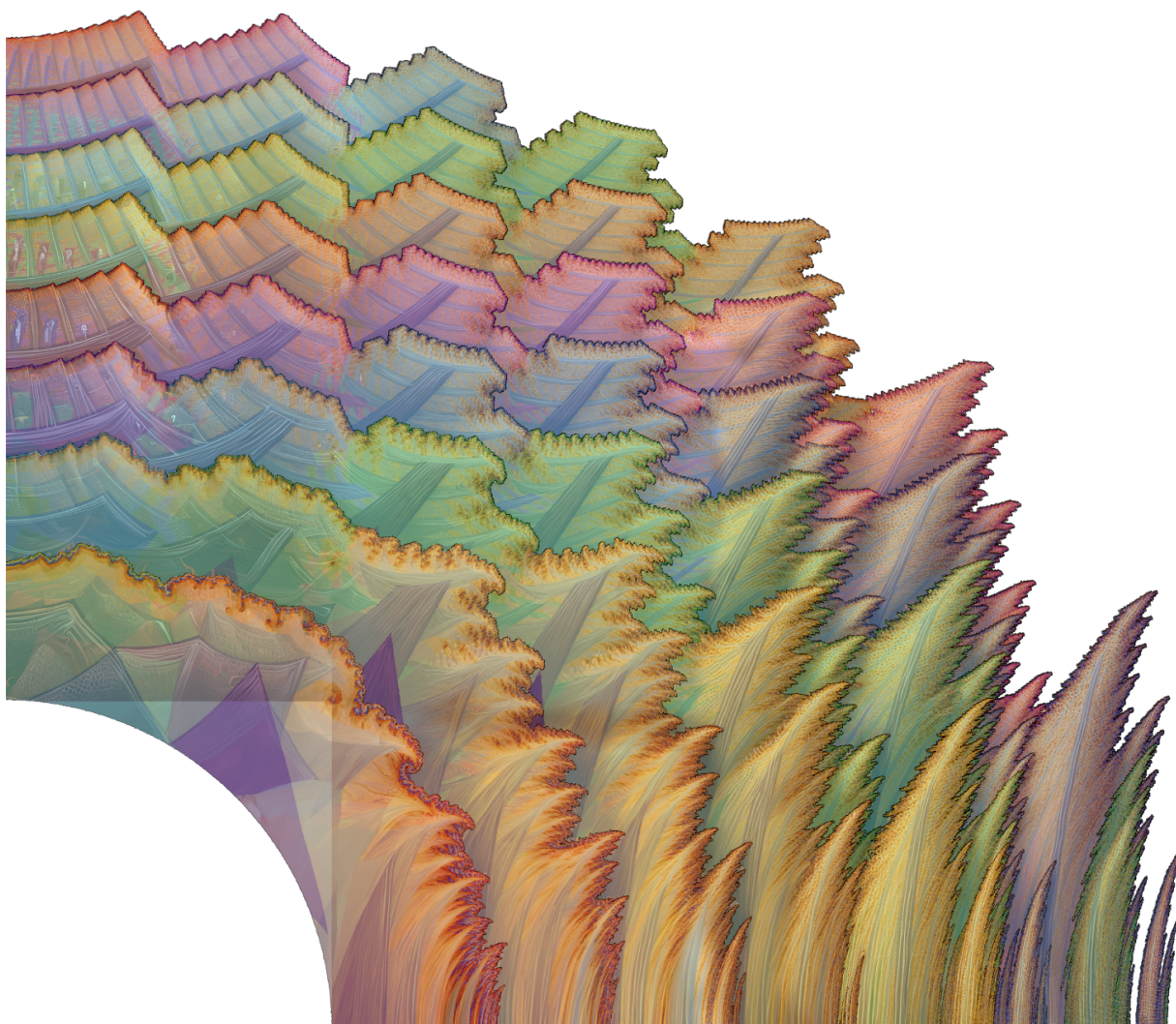


# Collinear Fractals and Connectedness Loci

Topology, Finite Capture,  
and Restricted Polynomial Roots

Bernat Espigulé Pons







DOCTORAL THESIS

# Collinear Fractals and Connectedness Loci: Topology, Finite Capture, and Restricted Polynomial Roots

Bernat Espigulé Pons

|                            |                                                  |
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# Publications and Related Manuscripts

The main chapters incorporate and extend the published article and joint finite-capture preprint listed first. The remaining works develop related questions discussed in Appendix C. Their results are attributed there and do not enter the proofs of the main chapters.

**Published article.** B. Espigule, D. Juher, and J. Saldaña, *Collinear Fractals and Bandt's Conjecture*, *Fractal and Fractional* 8(12), article 725 (2024) [EJS24].

**Joint finite-capture preprint.** B. Espigule and D. Juher, *Finite capture and the closure of roots of restricted polynomials*, arXiv:2603.07397 (2026) [EJ26].

**Related manuscript on regular-closedness.** B. Espigule, *Regular closedness off the real axis for collinear self-similar connectedness loci*, unpublished manuscript (2026) [Esp26g]. This manuscript proves regular-closedness away from the real axis for every  $n \geq 2$  by a perturbative trap argument; see Section C.1.

**Related preprint on complementary components.** B. Espigule, *Infinitely many holes in connectedness loci for collinear affine iterated function systems*, arXiv:2606.01467 (2026) [Esp26e]. This work proves the existence of infinitely many holes for every  $n \geq 2$ , with the larger-alphabet cases extending the known binary result; see Appendix C.

**Related manuscript on collinear dendrites.** B. Espigule, *Contact multiplicity and a sharp six-map cutoff for collinear self-similar dendrites*, unpublished manuscript (2026) [Esp26c]. It gives exact non-real dendrite examples for every  $2 \leq n \leq 6$  and proves that no non-real parameter yields a dendrite for  $n \geq 7$ . The contact-multiplicity argument imposes no separation or finite-intersection hypothesis; see Section C.5.

**Related manuscript on five-symbol double zeros.** B. Espigule, *The five-symbol double-zero set and the Tribonacci constant*, unpublished manuscript (2026) [Esp26h]. It determines the five-symbol double-zero set exactly, resolving the numerical observation in [SS06, Remark 6.4], and identifies the ternary antenna endpoint; see Section C.4.

**Related manuscript on sixth-cyclotomic antenna thresholds.** B. Espigule, *Birth, activation, and exact antenna thresholds in higher-order collinear connectedness loci*, unpublished manuscript (2026) [Esp26a]. Together with the preceding  $q = 2$  result, it determines the positive-real antenna endpoint at every order  $H_q = q^2 - q + 1$ ,  $q \geq 2$ . The stronger local classification and its range are stated in Section C.4.

**Related manuscript on quadratic capture.** B. Espigule, *Logarithmic capture and bounded-height approximation at quadratic parameters of collinear fractals*, unpublished manuscript (2026) [Esp26f]. It identifies interior parameters that are isolated within the set missed by the canonical trap and determines the logarithmic growth of the shortest capture depth near them; see Section C.2.

**Related manuscript on transversality and dimension.** B. Espigule, *Explicit transversality and eventual exponential separation for collinear complex self-similar measures*, unpublished manuscript (2026) [Esp26d]. It obtains an explicit non-real double-zero exclusion radius and expected-dimension results outside a zero-dimensional exceptional set in the corresponding strict off-axis region.

# Research Software

*The purpose of computing is insight, not numbers.*

— Richard Hamming

A companion implementation is available as open-access research software: *Collinear Fractals GPU: companion software for collinear fractals, finite capture, and restricted polynomial roots* [Esp26b], at <https://github.com/espigule/collinear-fractals-gpu>.

The browser interface links parameter-plane views with the corresponding dynamical attractors. Its floating-point inverse search prunes branches using the enclosure, detects trap entry, and exports finite words and search data. These outputs support independent verification of the inequalities required for an interior or exterior certificate.

The source code is distributed under the Apache License 2.0.



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# List of Notation and Abbreviations

| Symbol                                         | Meaning                                                                                                                                                                                             |
|------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathbb{C}, \mathbb{R}, \mathbb{N}$           | Complex numbers, real numbers, and positive integers.                                                                                                                                               |
| $\mathbb{N}_0$                                 | Nonnegative integers $\{0, 1, 2, \dots\}$ .                                                                                                                                                         |
| $\mathbb{D}, \overline{\mathbb{D}}$            | Open and closed unit disk in the complex plane.                                                                                                                                                     |
| $\text{int } X, \overline{X}, \partial X$      | Interior, closure, and boundary in the relevant ambient space; for parameter loci, unless otherwise stated, the ambient space is $\mathbb{C} \setminus \overline{\mathbb{D}}$ .                     |
| $A_m, A_n, A_N$                                | Symmetric collinear alphabets $A_m = \{-m+1, -m+3, \dots, m-1\}$ ; $A_n$ is the original alphabet and $A_N$ is the difference alphabet.                                                             |
| $E(c, m), E(c, n)$                             | Collinear attractor with alphabet $A_m$ ; in particular $E(c, n)$ denotes the original $n$ -ary attractor.                                                                                          |
| $N = 2n - 1$                                   | Difference-alphabet index; $A_n - A_n = A_N$ and $E(c, n) - E(c, n) = E(c, N)$ .                                                                                                                    |
| $D_n$                                          | Restricted collinear digit set $\{-n+1, -n+2, \dots, n-1\}$ with $2D_n = A_N$ .                                                                                                                     |
| $\mathcal{R}_n$                                | Set of roots of monic polynomials whose non-leading coefficients lie in $D_n$ .                                                                                                                     |
| $\mathcal{M}_n$                                | Collinear connectedness locus; equivalently, the locus of bases $c$ for which 1 has a signed $D_n$ -expansion, with $\mathcal{M}_n = \overline{\mathcal{R}_n} \setminus \overline{\mathbb{D}}$ .    |
| $2c$                                           | Marked point whose membership in $E(c, N)$ encodes $c \in \mathcal{M}_n$ ; normalizing its address by $2c$ gives the fixed represented value 1.                                                     |
| $\mathcal{M}_2^0$                              | Binary zero-section / Littlewood-root sublocus of $\mathcal{M}_2$ .                                                                                                                                 |
| $\varepsilon, A_m^k, A_m^{\leq k}$             | Empty word; words of length $k$ ; words of length at most $k$ .                                                                                                                                     |
| $u \preceq v, uv$                              | Prefix relation and concatenation of finite words.                                                                                                                                                  |
| $\pi_{c,m}$                                    | Address map $\pi_{c,m}(\omega) = \sum_{j \geq 0} a_j c^{-j}$ for $\omega = a_0 a_1 \dots \in A_m^{\mathbb{N}_0}$ .                                                                                  |
| $f_t, g_t, f_u, g_u$                           | Forward and inverse branches, and their word compositions; $g_u = f_u^{-1}$ .                                                                                                                       |
| $\mathcal{H}_{\mathcal{F}}, \mathcal{H}_{c,m}$ | Hutchinson operator of a finite system, and of the collinear alphabet $A_m$ .                                                                                                                       |
| $\ell_s, \ell_v, \mathcal{P}_c(S, V)$          | Slanted and vertical canonical coordinates, and the corresponding canonical parallelogram.                                                                                                          |
| $\mathcal{C}(c, N)$                            | Canonical trap; for $c \in \mathcal{X}_n \setminus \mathbb{R}$ , a nonempty open parallelogram contained in $\text{int } E(c, N)$ ; Definition 6.10.                                                |
| $\mathcal{E}(c, N)$                            | Canonical enclosure; for a non-real exterior parameter, the smallest closed parallelogram in the canonical directions that contains $E(c, N)$ ; Definition 6.13.                                    |
| $\mathcal{X}_n$                                | Open two-disk parameter lens $\{c \in \mathbb{C} \setminus \overline{\mathbb{D}}:  c-1  < \sqrt{2n},  c+1  < \sqrt{2n}\}$ , equivalently $\rho^2 + 2 x  < 2n-1$ for $c = x + iy$ and $\rho =  c $ . |
| $\Theta_k(n), \Theta_{\text{fin}}(n)$          | Parameters captured in at most $k$ inverse steps, and finite-capture locus $\bigcup_{k \geq 0} \Theta_k(n)$ .                                                                                       |
| $\Xi_k(n)$                                     | Survival set at depth $k$ : parameters admitting a length- $k$ inverse word whose every prefix iterate remains in $\mathcal{E}(c, N)$ ; see Definition 6.19 and Definition 6.20.                    |
| $\overline{\Theta_{\text{fin}}(n)}$            | Closure of the finite-capture locus.                                                                                                                                                                |
| IFS                                            | Iterated function system.                                                                                                                                                                           |
| GPU                                            | Graphics processing unit.                                                                                                                                                                           |



# Preface

My earlier work on self-contacting fractal trees began with the geometry of branches meeting at their tips. Writing those contacts as equations between addresses made it possible to vary the geometry while retaining prescribed identifications. That experience led me to the collinear family studied here: equally spaced translations preserve a simple difference structure even as the number of maps increases. The question is how these pieces join, and how their contacts organize the space of parameters.

A complex parameter controls contraction and rotation. As it varies, contacts appear and disappear across infinitely many scales. Equal spacing brings this geometry into direct contact with arithmetic. Connectedness is equivalent to an overlap between neighboring pieces; subtracting their addresses produces a power series with bounded integer coefficients. After normalization, the same equation represents the fixed value 1 in a complex base. Its finite truncations give monic integer polynomials. The connectedness locus and the exterior closure of their roots are therefore two descriptions of the same parameter space.

The main task of this thesis is to extract finite, stable information from that infinite equation. An explicit inner region guarantees that an address prefix can be continued indefinitely. A complementary outer region allows impossible continuations to be discarded. These constructions lead to *finite capture*: a finite inverse orbit entering the inner region certifies an interior parameter. Exhaustion of every branch within the outer region certifies nonmembership. The proofs explain when these tests succeed and what their finite witnesses establish.

The captured interior also has a geometric organization. It unfolds layer by layer as longer inverse words are allowed. Within the non-real parameter lens, each finite stage is open and semialgebraic in each quadrant, and its boundary is absorbed within at most two further steps. The enclosure also controls the inverse search: for  $n \geq 25$ , each node of the exact tree has at most three admissible continuations throughout the specified parameter region containing the non-real boundary. Together, the covering and finite-capture arguments prove approximation by interior parameters for  $n \geq 20$ , the sharp threshold at which the lens contains the whole non-real locus.

Computation helped reveal this structure. The companion software makes the parameter and dynamical planes explorable, displays successive capture sets, and exports finite words for verification. Pictures suggested the questions; the proofs and certificate inequalities determine which conclusions follow. The method's limitations are also part of the account: finite capture is dense in the connectedness locus within the non-real lens but can miss interior parameters, and smaller alphabets require additional geometry outside the canonical lens.

The mathematical core draws on the joint article with David Juher and Joan Saldaña and the subsequent finite-capture preprint with David Juher, listed under *Publications and Related Manuscripts*. This monograph develops their results in a common notation and in the order of their mathematical dependencies. Appendix B records that provenance; Appendix C discusses separate continuations. The aim throughout is to make the movement between geometry, addresses, polynomial roots, and finite computation explicit.



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# Abstract

This thesis relates connectedness of collinear self-similar sets to roots of integer polynomials with bounded coefficients. For  $n \geq 2$ , we consider  $n$  planar similarities with equally spaced translations on a line and common multiplier  $1/c$ , where  $|c| > 1$ . Their connectedness locus  $\mathcal{M}_n$  consists of parameters with a connected attractor.

Each  $\mathcal{M}_n$  is connected and locally connected. Outside the closed unit disk, it coincides with the closure of the set of roots of monic integer polynomials whose non-leading coefficients have absolute value at most  $n - 1$ . Membership in  $\mathcal{M}_n$  is also characterized by an expansion of 1 in base  $c$  with digits in  $\{-n + 1, \dots, n - 1\}$ .

To obtain finite certificates from this relation, we develop *finite capture*. We reduce connectedness to membership of a distinguished point in the difference attractor, the set of differences between pairs of points of the original attractor. A finite sequence of inverse maps sending this point into an explicit open self-covering parallelogram in the difference attractor certifies that the parameter lies in the interior of  $\mathcal{M}_n$ .

For every  $n \geq 2$ , the captured parameters are dense in the non-real part of  $\mathcal{M}_n$  within an explicit lens-shaped parameter region. We prove that this lens contains the entire non-real locus exactly when  $n \geq 20$ . Consequently, for  $n \geq 20$ , every non-real parameter in the locus is a limit of interior parameters admitting finite certificates.

Allowing longer inverse orbits yields an increasing sequence of open sets, semialgebraic in each quadrant. Within the non-real lens, the closure of the set captured by depth  $k$  is contained in the set captured by depth  $k + 2$ . For  $n \geq 25$ , a complementary outer enclosure reduces the  $2n - 1$  possible inverse branches to at most three admissible continuations from each node of the exact search, throughout a parameter region containing the non-real boundary. The accompanying software visualizes these constructions and exports certificate data for independent verification.

**Keywords:** collinear self-similar sets; connectedness loci; restricted polynomial roots; complex-base expansions; finite capture; semialgebraic sets.



# Resum

Aquesta tesi relaciona la connexitat dels conjunts autosimilars col·lineals amb les arrels de polinomis enters amb coeficients acotats. Per a  $n \geq 2$ , considerem  $n$  similituds del pla amb translacions igualment espaiades sobre una recta i multiplicador comú  $1/c$ , on  $|c| > 1$ . El conjunt de connexitat  $\mathcal{M}_n$  recull els paràmetres que donen un atractor connex.

Cada  $\mathcal{M}_n$  és connex i localment connex. Fora del disc unitat tancat, coincideix amb la clausura del conjunt d'arrels dels polinomis mòncics enters amb coeficients no principals de valor absolut menor o igual que  $n - 1$ . La pertinença a  $\mathcal{M}_n$  també es caracteritza per l'existència d'una expansió de 1 en base  $c$  amb dígit de  $\{-n + 1, \dots, n - 1\}$ .

Per obtenir certificats finits a partir d'aquesta relació, desenvolupem el mètode de *captura finita*. Reduïm la connexitat a la pertinença d'un punt distingit a l'atractor de diferències, format per les diferències entre parelles de punts de l'atractor original. Si una successió finita d'aplicacions inverses porta aquest punt a un paral·lelogram obert explícit recobert per les seves imatges sota el sistema de diferències, queda certificat que el paràmetre pertany a l'interior de  $\mathcal{M}_n$ .

Per a tot  $n \geq 2$ , els paràmetres capturats són densos en la part no real de  $\mathcal{M}_n$  situada dins d'una regió explícita de paràmetres en forma de lent. Demostrem que aquesta lent conté tota la part no real del conjunt si i només si  $n \geq 20$ . Per tant, per a  $n \geq 20$ , tot punt no real de  $\mathcal{M}_n$  és límit de paràmetres interiors que admeten certificats finits.

En augmentar el nombre de passos inversos admesos, obtenim una successió creixent de conjunts oberts, semialgebraics en cada quadrant. Dins la part no real de la lent, tot punt de la clausura del conjunt capturat en com a màxim  $k$  passos queda capturat en com a màxim  $k + 2$  passos. Per a  $n \geq 25$ , una envolupant exterior complementària redueix les  $2n - 1$  branques inverses possibles a un màxim de tres continuacions admissibles des de cada node de la cerca exacta, en una regió de paràmetres que conté la frontera no real. El programari que acompanya la tesi permet visualitzar aquestes construccions i exportar les dades necessàries per verificar els certificats de manera independent.

**Paraules clau:** conjunts autosimilars col·lineals; conjunts de connexitat; arrels de polinomis amb coeficients restringits; expansions en bases complexes; captura finita; conjunts semialgebraics.



# Resumen

Esta tesis relaciona la conexidad de conjuntos autosimilares colineales con las raíces de polinomios enteros de coeficientes acotados. Para  $n \geq 2$ , consideramos  $n$  similitudes del plano con traslaciones igualmente espaciadas sobre una recta y multiplicador común  $1/c$ , donde  $|c| > 1$ . El conjunto de conexidad  $\mathcal{M}_n$  reúne los parámetros que dan lugar a un atractor conexo.

Cada  $\mathcal{M}_n$  es conexo y localmente conexo. Fuera del disco unidad cerrado, coincide con la clausura del conjunto de raíces de los polinomios mónicos enteros cuyos coeficientes no principales tienen valor absoluto a lo sumo  $n-1$ . La pertenencia a  $\mathcal{M}_n$  también se caracteriza por la existencia de una expansión de 1 en base  $c$  con dígitos en  $\{-n+1, \dots, n-1\}$ .

Para obtener certificados finitos a partir de esta relación, desarrollamos el método de *captura finita*. Reducimos la conexidad a la pertenencia de un punto distinguido al atractor de diferencias, formado por las diferencias entre pares de puntos del atractor original. Si una sucesión finita de aplicaciones inversas lleva este punto a un paralelogramo abierto explícito, cubierto por sus imágenes bajo el sistema de diferencias, queda certificado que el parámetro pertenece al interior de  $\mathcal{M}_n$ .

Para todo  $n \geq 2$ , los parámetros capturados son densos en la parte no real de  $\mathcal{M}_n$  situada en una región explícita con forma de lente. Demostramos que esta lente contiene toda la parte no real del conjunto si y solo si  $n \geq 20$ . En consecuencia, para  $n \geq 20$ , todo parámetro no real de  $\mathcal{M}_n$  es límite de parámetros interiores que admiten certificados finitos.

Al aumentar el número de pasos inversos permitidos, obtenemos una sucesión creciente de conjuntos abiertos, semialgebraicos en cada cuadrante. Dentro de la parte no real de la lente, todo punto de la clausura del conjunto capturado en a lo sumo  $k$  pasos queda capturado en a lo sumo  $k+2$  pasos. Para  $n \geq 25$ , una envolvente exterior complementaria reduce las  $2n-1$  ramas inversas posibles a un máximo de tres continuaciones admisibles desde cada nodo de la búsqueda exacta, en una región de parámetros que contiene la frontera no real. El software que acompaña a la tesis permite visualizar estas construcciones y exportar los datos necesarios para verificar los certificados de forma independiente.

**Palabras clave:** conjuntos autosimilares colineales; conjuntos de conexidad; raíces de polinomios con coeficientes restringidos; expansiones en bases complejas; captura finita; conjuntos semialgebraicos.



PART I.

# Orientation

Chapters 1 and 2



# Introduction, Objectives, and Roadmap

## 1.1. Motivation

When several contracted copies of a set are translated to equally spaced positions on a line, their contacts determine whether the resulting attractor is connected. A common complex multiplier controls both contraction and rotation. Varying it produces a parameter space whose geometry records contacts at every scale.

The classical model for such a parameter space is the quadratic Mandelbrot set: for  $f_c(z) = z^2 + c$ , boundedness of the critical orbit is equivalent to connectedness of the Julia set. Affine iterated function systems also have connectedness loci, although their maps need not be inverse branches of a single global map. The collinear family retains enough algebraic structure to relate this topological question to polynomial roots and digit expansions.

The binary problem originates in Barnsley and Harrington and in Hardin's thesis [BH85; Har85]. Bousch established connectedness and local connectedness of its locus [Bou93]. Bandt's inverse-orbit study led to the question whether every non-real connectedness parameter is a limit of interior parameters [Ban02]. Solomyak and Xu obtained a regional result by covering arguments [SX03], and Calegari, Koch, and Walker proved the full binary statement and infinitely many holes [CKW17]. The broader history and its methodological connections are developed in Section 1.3.

This thesis studies the collinear attractors  $E(c, n)$  uniformly in the number  $n$  of maps, building on [EJS24]. Their connectedness loci  $\mathcal{M}_n$  admit several equivalent descriptions. Bousch's consecutive-digit criterion reduces connectedness to neighboring overlap [Bou92, Chapter 1, p. 41]. Subtracting the addresses of an overlap point yields a difference attractor of the same collinear type and a power-series equation with bounded integer coefficients. After normalization, the equation represents 1 in base  $c$  with digits from  $-(n-1)$  to  $n-1$ . Finite truncations then connect the locus to the closure of roots of monic restricted polynomials.

The central problem is to turn this infinite address condition into finite, stable information. The thesis constructs an explicit covering geometry and an inverse search with complementary entry and escape tests. Successful entry provides a finite witness for an interior parameter. Organizing those witnesses by their length reveals open semialgebraic capture layers, a two-step closure property, and a branching bound that becomes independent of the alphabet size. The closure of their union recovers the non-real locus within the parameter lens, and the sharp lens-containment threshold makes this recovery global for  $n \geq 20$ .

## 1.2. Objects and Main Results

### 1.2.1. The basic objects

Throughout the thesis,  $n \geq 2$  and  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , so  $|c| > 1$ . The original digit alphabet is

$$A_n = \{-n+1, -n+3, \dots, n-3, n-1\}.$$

The  $n$ -ary collinear fractal is the unique nonempty compact set satisfying

$$E(c, n) = \bigcup_{t \in A_n} \left( t + \frac{E(c, n)}{c} \right).$$

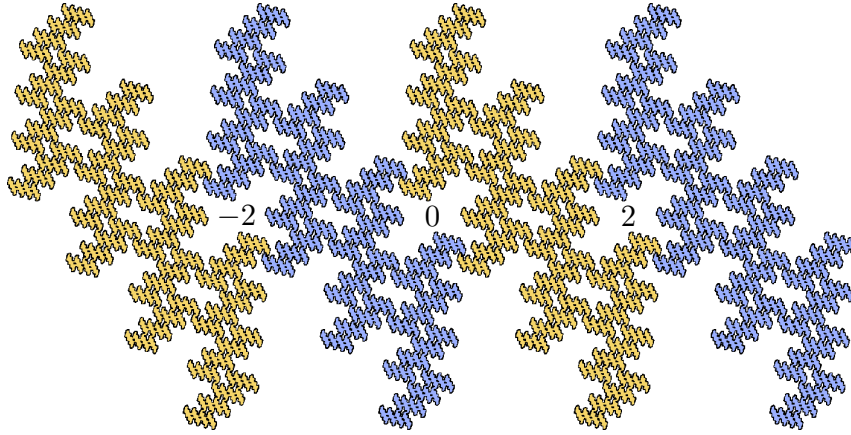
Equivalently,

$$E(c, n) = \left\{ \sum_{k=0}^{\infty} a_k c^{-k} : a_k \in A_n \right\}.$$

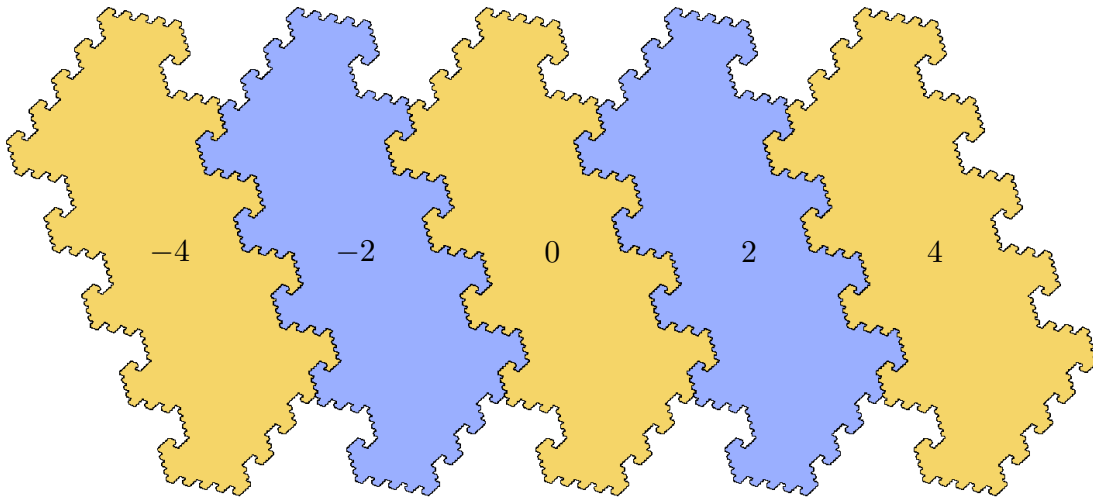
Its first-level pieces are

$$E_t(c, n) = t + \frac{E(c, n)}{c}, \quad t \in A_n.$$

Figures 1.1 and 1.2 show two geometries produced by the same construction: neighboring overlaps in a four-map attractor and a five-map attractor with nonempty interior. Both pictures lie in the dynamical plane, whose variable is  $z$ ; the parameter  $c$  determines the common scaling and rotation.



**Figure 1.1.** Example of  $E(c, 4)$  for  $c = (3 + i\sqrt{11})/2$ , illustrating intersections between neighboring first-level pieces, whose centers are the digits  $A_4 = \{-3, -1, 1, 3\}$ . The displayed values  $-2, 0, 2$  mark the real-axis midpoints between consecutive first-level centers.

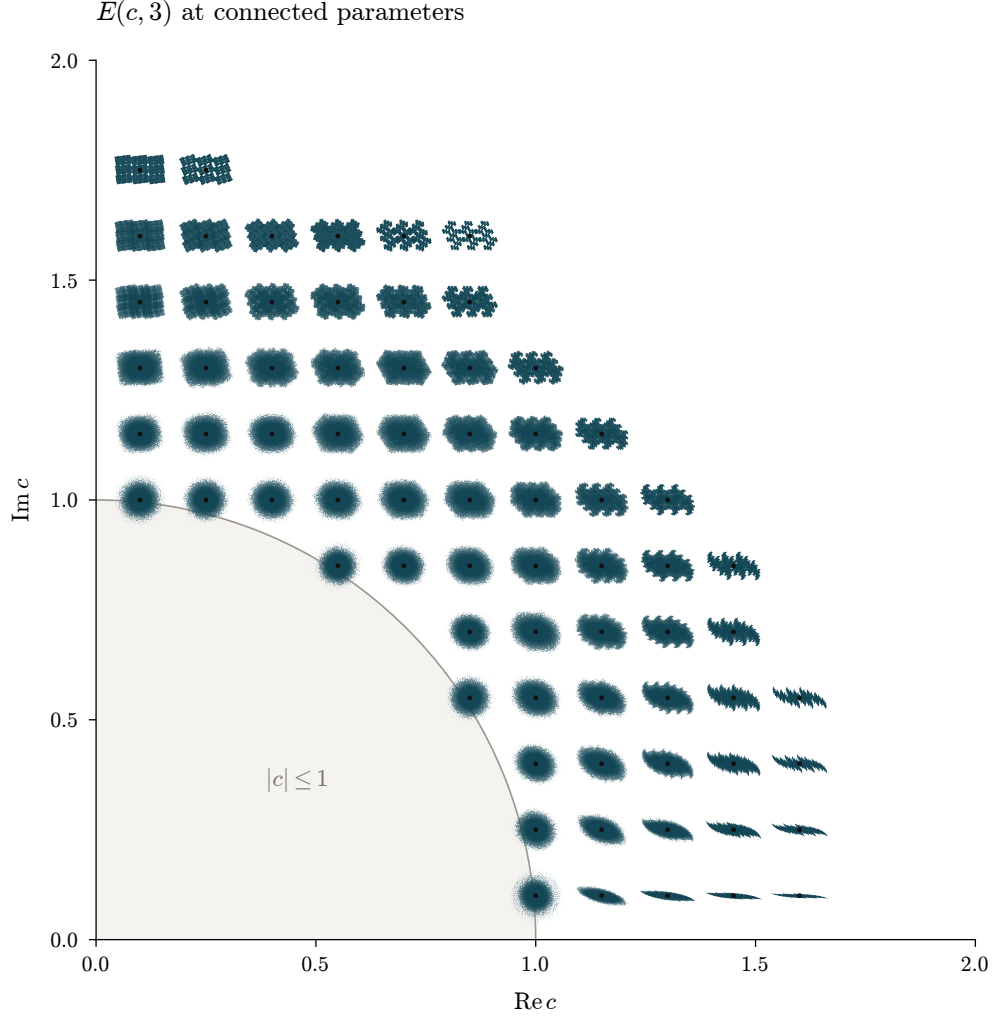


**Figure 1.2.** The attractor  $E(c, 5)$  for  $c = 1 + 2i$  has nonempty interior. Its five first-level pieces have centers in  $A_5$ .

The connectedness locus is

$$\mathcal{M}_n := \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : E(c, n) \text{ is connected} \right\}.$$

Figure 1.3 makes this definition visible by replacing selected parameter points with miniature pictures of  $E(c, 3)$  and retaining connected examples. A miniature's position records the parameter; its shape belongs to the dynamical plane.



**Figure 1.3.** Each miniature represents  $E(c, 3)$  at the parameter  $c$  marked by its black dot. The parameters lie on a square grid of spacing 0.15; connectedness is certified by the inner-annulus inclusion or finite capture (Proposition 3.11, and Theorem 6.15). The attractors are rendered by finite point samples. Each has its own isotropic display scale, with orientation unchanged. Only certified connected parameters are shown; omitted grid positions need not be exterior to  $\mathcal{M}_3$ .

Connectedness is equivalent to overlap of two neighboring first-level pieces (Proposition 3.3). Translating both pieces by the negative of the smaller digit gives:

$$c \in \mathcal{M}_n \iff \frac{E(c, n)}{c} \cap \left( 2 + \frac{E(c, n)}{c} \right) \neq \emptyset.$$

For the algebraic formulation, define

$$D_n = \{-n + 1, -n + 2, \dots, n - 1\}$$

and let  $\mathcal{R}_n$  be the roots of all monic polynomials

$$P(z) = z^q + \sum_{j=0}^{q-1} d_j z^j, \quad d_j \in D_n, \quad q \geq 1.$$

The index  $n$  refers to the original attractor, whereas

$$N := 2n - 1$$

indexes the difference attractor. Since

$$A_n - A_n = A_N = 2D_n,$$

the address representation gives

$$E(c, n) - E(c, n) = E(c, N).$$

We call  $E(c, N)$  the *difference attractor* and  $2c$  the *marked point*; see Definition 2.6. The term “marked point” means only that  $2c$  is the distinguished point whose membership in the difference attractor is tested.

The geometric arguments use the two-disk parameter lens

$$\mathcal{X}_n := \{c \in \mathbb{C} \setminus \mathbb{D} : |c - 1| < \sqrt{2n} \text{ and } |c + 1| < \sqrt{2n}\}.$$

For non-real parameters,  $\mathcal{X}_n \setminus \mathbb{R}$  is exactly the domain on which the canonical trap of Definition 6.10 is both nonempty and self-covering; see Corollary 6.11.

### 1.2.2. Equivalent descriptions

The overlap, address, and numeration descriptions are equivalent. For every  $n \geq 2$ ,

$$c \in \mathcal{M}_n \iff 2c \in E(c, 2n - 1) \iff 1 = \sum_{k=1}^{\infty} \varepsilon_k c^{-k} \text{ for some } (\varepsilon_k)_{k \geq 1} \in D_n^{\mathbb{N}};$$

see Proposition 3.3, Proposition 3.4, Remark 3.5, and Proposition 5.2. Thus  $\mathcal{M}_n$  is precisely the locus of bases  $c$ , with  $|c| > 1$ , for which the fixed value 1 admits at least one signed  $D_n$ -expansion. Moreover,

$$\mathcal{M}_n = \overline{\mathcal{R}_n} \setminus \mathbb{D};$$

see Theorem 5.5.

### 1.2.3. Bounds and topology

The bounds separate the real trace from the non-real geometry. If  $c \in \mathcal{M}_n \setminus \mathbb{R}$ , then

$$|c| < 1 + \sqrt{n - 1},$$

whereas the real trace is exactly

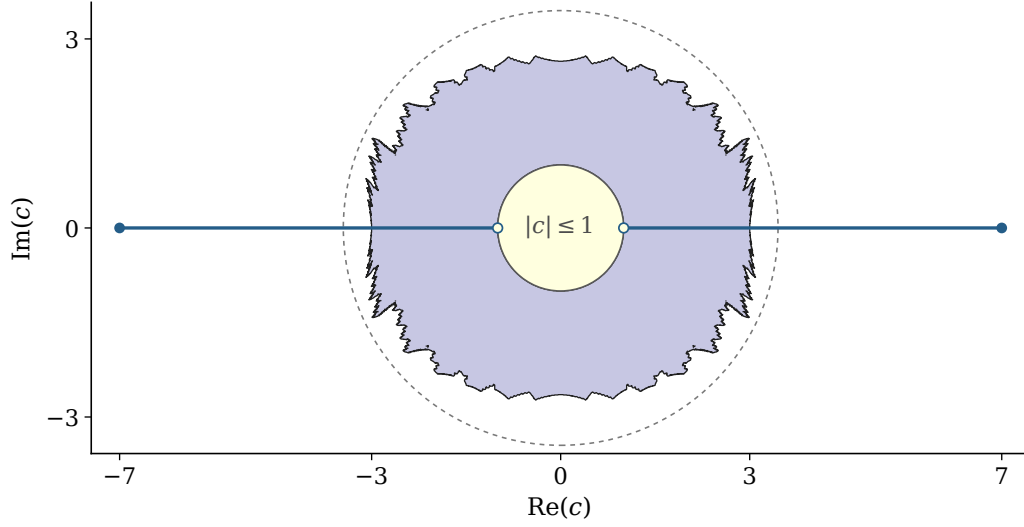
$$\mathcal{M}_n \cap \mathbb{R} = [-n, -1) \cup (1, n].$$

The non-real estimate follows from Proposition 3.9; the real classification is proved in Proposition 3.11. In the opposite direction,

$$\{c \in \mathbb{C}: 1 < |c| < \sqrt{n}\} \subset \mathcal{M}_n$$

by Proposition 3.11. Finally,  $\mathcal{M}_n$  is connected and locally connected for every  $n \geq 2$  (Theorem 3.13).

Each point in Figure 1.4 represents a parameter  $c$ . The full-plane view combines the exact real intervals with a finite-depth rendering of the non-real region.



**Figure 1.4.** Parameter-plane view of  $\mathcal{M}_7$ . The non-real shading is a finite-depth numerical approximation and includes unresolved parameters; its outline and apparent connectivity are not certified. The blue segments show the exact real trace  $[-7, -1) \cup (1, 7]$ ; open circles exclude  $\pm 1$ , and filled circles include  $\pm 7$ . The closed unit disk is excluded, and the dashed circle marks  $|c| = 1 + \sqrt{6}$ .

#### 1.2.4. Regular-closedness away from the real axis and finite capture

As formalized in Definition 2.1, the question whether  $\mathcal{M}_n$  is regular-closed away from the real axis is whether

$$\mathcal{M}_n \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

Equivalently, it asks whether every non-real point of  $\mathcal{M}_n$  can be approximated by interior points.

The covering argument proves, for every  $n \geq 2$ , that

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

It also proves

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n \quad (n \geq 21),$$

and hence regular-closedness away from the real axis for  $n \geq 21$ . The precise result is Theorem 4.5.

Chapter 6 introduces an open canonical trap  $\mathcal{C}(c, N)$  and a closed canonical enclosure  $\mathcal{E}(c, N)$ , formally defined in Definitions 6.10 and 6.13. They satisfy

$$\mathcal{C}(c, N) \subset \text{int } E(c, N) \subset E(c, N) \subseteq \mathcal{E}(c, N)$$

on the domain specified in Theorem 6.15. If a finite inverse word sends the marked point into the trap, it gives an interior certificate. An inverse word is called *enclosure-admissible* when every prefix iterate remains in the canonical enclosure; see Definition 6.19. If all enclosure-admissible words are exhausted at some finite depth, this gives an exterior certificate. If neither event occurs at the chosen depth, the computation remains inconclusive.

The corresponding finite-capture set  $\Theta_{\text{fin}}(n)$ , defined in Definition 6.17, recovers the non-real locus inside the lens:

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} = \overline{\Theta_{\text{fin}}(n)} \cap (\mathcal{X}_n \setminus \mathbb{R});$$

see Theorem 6.39. The sharp lens-containment result is

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n \quad (n \geq 20),$$

whereas for  $2 \leq n \leq 19$  there are non-real parameters of  $\mathcal{M}_n$  outside  $\mathcal{X}_n$ ; see Theorem 6.44. Consequently, for  $n \geq 20$ ,

$$\mathcal{M}_n = \overline{\Theta_{\text{fin}}(n)} \cup (\partial \mathcal{M}_n \cap \mathbb{R});$$

see Corollary 6.45.

**Capture depth and the number of inverse branches.** For  $k \geq 0$ , let  $\Theta_k(n)$  be the set of parameters  $c \in \mathcal{X}_n \setminus \mathbb{R}$  for which an inverse word of length at most  $k$  sends the marked point  $2c$  into the canonical trap; see Definition 6.17. These sets increase with  $k$ . Each is open and, in every open quadrant, a finite union of open semialgebraic sets (Lemmas 6.34 and 6.37). Moreover,

$$\partial_{\mathcal{X}_n \setminus \mathbb{R}} \Theta_k(n) \subset \Theta_{k+2}(n) \setminus \Theta_k(n).$$

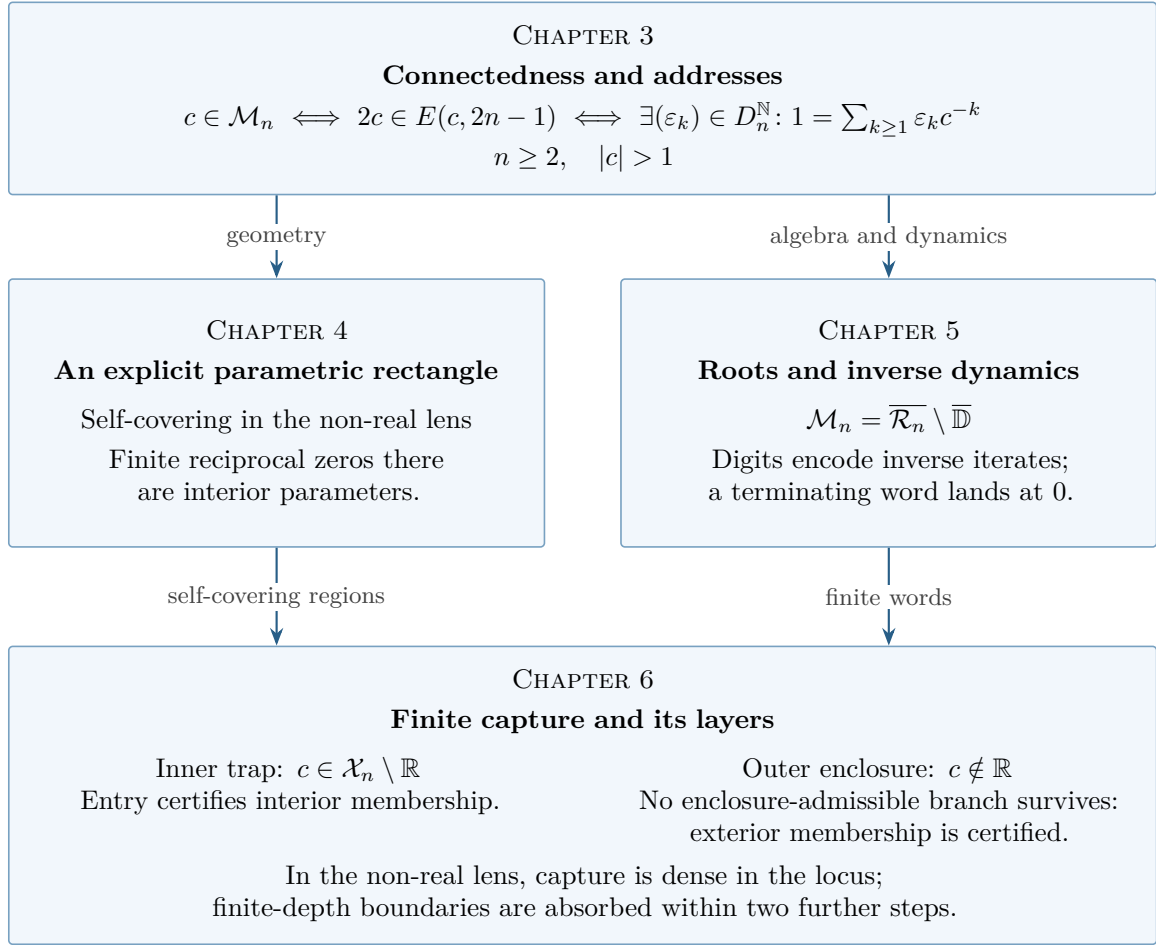
Thus every boundary point of a capture set that remains in the non-real part of the lens is captured when the allowed depth is increased by at most two steps (Corollary 6.36).

The enclosure also controls how many branches the inverse search retains. The difference alphabet has  $2n - 1$  digits, giving  $(2n - 1)^k$  possible words of length  $k$ . For non-real  $c$  with  $|c| \geq \sqrt{n}$ , however, each node of the exact tree of branches that remain in the canonical enclosure has at most

$$b_n = \left\lceil \frac{2(\sqrt{n} + 1)}{\sqrt{n} - 1} \right\rceil$$

admissible continuations (Theorem 6.28). For  $n \geq 25$ , this bound is three, so at most  $3^k$  nodes survive at depth  $k$ . The bound applies throughout a region containing the non-real boundary.

Figure 1.5 locates these arguments in Chapters 3–6.



**Figure 1.5.** The address equation links the geometric construction in Chapter 4 with the root closure and inverse dynamics in Chapter 5. Chapter 6 uses self-covering regions and finite words to obtain certificates and organize capture by depth.

### 1.3. Related Work

The connectedness problem links the geometry of overlapping pieces with equations between their symbolic addresses. For the collinear family, equally spaced digits make the difference attractor collinear as well; the resulting address equation describes both a restricted root closure and a locus of complex bases. This section traces these connections, the geometric methods used to obtain finite information from them, and the related study of the individual attractors.

#### 1.3.1. Connectedness, restricted roots, and complex bases

**From binary similarities to collinear alphabets.** The binary parameter problem originates in the work of Barnsley and Harrington and in Hardin’s thesis [BH85; Har85]. Their contraction parameter lies in the unit disk. After  $\lambda = 1/c$  and affine normalization, the connectedness locus is  $\mathcal{M}_2$  for the alphabet  $A_2 = \{-1, 1\}$ . Barnsley and Harrington emphasized the complementary disconnectedness set. In the connectedness convention, Bousch established connectedness, local connectedness, and Hölder path connectedness [Bou93]. Bandt developed inverse-orbit and escape-time methods, proved the existence of a hole, and conjectured density of the interior away from the real axis [Ban02]. Solomyak and Xu proved this density on a substantial region [SX03]; Calegari, Koch, and Walker later proved the full binary conjecture and the existence of infinitely many holes [CKW17].

Bousch also considered consecutive collinear alphabets  $\{0, 1, \dots, n-1\}$  for every  $n \geq 2$ , proving that connectedness is equivalent to overlap between neighboring first-level pieces [Bou92, Chapter 1, p. 41]. This is the family  $E(c, n)$  after reciprocal and affine changes of coordinates. Calegari, Koch, and Walker subsequently studied its iterated difference systems: starting from the binary alphabet gives the orders  $n = 2^k + 1$ ,  $k \geq 0$ , with

$$E(c, 2^k + 1) - E(c, 2^k + 1) = E(c, 2^{k+1} + 1).$$

They established connectedness and marked-point relations and posed questions about these systems [CKW17, Section 6.1.1, Proposition 6.1.3, and Question 6.1.5]. The author’s joint work with Juher and Saldaña studies the collinear loci uniformly in  $n$ , and the subsequent joint work with Juher develops their restricted-root and finite-capture descriptions [EJS24; EJ26].

**Restricted polynomial and power-series zeros.** The neighboring-overlap criterion gives the restricted reciprocal-series equation. Truncation and root approximation then identify the connectedness locus with the closure, outside the closed unit disk, of the roots of monic polynomials with the prescribed coefficient bound (Theorem 5.5). In the binary case, Indlekofer, Járαι, and Kátai explicitly obtain the reciprocal root-closure identity by Rouché approximation [IJK95, Section 7]. The explicit formulation and its geometric development in [EJ26] build on these overlap and approximation antecedents.

The broader theory asks how coefficient restrictions govern root location, multiplicity, and the topology of root closures. Odlyzko and Poonen studied the closure of the nontrivial zeros of 0, 1-polynomials, proving path connectedness and obtaining root bounds [OP93]. Beauoup, Borwein, Boyd, and Pinner studied extremal and multiple zeros for power series with coefficients in real intervals [Bea+98a; Bea+98b]; their bounds apply to the continuous-coefficient relaxation used here. Borwein, Erdélyi, and Littmann obtained quantitative results near the unit circle and analytic restrictions arising from finite coefficient sets [BEL08]. Baker and Yu give annular inclusion and density results for sufficiently dense

finite alphabets on the unit circle [BY18]. Kewarth uses bounded coefficient recursions to study root depth, connectedness, and partial recovery of a symmetric integer alphabet from its root set [Kew26]. Hokken shows that, for integer alphabets containing  $\{\pm 1\}$ , requiring a square discriminant, possibly zero, preserves the closure of the zero set; his polynomial family has nonzero constant term and does not impose monicity [Hok26].

The experimental study of root geometry includes Christensen’s plots, algebraic starscapes, and arithmetic visualizations [Chr06; HST22; DX22]. Bohemian matrices provide a related finite-alphabet spectral setting [Sen21]; broader experimental and expository accounts include [Bai+07; BCD23].

**The zero-section, Littlewood roots, and entropy.** The binary zero-section is the distinguished sublocus

$$\mathcal{M}_2^0 = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : 0 \in E(c, 2)\} \subset \mathcal{M}_2.$$

In reciprocal coordinates it is described by zeros of power series with coefficients in  $\{-1, +1\}$ , as studied by Bousch [Bou88; Bou93]. This Littlewood zero locus also occurs in Thurston’s programme on entropy. Tiozzo identifies it with the part inside the unit disk of Thurston’s set, the closure of the Galois conjugates of growth rates of critically periodic unimodal interval maps [Thu14; Tio20]. The Master Teapot retains the growth rate as an additional real coordinate. For critically periodic unimodal maps, Bray, Davis, Lindsey, and Wu prove connectedness of the teapot and upward persistence of its disk slices [Bra+21]. Lindsey and Wu describe those slices through symbolic admissibility conditions and affine systems, obtaining finite exclusion algorithms [LW23]; Lindsey, Tiozzo, and Wu extend the programme to principal veins of the quadratic Mandelbrot set [LTW25]. For postcritically finite tent-map slopes in  $(\sqrt{2}, 2)$ , Wu realizes the disk slices of the unimodal Master Teapot as marked-point loci for graph-directed affine systems built from reversed Markov transition graphs [Wu26, Theorem 2.5].

The zero-section also has its own statistical and local geometry. Ang constructs a natural parameter measure [Ang23]. Under a simple-root hypothesis, Michelen and Yakir show that suitably rescaled Littlewood-root clusters sharing a long prefix converge to the corresponding binary self-similar attractor, and establish a random counterpart [MY26].

**Complex bases and bounded remainders.** The numeration interpretation of  $\mathcal{M}_n$  fixes the value 1 and the alphabet  $D_n$  while the base  $c$  varies. Rényi–Parry theory supplies the classical setting of greedy expansions and their admissible languages [Rén57; Par60]; negative bases and complex bases lead to further digit systems [IS09; LS12; Sur20]. Surer’s complex integer-digit constructions use a parallelogram domain and bounded remainder dynamics, giving a geometric continuation mechanism closely related to the inverse iterations used here. Frougny and Pelantová connect discreteness of digit spectra with finite-automaton recognition of zero representations [FP18]. These results place the inverse states of the marked-point problem within a broader theory of digit expansions.

A terminating signed expansion of 1 gives a monic restricted polynomial after clearing denominators. Classical finiteness properties of  $\beta$ -expansions and recent work on greedy expansions in Salem-number families provide arithmetic context [FS92; HO26]. In the thesis’s inverse dynamics, termination is exact landing  $g_u(2c) = 0$ ; finite capture allows a prefix to enter an open self-covering region and continue indefinitely. This distinction connects finite expansions to the stable certificates developed here.

### 1.3.2. Geometric methods and finite capture

**Polygonal digit geometry.** Changing the placement of the translations gives the regular polygonal family considered by Bousch [Bou92, Chapter 1, p. 42] and developed by Bandt and Hung [BH08a; Hun07]. Its connectedness loci  $\mathcal{M}_n$ , parametrized by  $\lambda \in \mathbb{D} \setminus \{0\}$ , are connected by Bousch’s framework and locally connected by Nakajima’s power-series zero-set criterion [Bou92; Nak24]. The latter criterion also underlies the local-connectedness argument for the collinear loci in Theorem 3.13. Bandt and Hung prove regular-closedness for  $n \geq 3$ ,  $n \neq 4$ ; Himeki and Ishii settle  $n = 4$  [BH08a; HI20]. Calegari and Walker characterize the points of fractal  $n$ -gons on the boundaries of their convex hulls, refining the analysis used in these interior-density arguments [CW19].

**Self-affine pairs and multiple-zero limits.** Changing the common linear part gives another extension of the binary problem. Shmerkin and Solomyak characterize connectedness of nondegenerate homogeneous self-affine pairs by a single restricted power series vanishing at the eigenvalues, with derivative conditions for nontrivial Jordan blocks [SS06]. For a complex eigenvalue this is a complex zero condition; for two distinct real eigenvalues it gives

$$\mathcal{N} = \{(\gamma, \lambda) \in (-1, 1)^2 : f(\gamma) = f(\lambda) = 0 \text{ for some } f \in \mathcal{B}\},$$

where

$$\mathcal{B} = \left\{ 1 + \sum_{k \geq 1} a_k x^k : a_k \in \{-1, 0, 1\} \right\}.$$

The definition includes the degenerate diagonal. Solomyak obtains bounds, regions of \*-transversality, locally connected portions, and algebraic cusp corners for this locus [Sol15]. Hare and Sidorov study the positive diagonal setting through  $T_{\pm}(x, y) = ((x \pm 1)/\beta_1, (y \pm 1)/\beta_2)$ , with  $\beta_1, \beta_2 > 1$ : these are simultaneous signed expansions after  $\gamma = 1/\beta_1$ ,  $\lambda = 1/\beta_2$  [HS17]. Their work on  $T_{\pm}(v) = Mv \pm u$  gives attractor-interior and uniqueness results under hypotheses on the eigenvalue configuration [HS16].

Multiple zeros connect this spectral viewpoint to real-axis geometry. When conjugate non-real zeros coalesce at a real parameter, the limiting relation has a double zero. The same necessary condition arises when distinct real eigenvalues of connected pairs coalesce; for a planar nontrivial Jordan block, the double-zero condition characterizes connectedness [SS06]. For the ternary collinear family, the reciprocal coefficients are  $\{-2, -1, 0, 1, 2\}$ , so the five-symbol double-zero problem of [SS06, Section 6] directly constrains real-axis accumulation. The companion manuscripts determine its Tribonacci threshold and the positive-real antenna thresholds at the orders  $H_q = q^2 - q + 1$  [Esp26h; Esp26a]. Their precise conclusions, including the distinction between a double zero and accumulation by non-real parameters, are discussed in Section C.4.

**Covering and crossing certificates.** The covering method turns a finite address prefix into a guaranteed infinite continuation. Indlekofer, J  rai, and K  tai use a disk covered by iterated maps of the difference system and a finite inverse word entering the disk to certify connectedness [IJK95, Lemma 7 and pp. 422–424]. Solomyak and Xu develop a rectangle-based argument and combine it with root approximation [SX03]; Bandt and Hung use a self-covering disk for polygonal systems [BH08a]. The author’s joint work with Juher and Salda  a supplies an explicit parametric rectangle construction for every collinear arity  $n \geq 2$ , extending the Solomyak–Xu method [EJS24].

Calegari–Koch–Walker traps certify overlap through stable crossing configurations in cylinder neighborhoods: nonzero algebraic intersection, together with short-hop paths, rules out a positive gap between the first-level pieces [CKW17, Definition 7.1.3 and Proposition 7.1.6]. Himeki and Ishii use this crossing method for polygonal systems [HI20]. Rosler adapts it to self-affine pairs and proves that every non-isolated point of  $\mathcal{N}$  outside the diagonal belongs to  $\overline{\text{int } \mathcal{N}}$  [Ros25, Theorems 4.4 and 6.5].

Finite capture develops the covering route into a geometric filtration of parameter space [EJ26]. For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , its trap is an open self-covering parallelogram in  $E(c, 2n - 1)$  (Theorem 6.15). A finite inverse word carrying  $2c$  into it supplies a bounded infinite continuation; the strict entry condition persists under parameter perturbation. The canonical enclosure gives the complementary exclusion test when all admissible branches are exhausted at finite depth. The construction also organizes the witnesses by capture depth: the resulting sets are open and quadrantwise semialgebraic, their boundaries inside the non-real lens are absorbed within two further steps, and the closure of their union recovers the non-real locus there. The sharp lens-containment threshold  $n \geq 20$  makes this recovery global off the real axis.

**Bounding geometry.** Bounding sets make inverse search effective. Vass studies convex hulls and reconstructs them in specified classes from extremal addresses [Vas13; Vas18]. The thesis uses two related bounds: the convex-hull relaxation of the marked-point test and the exact support enclosure in two canonical directions (Remark 3.10 and Definition 6.13).

**Local geometry and accessibility.** Local parameter geometry concerns the behavior near an individual boundary point. Solomyak studies local geometry and asymptotic similarity between parameter loci and dynamical sets [Sol04; Sol05]. Using this correspondence and eventually periodic restricted-series relations, Silvestri and Pérez obtain sufficient conditions for boundary accessibility in  $\mathcal{M}_2$  and its zero-section [SP23]; see also [Sil19]. This is the background for the higher-arity local-geometry questions in Section 7.6.

### 1.3.3. Address structure and arithmetic geometry

**Address identifications and boundary dynamics.** At a fixed parameter, overlap determines which symbolic addresses represent the same point. Troshin’s dissertation and subsequent chapter study this question in the binary family [Tro09; Tro11]. For its attractor  $A(q)$  with digits 0, 1, the normalization is

$$q = 1/c, \quad E(c, 2) = 2A(q) - \frac{1}{1 - q}.$$

Building on Barnsley’s top-address formalism [Bar06], Troshin describes code identifications in specified overlap regimes, including cases with a unique normalized restricted-series relation. Bandt and Hung’s overlaps compatible with the open set condition belong to this symbolic-topological setting [BH08b].

Such identifications can also support dynamical models. Eroğlu, Rohde, and Solomyak obtain quasisymmetric conjugacies with quadratic dynamics for postcritically finite dendritic similarity-pair attractors, within a broader finite-overlap theory [ERS10]. Calegari and Walker associate a lamination and a constant-slope model to the boundary action on the filled attractor of a boundary similarity pair. Their complete description of cut points through an inclusion graph is conditional on Conjecture A [CW26].

**Language restrictions and geometric realizations.** Restricting addresses changes both the attractor and the parameter equations. Sirvent’s thesis studies geometric realizations of Pisot substitutions, boundary identifications, and associated measured laminations [Sir93]. Sirvent and Thuswaldner’s Fibonacci–Mandelbrot set uses the language with no two consecutive ones; its block presentation is

$$z \mapsto \lambda z, \quad z \mapsto 1 + \lambda^2 z.$$

Its connectedness locus is contained in the full binary locus, and after inversion gives a sublocus of  $\mathcal{M}_2$  [ST15]. Sirvent develops further examples through direct and reversed valuations of  $k$ -Fibonacci,  $k$ -Pell, and  $k$ -metallic languages [Sir26]. An address restriction induces a constrained difference language. This motivates the language-restricted base loci and questions about finite certificate languages formulated in Section C.3.

**Self-affine tiles and collinear digits.** Integral self-affine tile theory connects digit expansions with lattices and tilings [Ban91; LW96]. It meets the present family at arithmetic parameters: if  $c^2 - ac + n = 0$ , with  $a \in \mathbb{Z}$  and  $c \notin \mathbb{R}$ , multiplication by  $c$  has integer matrix  $\begin{pmatrix} 0 & -n \\ 1 & a \end{pmatrix}$  on  $\mathbb{Z} + \mathbb{Z}c$ , and  $E(c, n)$  is affinely equivalent to the corresponding tile with consecutive digits. Connectedness, disk-likeness, cut points, and interior components in this setting are studied by Kirat and Lau, Leung and Lau, and Akiyama and collaborators [KL00; LL07; AT00; ALT21]. Neighbor intersections provide finite combinatorial descriptions of tile boundaries and topology.

For binary complex-base tiles, such transition graphs already appear in the work of Indlekofer, Kátai, and Racsó [IKR93, Sections 3–4].

In higher dimensions, Zaitseva and Protasov study two-digit self-affine attractors and affine equivalence for arithmetic-progression digits with a fixed expanding matrix [ZP22, Section 4]. Thuswaldner and Zhang prove spherical-boundary and ball-homeomorphism results for specified classes of three-dimensional collinear-digit tiles [TZ20; TZ24]. Mekhontsev develops an algebraic framework for computational exploration, and Bandt and Mekhontsev construct rep-tiles with holes [Mek19; BM20]. Fathauer’s iterated-polyform constructions provide a complementary geometric approach, with bar-polyomino examples related to collinear-digit tiles [Fat25; Fat26].

#### 1.3.4. Fractal trees and geometric motivation

**From binary symmetry to prescribed contacts.** Mirror-symmetric branching gives a different way to vary the geometry of a two-map system. Barnsley and Hardin obtain a piecewise-smooth boundary description for a conjugate-multiplier binary family and also give an early finite-depth plotting procedure for the disconnectedness set of the common-multiplier binary family [BH89]. Wolfram discusses symmetric binary trees and their parameter spaces [Wol02, Chap. 8, Sect. 6, pp. 405–407]; the extended note “Parameter space sets” gives complex-parameter equations, algebraic boundary curves, and a pruning procedure [Wol02, p. 1006]. Mandelbrot and Frame study self-contacting tree canopies and shortest paths [MF99], while Taylor develops computational topology and topological barcodes for symmetric binary fractal trees [Tay05; Tay09].

Higher-arity examples occur in Wolfram’s ternary and quaternary trees [Wol02, Chap. 8, Sect. 6, p. 402] and in Pagon’s constructions [Pag03]. Motivated in part by these works, the author derives systematic self-contact equations for mirror-symmetric trees with equally spaced branches [Esp13]. The programme continues in the author’s master’s thesis at the Universitat de Barcelona with complex trees [Esp18]. Equations between branch multipliers

preserve prescribed tip-to-tip address identifications while allowing the geometry to move beyond mirror symmetry. On stable parameter regions with no additional identifications, the tipsets retain their topological type. The fern-like ternary family at Bridges 2019 illustrates this construction [Esp19].

Working with these higher-arity families developed the author's intuition for using address equations to organize geometric parameter spaces. That experience led to recognizing the potential of the collinear family: its equally spaced digits preserve a simple difference structure while the arity varies. This provided the initial impetus for the present thesis.

### 1.3.5. Measures and analytic questions

**Transversality, separation, and self-similar measures.** The same address differences that encode connectedness also control near coincidences between cylinder maps and enter the study of the natural self-similar measures. Peres and Schlag combine quantitative transversality with regularity and energy estimates to bound exceptional parameter sets [PS00]; Solomyak surveys this method for iterated function systems [Sol23]. Hochman's entropy theory relates dimension loss to super-exponential cylinder concentration; in higher dimensions invariant subspaces provide an additional alternative [Hoc14; Hoc15]. Varjú surveys the role of entropy rates and exact overlaps in real self-similar measure theory [Var25].

Solomyak and Xu connect the binary locus with complex Bernoulli convolutions [SX03]. Shmerkin and Solomyak's later fixed-translation theorem applies directly to every collinear alphabet  $A_n$ : for non-real  $c$  with  $1 < |c| < \sqrt{n}$ , the equal-weight self-similar measure is absolutely continuous, with a density in some  $L^q$ ,  $q > 1$ , outside an exceptional set of Hausdorff dimension zero [SS16, Theorem B]. Corso and Shmerkin obtain  $L^q$ -dimension results under separation and rotation hypotheses; their planar specialization assumes irrational rotation and exponential separation [CS24].

**Analysis on the attractors.** Analytic questions concern additional structures on the sets. Deheuvels studies Sobolev spaces, traces, and extension operators on ramified domains with self-similar boundaries [Deh13; Deh16]. Dirichlet forms, Laplacians, and spectral problems provide another setting [Kig01; Str06; DL23], while fractal zeta functions relate scaling to tube formulas and Minkowski measurability [LF13; LRŽ17]. For collinear attractors, these directions raise questions about how changes in overlap and topology affect measures, spectra, and boundary scaling.

## 1.4. Objectives

The six objectives follow the passage from connectedness to finite certification.

- (O1) Establish the exact correspondence among connectedness of  $E(c, n)$ , signed  $D_n$ -expansions of the fixed value 1 in the variable base  $c$ , restricted reciprocal-series equations, and membership in the exterior closure of the roots of monic polynomials whose non-leading coefficients lie in  $D_n$ .
- (O2) Determine basic geometric and topological properties of  $\mathcal{M}_n$ , including nesting, inner and outer bounds, connectedness, and local connectedness.
- (O3) Construct a self-covering rectangle and use it to prove that non-real parameters in  $\mathcal{M}_n \cap \mathcal{X}_n$  are limits of interior parameters and to obtain the resulting theorem for

$$n \geq 21.$$

- (O4) Construct explicit inner and outer regions for the difference attractor  $E(c, 2n - 1)$ , giving finite certificates for membership and nonmembership of the marked point  $2c$ .
- (O5) Develop the finite-capture filtration; prove its fixed-depth openness and semialgebraicity, the two-step absorption of every finite-depth boundary within  $\mathcal{X}_n \setminus \mathbb{R}$ , and recovery of the non-real locus inside the lens from the closure of finite capture; and determine the sharp lens-containment threshold  $n \geq 20$ .
- (O6) Establish explicit pruning and uniform branching bounds for the exact inverse tree—reducing the naive  $(2n - 1)$ -ary search to at most three admissible continuations per node for  $n \geq 25$  in the non-real boundary regime—and provide reproducible software for exploring the attractors and parameter loci and exporting finite words and search data for independent verification.

## 1.5. Methodological Overview

The proofs combine overlap geometry, approximation of restricted series, and bounded inverse dynamics. The address relation links these arguments; polynomial approximation connects finite identities to the full locus.

The neighboring-piece criterion is the geometric starting point. Subtracting addresses turns its overlap condition into the marked-point test  $2c \in E(c, 2n - 1)$ , hence into a signed expansion of 1 in base  $c$ . Terminating expansions give monic polynomial relations. Rouché approximation of general expansions, together with compactness of the coefficient space, identifies  $\mathcal{M}_n$  with the part of the root closure outside the closed unit disk (Propositions 5.2 and 3.3, and Theorem 5.5).

The rectangle construction supplies bounded continuation. Its self-covering property places it inside the difference attractor; polynomial relations then provide entry points, and strict inequalities persist under perturbation. This proves interior approximation in the lens and the first global threshold  $n \geq 21$  (Theorem 4.5).

Canonical coordinates turn this geometry into explicit inner and outer tests. They also make each fixed-depth capture condition a finite system of inequalities. The two-step theorem controls limits of bounded-depth witnesses within the non-real lens. Separately, terminating relations in that region give trap-entry witnesses; their density recovers the non-real locus inside the lens from the closure of all finite-capture sets (Lemma 6.37, Corollary 6.36, and Theorem 6.39).

Finally, the outer enclosure has two further uses. Its admissible-digit interval bounds the branching of the inverse tree (Theorem 6.28), and its support inequalities settle the borderline case  $n = 20$ . Together with the common disk bound for  $n \geq 21$  and the certified off-lens roots for  $2 \leq n \leq 19$ , this proves sharp lens containment (Corollary 6.42, Proposition 6.43, Proposition A.2, and Theorem 6.44).

## 1.6. Computational Infrastructure and Reproducibility

The companion implementation, *Collinear Fractals GPU*, links interactive views of  $E(c, n)$ ,  $\mathcal{M}_n$ , and the finite-capture sets [Esp26b]. The browser interface is available at <https://complextrees.com/collinear-fractals-gpu/>.

The implementation searches inverse words in canonical coordinates, prunes them against the enclosure, tests trap entry, and exports the surviving words and search data. It supports comparison of parameter-space magnifications with dynamical attractors and was used to locate candidate boundary phenomena. The certificate conditions in Section 2.10 specify which inequalities must be verified to turn a candidate into a proof.

The mathematical branching bound concerns the exact enclosure. An enlarged working enclosure used to account for rounding may retain more branches. Accordingly, the thesis separates the exact inverse tree and its proved bounds from the implementation used to explore it.

## 1.7. Structure of the Thesis

Chapter 2 gives the common background and methodology. Chapter 3 develops the collinear attractor and its connectedness locus.

Chapter 4 develops the covering-property method and the rectangle-covering proof of regular-closedness away from the real axis for  $n \geq 21$ . Chapter 5 relates restricted polynomial roots to reciprocal series and the marked-point criterion.

Chapter 6 develops the trap and enclosure, finite search, capture layers, and the sharp lens-containment theorem. Chapter 7 explains how the methods fit together, and Chapter 8 summarizes the main conclusions. The appendices collect the finite certificates used in the sharpness argument, the provenance of the principal results, and an outlook on related questions and separate continuations.



# Background and Methodology

Connectedness begins as a question about intersections between pieces of an attractor. For equally spaced collinear digits, these intersections can be expressed as membership of a marked point in a difference attractor, and then as an equation between digit expansions. This chapter develops that dictionary and introduces the geometric tools that turn it into finite certificates. The later chapters establish the corresponding characterizations and carry out the covering and finite-capture constructions.

## 2.1. Complex Notation and Topological Language

We write  $\mathbb{C}$  for the complex plane,  $\mathbb{R}$  for the real line,  $\mathbb{N}$  for the positive integers,  $\mathbb{D}$  for the open unit disk, and  $\overline{\mathbb{D}}$  for its closure. If  $X \subset \mathbb{C}$ , then  $\overline{X}$ ,  $\text{int}(X)$ , and  $\partial X$  denote its closure, interior, and boundary in the ambient space under discussion.

The parameters lie in the exterior of the closed unit disk,

$$\mathbb{C} \setminus \overline{\mathbb{D}} = \{c \in \mathbb{C} : |c| > 1\}.$$

Closure, interior, and boundary of a parameter locus are taken relative to this space unless stated otherwise. For attractors and other subsets of the dynamical plane, the ambient space is  $\mathbb{C}$ .

**Definition 2.1** (Regular-closed set). A subset  $X$  of a topological space is regular-closed if

$$X = \overline{\text{int}(X)},$$

where closure and interior are taken in that space.

For a relatively closed parameter locus  $L \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ , we say that  $L$  is *regular-closed away from the real axis* if  $L \setminus \mathbb{R}$  is regular-closed in the relative space

$$(\mathbb{C} \setminus \overline{\mathbb{D}}) \setminus \mathbb{R}.$$

Equivalently, because  $L$  is relatively closed,

$$L \setminus \mathbb{R} \subseteq \overline{\text{int}(L)},$$

where closure and interior are taken in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ .

**Lemma 2.2** (Rectangle crossing for plane continua). *Let  $R = [a, b] \times [u, v]$  be a nondegenerate closed rectangle. If  $H, V \subset R$  are compact connected sets such that  $H$  meets both vertical sides of  $R$  and  $V$  meets both horizontal sides, then  $H \cap V \neq \emptyset$ .*

This crossing principle lets connected sets play the role of paths in the geometric arguments below. It follows from the Phragmén–Brouwer separation property, equivalently from the Jordan curve theorem; see Pascoletti and Zanolin [PZ13, Lemma 7]. Compactness and connectedness suffice, even when no path joining the opposite sides is available.

## 2.2. Affine Iterated Function Systems

An iterated function system (IFS) on  $\mathbb{C}$  is a finite family of contractions  $\mathcal{F} = \{f_1, \dots, f_m\}$ . Its action on sets is described by a single operator.

**Definition 2.3** (Hutchinson operator). The Hutchinson operator associated with  $\mathcal{F}$  is the set map

$$\mathcal{H}_{\mathcal{F}}(U) = \bigcup_{j=1}^m f_j(U)$$

for  $U \subset \mathbb{C}$ . On the space of nonempty compact subsets of  $\mathbb{C}$ , equipped with the Hausdorff metric, it is a contraction.

Hutchinson's theorem [Hut81] states that  $\mathcal{H}_{\mathcal{F}}$  has a unique nonempty compact fixed point and that, for every nonempty compact set  $A$ , the iterates  $\mathcal{H}_{\mathcal{F}}^k(A)$  converge to that fixed point in the Hausdorff metric.

**Definition 2.4** (Attractor). The unique nonempty compact set  $K$  satisfying

$$K = \mathcal{H}_{\mathcal{F}}(K) = \bigcup_{j=1}^m f_j(K)$$

is called the attractor of  $\mathcal{F}$ .

The affine systems studied here share a common complex linear part and have collinear translation vectors. Their first-level pieces are therefore translates of one contracted and rotated copy of the attractor. This common geometry is what makes differences of addresses useful for connectedness.

## 2.3. The Collinear Family and Its Connectedness Locus

For an alphabet of cardinality  $m \geq 2$ , let

$$A_m = \{-m+1, -m+3, \dots, m-3, m-1\}.$$

Thus  $A_m$  is the symmetric arithmetic progression of  $m$  real digits with spacing 2. For  $c \in \mathbb{C} \setminus \mathbb{D}$  and  $t \in A_m$ , define

$$f_t(z) = t + \frac{z}{c}.$$

The corresponding Hutchinson operator is

$$\mathcal{H}_{c,m}(U) = \bigcup_{t \in A_m} \left( t + \frac{U}{c} \right).$$

Its attractor is denoted by  $E(c, m)$ ; equivalently,

$$E(c, m) = \bigcup_{t \in A_m} \left( t + \frac{E(c, m)}{c} \right).$$

In the terminology of Definition 3.1, the sets

$$t + \frac{E(c, m)}{c}, \quad t \in A_m,$$

are the first-level pieces of  $E(c, m)$ .

We study connectedness by fixing the cardinality  $n \geq 2$  and varying  $c$ . The corresponding parameter locus is

$$\mathcal{M}_n := \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : E(c, n) \text{ is connected}\}.$$

This is the connectedness locus of Definition 3.2.

Regular-closedness away from the real axis means that every non-real connectedness parameter can be approximated by interior parameters. Such an approximation replaces a possibly unstable overlap by connectedness that persists throughout a parameter neighborhood. For  $n = 2$ , the assertion is Bandt's conjecture [Ban02]; it was partially resolved by Solomyak and Xu [SX03] and proved in full by Calegari, Koch, and Walker [CKW17].

The covering construction in Theorem 4.5 proves regular-closedness away from the real axis for  $n \geq 21$ . The finite-capture analysis gives the sharper lens-containment threshold  $n \geq 20$  in Theorem 6.44. Together, these two results also give regular-closedness away from the real axis for  $n = 20$ . An unpublished companion manuscript establishes regular-closedness away from the real axis for every  $n \geq 2$  by a perturbative trap argument [Esp26g, Theorem 1.1].

## 2.4. Addresses and Difference Attractors

Addresses record the successive pieces containing a point. For  $m \geq 2$  and  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , removing the first digit of an address uses the corresponding inverse branch

$$g_t(z) = c(z - t) = f_t^{-1}(z), \quad t \in A_m.$$

**Definition 2.5** (Words, addresses, and composed branches). Let  $A_m^0 = \{\varepsilon\}$ , where  $\varepsilon$  is the empty word, and set

$$f_\varepsilon = g_\varepsilon = \text{id}.$$

For a word

$$u = t_0 t_1 \cdots t_{k-1} \in A_m^k, \quad |u| = k,$$

define

$$f_u = f_{t_0} \circ f_{t_1} \circ \cdots \circ f_{t_{k-1}}, \quad g_u = g_{t_{k-1}} \circ \cdots \circ g_{t_1} \circ g_{t_0} = f_u^{-1}.$$

If  $u$  and  $v$  are finite words and  $uv$  denotes their concatenation, then

$$f_{uv} = f_u \circ f_v, \quad g_{uv} = g_v \circ g_u.$$

Finally,

$$A_m^{\leq k} = \bigcup_{j=0}^k A_m^j.$$

The first digit of  $u$  selects the first-level piece, the second selects a piece within it, and so on:

$$f_u(z) = t_0 + \frac{t_1}{c} + \cdots + \frac{t_{k-1}}{c^{k-1}} + \frac{z}{c^k}.$$

For an infinite word

$$\omega = a_0 a_1 a_2 \cdots \in A_m^{\mathbb{N}_0}, \quad \mathbb{N}_0 = \{0, 1, 2, \dots\},$$

define the address map

$$\pi_{c,m}(\omega) = \sum_{j=0}^{\infty} a_j c^{-j}.$$

The index 0 records the first translation, before any contraction is applied. The address representation of the attractor is

$$E(c, m) = \pi_{c,m}(A_m^{\mathbb{N}_0}) = \left\{ \sum_{j=0}^{\infty} a_j c^{-j} : a_j \in A_m \right\}.$$

Prepending a finite word applies its forward branch: for  $u \in A_m^k$ ,

$$f_u(\pi_{c,m}(\omega)) = \pi_{c,m}(u\omega).$$

We now return to the fixed original alphabet  $A_n$  and set

$$N = 2n - 1, \quad D_n = \{-n + 1, -n + 2, \dots, n - 1\}.$$

We use  $n$  for the original attractor and  $N$  for its difference attractor;  $m$ , when used as an alphabet index, allows either cardinality. Subtracting addresses digit by digit, the identities

$$A_n - A_n = A_N, \quad A_N = 2D_n$$

give the difference-attractor identity

$$E(c, n) - E(c, n) = E(c, N).$$

**Definition 2.6** (Marked point). For the difference attractor  $E(c, N)$ , where  $N = 2n - 1$ , the parameter-dependent point  $2c$  is called the marked point.

An overlap between neighboring pieces requires two points of  $E(c, n)$  whose difference is  $2c$ . Thus the neighboring-piece criterion Proposition 3.3 becomes

$$c \in \mathcal{M}_n \iff 2c \in E(c, N), \quad N = 2n - 1,$$

proved formally in Proposition 5.2.

**Definition 2.7** (Restricted polynomial root set). For  $n \geq 2$ , let

$$\mathcal{R}_n = \left\{ c \in \mathbb{C} : c^q + \sum_{j=0}^{q-1} d_j c^j = 0 \text{ for some } q \geq 1 \text{ and } d_j \in D_n \right\}.$$

These are the roots of monic integer polynomials whose non-leading coefficients have absolute value at most  $n - 1$ . The next section explains why their closure outside the closed unit disk describes the same connectedness locus.

## 2.5. Reciprocal Series

A polynomial relation can be written as a terminating digit expansion. Dividing

$$c^q + \sum_{j=0}^{q-1} d_j c^j = 0$$

by  $c^q$  gives the finite reciprocal equation

$$1 + \sum_{k=1}^q d_{q-k} c^{-k} = 0.$$

For an infinite digit sequence  $(d_k)_{k \geq 1} \in D_n^{\mathbb{N}}$ , the reciprocal series converges absolutely because  $|c| > 1$ . The symmetry  $D_n = -D_n$  identifies a zero equation with a signed expansion:

$$1 + \sum_{k=1}^{\infty} d_k c^{-k} = 0 \iff 1 = \sum_{k=1}^{\infty} \varepsilon_k c^{-k}, \quad \varepsilon_k = -d_k \in D_n.$$

We use these two forms interchangeably. The represented value 1 and the alphabet are fixed, while the complex base varies; any digit sequence giving the expansion is allowed.

The marked-point condition  $2c \in E(c, N)$  gives the same equation: divide an address of  $2c$  by  $2c$  and use  $A_N = 2D_n$ , then change the signs of the digits. This division explains why reciprocal series start at index 1, while attractor addresses start at index 0. An address equality between neighboring pieces therefore becomes a restricted-series zero, as formalized in Proposition 3.4.

A zero of a finite reciprocal expression is called a *finite truncation zero*; multiplying by  $c^q$  identifies it with a root in  $\mathcal{R}_n$ . Rouché's theorem connects these finite equations to infinite expansions: zeros of the truncations approximate every zero of the infinite series (Lemma 4.4). It also supplies a finite verification tool. A *certified root disk* is a specified disk for which a strict boundary inequality establishes the required root count; the test used here is Lemma A.1.

Together with the address characterization, this approximation yields

$$\mathcal{M}_n = \overline{\mathcal{R}_n} \setminus \overline{\mathbb{D}}$$

as proved in Theorem 5.5.

## 2.6. The Two-Disk Parameter Lens

The covering constructions work in a parameter region determined by the interaction between the contraction and the digit spacing.

**Definition 2.8** (Two-disk parameter lens). For  $n \geq 2$ , define

$$\mathcal{X}_n = \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : |c - 1| < \sqrt{2n}, |c + 1| < \sqrt{2n} \right\}.$$

For  $c = x + iy \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , set  $\rho = |c|$  and  $N = 2n - 1$ . Then

$$c \in \mathcal{X}_n \iff \rho^2 + 2|x| < N.$$

For non-real parameters, this inequality is exactly the feasibility condition for the nearest-digit covering criterion (Proposition 6.9). The canonical trap of Definition 6.10 is both nonempty and self-covering precisely when  $c \in \mathcal{X}_n \setminus \mathbb{R}$ ; see Corollary 6.11.

The sharp containment theorem Theorem 6.44 states that

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n \quad \text{for } n \geq 20,$$

whereas for each  $2 \leq n \leq 19$  it supplies a non-real point of  $\mathcal{M}_n$  outside  $\mathcal{X}_n$ . Thus 20 is the exact threshold.

## 2.7. Covering Properties

A self-covering set is a region in which an inverse address can always be continued. For a finite IFS with Hutchinson operator  $\mathcal{H}_{\mathcal{F}}$  and attractor  $K$ , a nonempty set  $U \subset \mathbb{C}$  is called *self-covering* if

$$U \subset \mathcal{H}_{\mathcal{F}}(U).$$

For compact  $U$  and contracting similarities, this is the sub-self-similar condition introduced by Falconer [Fal95]. Every point of  $U$  then has at least one inverse image in  $U$ . If  $U$  is bounded, this continuation places it inside the attractor.

**Proposition 2.9** (Self-covering principle for finite IFSs). *If  $U$  is nonempty, bounded, and self-covering for  $\mathcal{F}$ , then*

$$\overline{U} \subset K.$$

*If, in addition,  $U$  is open, then  $U \subset \text{int } K$ .*

*Proof.* Put  $A = \overline{U}$ , which is nonempty and compact. Finiteness of the family and continuity of its maps allow us to take closures in  $U \subset \mathcal{H}_{\mathcal{F}}(U)$ , obtaining

$$A \subset \mathcal{H}_{\mathcal{F}}(A).$$

Monotonicity of the Hutchinson operator gives  $A \subset \mathcal{H}_{\mathcal{F}}^k(A)$  for every  $k \geq 1$ . Hutchinson's theorem gives  $\mathcal{H}_{\mathcal{F}}^k(A) \rightarrow K$  in the Hausdorff metric, so every point of  $A$  has distance zero from  $K$ . Thus  $A \subset K$ . If  $U$  is open, the inclusion  $U \subset K$  implies  $U \subset \text{int } K$ .  $\square$

For the collinear system, self-covering is the concrete inclusion

$$U \subset \bigcup_{t \in A_m} \left( t + \frac{U}{c} \right) = \mathcal{H}_{c,m}(U).$$

The rectangle construction uses this principle for a compact set (Lemma 4.1); the canonical trap uses it for a bounded open set (Lemma 6.4).

In Chapter 4, the argument has four steps:

1. construct the parameter-dependent rectangle  $R(c, N)$ , with  $N = 2n - 1$ ;
2. prove that  $R(c, N)$  is self-covering for  $c \in \mathcal{X}_n$ ;
3. apply Lemma 4.1 to obtain  $R(c, N) \subset E(c, N)$ ;
4. for non-real parameters, use the rectangle's interior to continue finite reciprocal equations as infinite address equations, producing nearby interior points of  $\mathcal{M}_n$ .

This proves density of interior parameters in the non-real part of  $\mathcal{M}_n \cap \mathcal{X}_n$ , as stated in Theorem 4.5. For  $n \geq 21$ , the radial bound of Proposition 3.9 places the whole non-real locus in the lens, giving regular-closedness away from the real axis.

## 2.8. Traps, Enclosures, and Finite Capture

Finite capture records how many address digits are needed before the marked point reaches a self-covering region. With  $N = 2n - 1$ , its inverse search starts at  $2c$  and uses

$$g_t(z) = c(z - t), \quad t \in A_N,$$

with the word-composition convention of Definition 2.5.

Two parallelograms give an inner target and an outer bound. The canonical trap  $\mathcal{C}(c, N)$  is open and self-covering for  $c \in \mathcal{X}_n \setminus \mathbb{R}$  (Definition 6.10); the closed canonical enclosure  $\mathcal{E}(c, N)$  contains the difference attractor for every non-real exterior parameter (Definition 6.13). On  $\mathcal{X}_n \setminus \mathbb{R}$ , Theorem 6.15 gives

$$\mathcal{C}(c, N) \subset \text{int } E(c, N) \subset E(c, N) \subseteq \mathcal{E}(c, N),$$

so the trap supplies an infinite continuation, while the enclosure bounds every possible continuation.

If a finite word  $u$  satisfies

$$g_u(2c) \in \mathcal{C}(c, N),$$

then applying  $f_u$  places  $2c$  in the interior of the difference attractor. The strict trap inequalities persist for this same word as  $c$  varies in a sufficiently small neighborhood. The marked-point characterization therefore proves  $c \in \text{int}(\mathcal{M}_n)$ .

To exclude membership, retain only the inverse words whose every prefix iterate stays in  $\mathcal{E}(c, N)$ . These are the enclosure-admissible words of Definition 6.19. If none survives to a finite depth, no address of  $2c$  exists, and  $c \notin \mathcal{M}_n$ . Thus one trap hit certifies interior membership, whereas extinction of all admissible branches certifies exclusion. A search that reaches neither event within its computational limits remains inconclusive.

Organizing trap hits by word length gives the finite-capture sets and capture time of Definition 6.17. Their closure behavior controls how the locus unfolds layer by layer: within  $\mathcal{X}_n \setminus \mathbb{R}$ , a limit of parameters captured by a fixed depth is captured within at most two additional steps (Theorem 6.35).

## 2.9. Equivalent Formulations and One-Sided Tests

The first six rows of Table 2.1 express the same membership condition, as proved in Chapters 3 and 5. The last row gives the finite tests of Chapter 6: they certify a conclusion when their stopping condition is met.

| Viewpoint            | Condition                                                                                              | Interpretation                                                                                                                                                                                                                                     |
|----------------------|--------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Original attractor   | $E(c, n)$ is connected                                                                                 | definition of $c \in \mathcal{M}_n$ .                                                                                                                                                                                                              |
| Symbolic addresses   | two neighboring first-level pieces intersect                                                           | overlap formulation of connectedness.                                                                                                                                                                                                              |
| Difference attractor | $2c \in E(c, 2n - 1)$                                                                                  | marked-point form of connectedness.                                                                                                                                                                                                                |
| Reciprocal series    | $1 + \sum_{k \geq 1} d_k c^{-k} = 0$ , with $(d_k)_{k \geq 1} \in D_n^{\mathbb{N}}$                    | analytic form of the same address equality.                                                                                                                                                                                                        |
| Numeration           | the signed-expansion form in Section 2.5                                                               | the fixed value 1 has a signed expansion in the variable base $c$ .                                                                                                                                                                                |
| Restricted roots     | $c \in \overline{\mathcal{R}}_n \setminus \overline{\mathbb{D}}$                                       | closure of bases admitting terminating signed expansions.                                                                                                                                                                                          |
| Finite certification | a trap hit, or extinction of all enclosure-admissible branches of Definition 6.19 at some finite depth | sufficient finite evidence for $c \in \text{int}(\mathcal{M}_n)$ , under $c \in \mathcal{X}_n \setminus \mathbb{R}$ , or for $c \notin \mathcal{M}_n$ , under $c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , respectively. |

**Table 2.1.** Equivalent descriptions of connectedness and the finite tests derived from them.

## 2.10. Finite Certificates

Fix  $n \geq 2$  and put  $N = 2n - 1$ . A *finite certificate* records enough verified data to check its conclusion independently: exact values or outward-rounded rational intervals, together with finitely many equalities or inequalities. The three formats used here correspond to trap entry, exhaustion of inverse addresses, and polynomial root isolation. The linear coordinates and half-widths appearing below are those of the canonical trap and enclosure.

1. A *trap-entry certificate* for  $c \in \mathcal{X}_n \setminus \mathbb{R}$  consists of a word  $u \in A_N^m$  and the two strict inequalities

$$|\ell_s(g_u(2c); c)| < S(c, N), \quad |\ell_v(g_u(2c))| < V(c, N).$$

These verify a trap hit. The capture implication Proposition 6.23 and openness of the fixed-word capture set (Lemma 6.34) give  $c \in \text{int}(\mathcal{M}_n)$ . An example is worked out in Example 6.18.

2. An *enclosure-extinction certificate* for  $c \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$  consists of a depth  $k$  and the complete finite inverse tree through that depth, with every possible next digit accounted for. The certificate verifies one of the strict inequalities

$$|\ell_s(g_{u'}(2c); c)| > \mathcal{S}_{\mathcal{E}}(c, N) \quad \text{or} \quad |\ell_v(g_{u'}(2c))| > \mathcal{V}_{\mathcal{E}}(c, N)$$

for a prefix  $u'$  of each discarded branch. If no word survives at depth  $k$ , equivalently  $c \notin \Xi_k(n)$ , then  $c \notin \mathcal{M}_n$  by Proposition 6.21.

3. A *root-isolation certificate* consists of a specified monic polynomial

$$P(z) = z^d + a_1 z^{d-1} + \cdots + a_d, \quad a_j \in D_n,$$

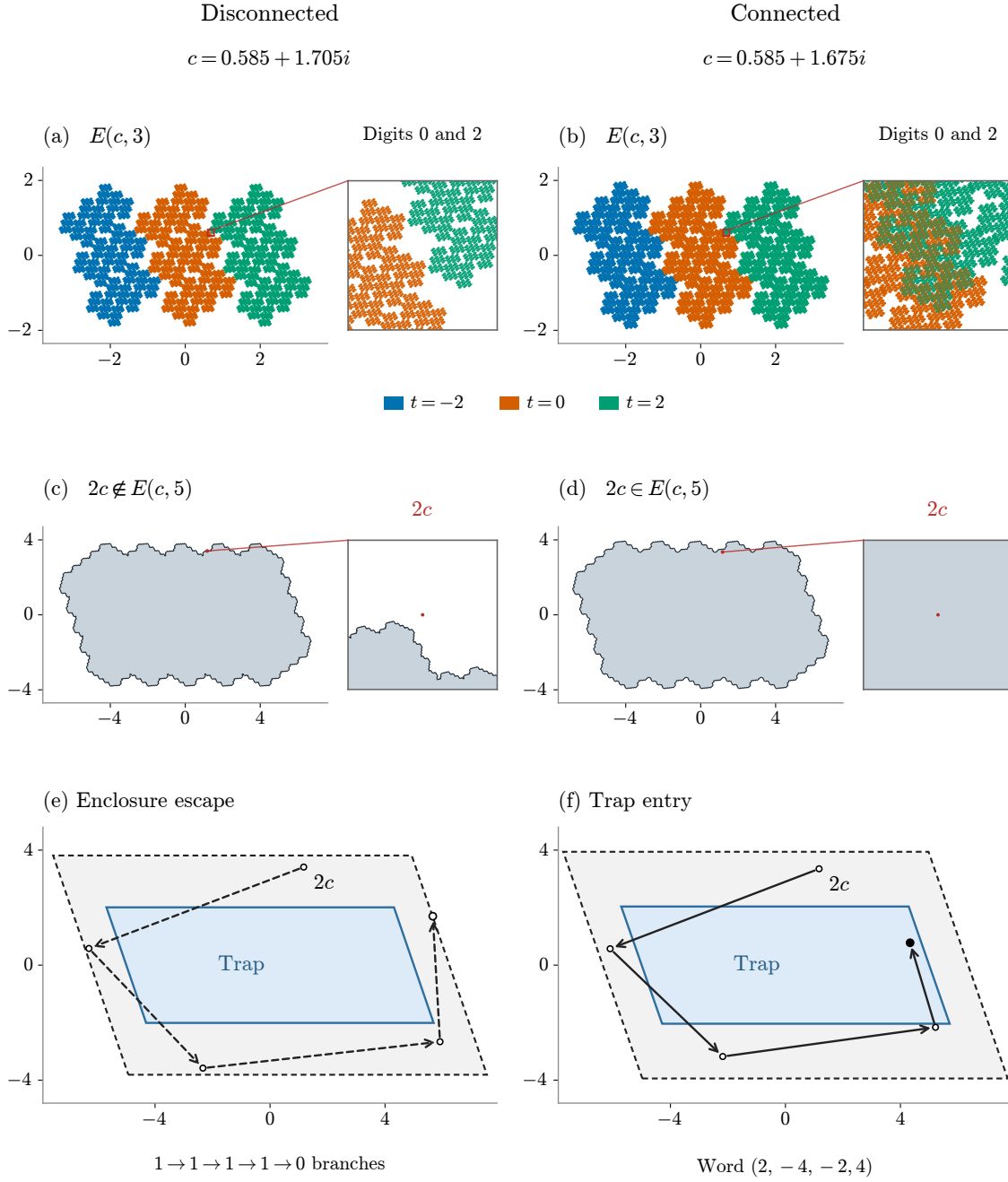
a closed disk  $\overline{B(z_0, r)}$ , and verified bounds giving the strict Rouché inequality

$$|P(z_0)| + \mathcal{R}_P(z_0, r) < |P'(z_0)|r.$$

The disk then contains exactly one zero of  $P$ , counted with multiplicity (Lemma A.1). Further inequalities, such as  $|\text{Im } z_0| > r$  and  $|z_0 + 1| - r > \sqrt{2n}$ , certify that this zero is non-real and lies outside  $\mathcal{X}_n$ . Such certificates are recorded in Table A.1.

To certify an entire parameter neighborhood, verify the inequalities uniformly there using exact rational or outward-rounded interval bounds. Floating-point plots and searches with depth or node limits locate candidate witnesses. A returned status **Interior** or **Exterior** becomes a certificate when accompanied by the strict trap inequalities or exhaustive enclosure checks above; **Undetermined** records that neither certificate has been obtained.

Figure 2.1 illustrates a trap-entry certificate and the rejection of a single inverse branch.



**Figure 2.1.** Connectedness and finite inverse search for  $n = 3$ ,  $N = 5$ , at the two parameters shown above. Top: first-level pieces  $t + c^{-1}E(c, 3)$ , with separate  $0.17 \times 0.17$  details of the digit-0 and digit-2 region. Middle: the difference attractor and  $0.028 \times 0.028$  details centred at the red point  $2c$ . Bottom: the word  $(2, -4, -2, 4)$  escapes on the left and enters the trap on the right; all left branches disappear by depth four. Rasters are numerical; exact arithmetic verifies the membership labels.



PART II.

# Connectedness Loci

Chapters 3 and 4



# Collinear Fractals and Connectedness Loci

Connectedness of a collinear attractor can be read from a single neighboring overlap. Taking differences then turns that geometric condition into a signed expansion of the number 1 in base  $c$ . This chapter establishes these equivalent descriptions and uses them to prove nesting, quantitative bounds, connectedness, and local connectedness of the parameter loci. The same overlap equation will later provide the starting point for finite capture.

## 3.1. The Collinear System

For  $n \geq 2$ , the collinear family uses the equally spaced alphabet

$$A_n = \{n + 1 - 2k : 1 \leq k \leq n\} = \{-n + 1, -n + 3, \dots, n - 3, n - 1\}.$$

For  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , consider the affine maps

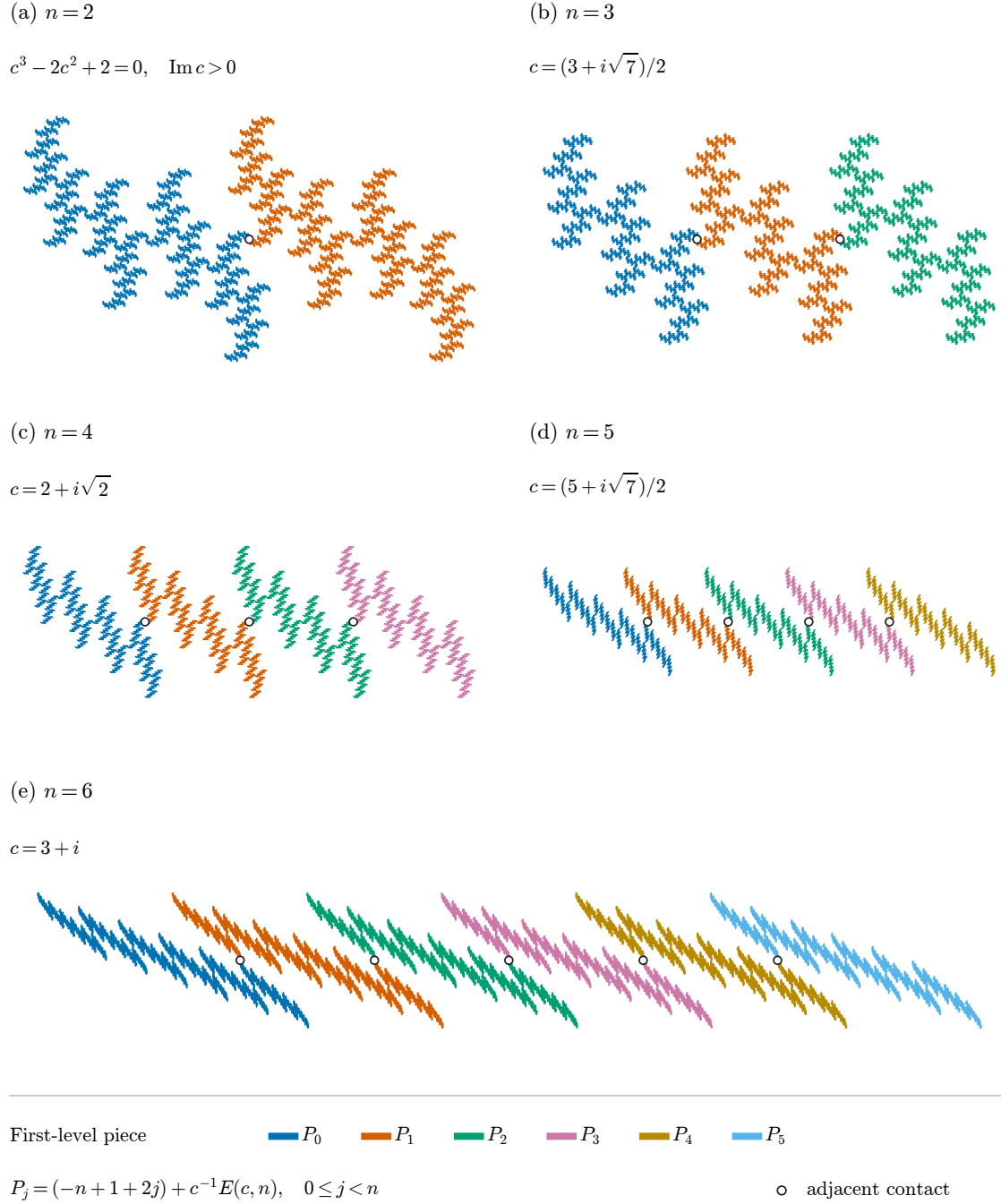
$$f_t(z) = t + c^{-1}z, \quad t \in A_n.$$

**Definition 3.1** (The  $n$ -ary collinear fractal). The  $n$ -ary collinear fractal  $E(c, n)$  is the unique nonempty compact set satisfying

$$E(c, n) = \bigcup_{t \in A_n} \left( t + \frac{E(c, n)}{c} \right).$$

The first-level pieces are  $E_t(c, n) = t + c^{-1}E(c, n)$ . Since consecutive digits in the ordered alphabet  $A_n$  differ by 2, we call  $E_t(c, n)$  and  $E_{t+2}(c, n)$  adjacent first-level pieces. The examples in Figures 1.1 and 1.2 show neighboring overlaps and nonempty interior. Single-point contacts lead to a different geometry.

A *dendrite* is a locally connected continuum containing no simple closed curve. Figure 3.1 shows a dendrite  $E(c, n)$  at an exact algebraic parameter for each of the five alphabet sizes  $2 \leq n \leq 6$ . Neighboring first-level pieces meet at a single point, and all other pairs are disjoint. The constructions and proofs are given in [Esp26c, Theorem 1.1]; the obstruction for  $n \geq 7$  is discussed in Section C.5.



**Figure 3.1.** Exact non-real dendrite examples from [Esp26c]. In (a),  $c$  is the unique upper-half-plane root of  $c^3 - 2c^2 + 2$ ; in (b)–(e),  $c = (n + i\sqrt{8(n-1) - n^2})/2$ . Colours identify the first-level pieces  $P_j = (-n + 1 + 2j) + c^{-1}E(c, n)$ . The circles mark the exact contacts  $P_j \cap P_{j+1} = \{-n + 2 + 2j\}$ ; non-neighboring pieces are disjoint. Each attractor is scaled independently while preserving its orientation and aspect ratio; the six-map example occupies the full-width lower panel. The renderings use finite digit prefixes; the dendrite assertion follows from the cited proofs.

### 3.2. Connectedness Through Neighboring Pieces

**Definition 3.2** (Connectedness locus for collinear fractals). The connectedness locus  $\mathcal{M}_n$  is

$$\mathcal{M}_n = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : E(c, n) \text{ is connected}\}.$$

Equal spacing makes every neighboring intersection a translate of the same one. Bousch's consecutive-digit criterion shows that this single intersection also characterizes connectedness [Bou92, Chapter 1, p. 41] in the present normalization. For  $s = 1/c$ , the affine map

$$h(z) = 2z - \frac{n-1}{1-s}$$

conjugates his maps  $z \mapsto sz + j$ ,  $j = 0, \dots, n-1$ , to the maps with alphabet  $A_n$ . An overlap gives a chain connecting the first-level pieces. For the converse, Lemma 2.2 rules out a smallest intersection gap larger than one.

**Proposition 3.3** (Neighbor-overlap criterion). *For every  $n \geq 2$ , the attractor  $E(c, n)$  is connected if and only if*

$$\frac{E(c, n)}{c} \cap \left(2 + \frac{E(c, n)}{c}\right) \neq \emptyset.$$

*Proof.* Put  $K = c^{-1}E(c, n)$ , and write

$$d_j = -n + 1 + 2j, \quad P_j = d_j + K, \quad 0 \leq j \leq n-1.$$

Assume first that  $K \cap (2 + K) \neq \emptyset$ . Since  $d_{j+1} = d_j + 2$ ,

$$P_j \cap P_{j+1} = d_j + (K \cap (2 + K)) \neq \emptyset, \quad 0 \leq j \leq n-2.$$

Thus the first-level connectivity graph contains the spanning chain

$$0 - 1 - \dots - (n-1).$$

The standard connectivity criterion for an IFS attractor [Hat85; BK91] now implies that  $E(c, n)$  is connected.

Conversely, assume that  $E(c, n)$  is connected. Then  $K$  and all the pieces  $P_j$  are nonempty compact connected sets. Moreover, the finite family  $\{P_j\}_{j=0}^{n-1}$  cannot be pairwise disjoint, because its union is the connected set  $E(c, n)$ . Hence

$$\mathcal{S} = \{q - p : 0 \leq p < q \leq n-1, P_p \cap P_q \neq \emptyset\}$$

is nonempty. Choose  $m = \min \mathcal{S}$  and indices  $p < p+m$  for which  $P_p \cap P_{p+m} \neq \emptyset$ .

Suppose first that  $c \notin \mathbb{R}$ . Set

$$x_- = \min_{z \in K} \operatorname{Re} z, \quad x_+ = \max_{z \in K} \operatorname{Re} z, \quad y_- = \min_{z \in K} \operatorname{Im} z, \quad y_+ = \max_{z \in K} \operatorname{Im} z.$$

We have  $y_- < y_+$ . Indeed,  $E(c, n)$  contains two points whose first digits differ by 2 and whose remaining digits agree; after multiplication by  $c^{-1}$ , the corresponding points of  $K$  differ by  $2/c$ , which has nonzero imaginary part.

If  $m \geq 2$ , define

$$R = [x_- + d_p, x_+ + d_{p+m}] \times [y_-, y_+], \quad H = P_p \cup P_{p+m}, \quad V = P_{p+1}.$$

The set  $H$  is a continuum because  $P_p \cap P_{p+m} \neq \emptyset$ ; it is contained in  $R$  and meets both vertical sides. The set  $V$  is also a continuum contained in  $R$ , and it meets both horizontal sides because all  $P_j$  are real translates of  $K$ . By Lemma 2.2,

$$P_{p+1} \cap (P_p \cup P_{p+m}) \neq \emptyset.$$

Consequently, either  $P_{p+1} \cap P_p \neq \emptyset$ , which gives an intersection of index gap 1, or  $P_{p+1} \cap P_{p+m} \neq \emptyset$ , which gives an intersection of index gap  $m - 1$ . Both alternatives contradict the minimality of  $m$ . Therefore  $m = 1$ .

If  $c \in \mathbb{R}$ , then  $E(c, n) \subset \mathbb{R}$ , and connectedness makes  $K$  a compact interval. An intersection  $P_p \cap P_{p+m} \neq \emptyset$  implies that the length of  $K$  is at least  $2m$ , hence at least 2. Therefore  $K \cap (2 + K) \neq \emptyset$  directly.

In the non-real case,  $m = 1$  gives  $P_p \cap P_{p+1} \neq \emptyset$  for some  $p$ ; translating by  $-d_p$  yields  $K \cap (2 + K) \neq \emptyset$ . Since every neighboring intersection is a translate of this one, all neighboring pieces intersect. The real case already gives the same conclusion.  $\square$

Taking differences preserves the collinear form and enlarges the alphabet:  $A_n - A_n = A_{2n-1}$ . With

$$D_n = \{-n + 1, -n + 2, \dots, n - 1\} = \frac{1}{2}A_{2n-1},$$

Proposition 3.3 therefore converts connectedness into a reciprocal-series equation with consecutive integer coefficients.

**Proposition 3.4** (Series characterization). *For every  $n \geq 2$ ,*

$$\mathcal{M}_n = \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : \exists (d_k)_{k \geq 1} \in D_n^{\mathbb{N}} \text{ such that } 1 + \sum_{k=1}^{\infty} d_k c^{-k} = 0 \right\}.$$

*Proof.* Applying the address representation

$$E(c, n) = \left\{ \sum_{k \geq 0} a_k c^{-k} : a_k \in A_n \right\},$$

we can write the difference set

$$E(c, n) - E(c, n) = \{p - q : p, q \in E(c, n)\}$$

in the form

$$E(c, n) - E(c, n) = \left\{ \sum_{k \geq 0} (a_k - a'_k) c^{-k} : a_k, a'_k \in A_n \right\}.$$

Each digit in  $A_{2n-1}$  can be chosen independently as a difference of two digits in  $A_n$ . The last set is therefore precisely  $E(c, 2n - 1) = E(c, N)$ , with  $N = 2n - 1$ :

$$E(c, n) - E(c, n) = E(c, N). \quad (\star)$$

By Proposition 3.3, connectedness is equivalent to

$$0 \in 2 + \frac{E(c, n) - E(c, n)}{c} = 2 + \frac{E(c, N)}{c} \quad \text{by } (\star).$$

Using the series representation for  $E(c, N)$ , this is equivalent to the existence of  $b_k \in A_N$  such that

$$0 = 2 + \sum_{k=1}^{\infty} b_k c^{-k}.$$

Dividing by 2 and writing  $d_k = b_k/2 \in D_n$  gives the stated reciprocal-series condition.  $\square$

**Remark 3.5** (Numeration interpretation). Membership requires only the existence of a signed  $D_n$ -expansion of 1 in base  $c$ . The expansion need not follow a greedy, lazy, or other prescribed digit-selection rule.

The same digit arithmetic describes how the attractors change with arity.

**Proposition 3.6** (Minkowski dilation cascade). *Let  $m, \ell \geq 2$  and  $|c| > 1$ . Then*

$$A_m + A_\ell = A_{m+\ell-1} \quad \text{and} \quad E(c, m) + E(c, \ell) = E(c, m + \ell - 1).$$

Consequently, for every integer  $r \geq 0$ ,

$$E(c, n + r) = E(c, n) + \bigoplus_{j=1}^r E(c, 2),$$

where  $\bigoplus_{j=1}^r E(c, 2)$  is the  $r$ -fold Minkowski sum, with the empty sum interpreted as  $\{0\}$  when  $r = 0$ .

*Proof.* Every element of  $A_m + A_\ell$  has the form

$$(-m + 1 + 2i) + (-\ell + 1 + 2j) = -(m + \ell) + 2 + 2(i + j),$$

with  $0 \leq i \leq m - 1$  and  $0 \leq j \leq \ell - 1$ . As  $i + j$  runs through all integers from 0 to  $m + \ell - 2$ , this proves the alphabet identity. Applying it independently to every digit in the address representation yields the attractor identity. The final formula follows by induction with  $\ell = 2$ .  $\square$

**Remark 3.7.** Here *dilation* means repeated Minkowski addition. This identity alone does not give nesting of the attractors: the added set  $E(c, 2)$  need not contain 0. Nesting of the parameter loci, however, follows from their signed coefficient alphabets.

**Proposition 3.8** (Nested loci). *For every  $n \geq 2$ ,  $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ .*

*Proof.* This follows directly from  $D_n \subset D_{n+1}$  in Proposition 3.4.  $\square$

Figure 3.2 illustrates the nesting by superimposing the loci  $\mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_{21}$  in the upper-right parameter plane.

### 3.3. Bounds and Basic Topology

To obtain an outer bound, first allow every real coefficient between the extreme digits. Let  $I_n = [1 - n, n - 1]$ , and define the convexity set

$$\mathcal{M}_n^* = \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : \exists (a_k)_{k \geq 1} \in I_n^{\mathbb{N}} \text{ such that } 1 + \sum_{k=1}^{\infty} a_k c^{-k} = 0 \right\}.$$

**Proposition 3.9** (Upper bound). *For every  $n \geq 2$ ,*

$$\mathcal{M}_n \subset \mathcal{M}_n^* \subset B(0, 1 + \sqrt{n-1}) \cup [-n, n].$$

*Proof.* The inclusion  $\mathcal{M}_n \subset \mathcal{M}_n^*$  follows directly from  $D_n \subset I_n$  and from Proposition 3.4.

Bounds for zeros with interval coefficients transfer to  $\mathcal{M}_n^*$  by inversion. Following Beaucoup, Borwein, Boyd, and Pinner, put  $g = n - 1$ , and let  $Z_g$  be the set of zeros in the open unit disk of power series of the form

$$1 + \sum_{k=1}^{\infty} a_k z^k, \quad a_k \in [-g, g].$$

By definition of  $\mathcal{M}_n^*$ ,

$$c \in \mathcal{M}_n^* \iff \lambda = \frac{1}{c} \in Z_{n-1}.$$

Let  $r_g(\varphi)$  denote the infimum of the moduli of points of  $Z_g$  with argument  $\varphi$ . The real-root estimate of [Bea+98b] gives

$$r_g(\varphi) \geq \frac{1}{g+1},$$

with equality on the positive real ray. After inversion and setting  $g = n - 1$ , this implies that the real trace of  $\mathcal{M}_n^*$  is contained in

$$[-n, n].$$

For  $g > 1$ , the non-real-root estimate of [Bea+98b] gives

$$r_g(\varphi) > \frac{1}{1 + \sqrt{g}} \quad (0 < \varphi < \pi).$$

Because the coefficients are real,  $Z_g$  is invariant under complex conjugation, so the same estimate holds for  $-\pi < \varphi < 0$ . For  $g = 1$ , if  $\lambda \in Z_1$ , the triangle inequality gives

$$1 \leq \sum_{k \geq 1} |a_k \lambda^k| \leq \frac{|\lambda|}{1 - |\lambda|},$$

hence  $|\lambda| \geq 1/2$ . Equality would force  $|a_1| = 1$  and  $a_1 \lambda$  to be real, hence  $\lambda \in \mathbb{R}$ . Therefore every non-real zero also satisfies  $|\lambda| > 1/2$  when  $g = 1$ . Again inverting and taking  $g = n - 1$ , we obtain

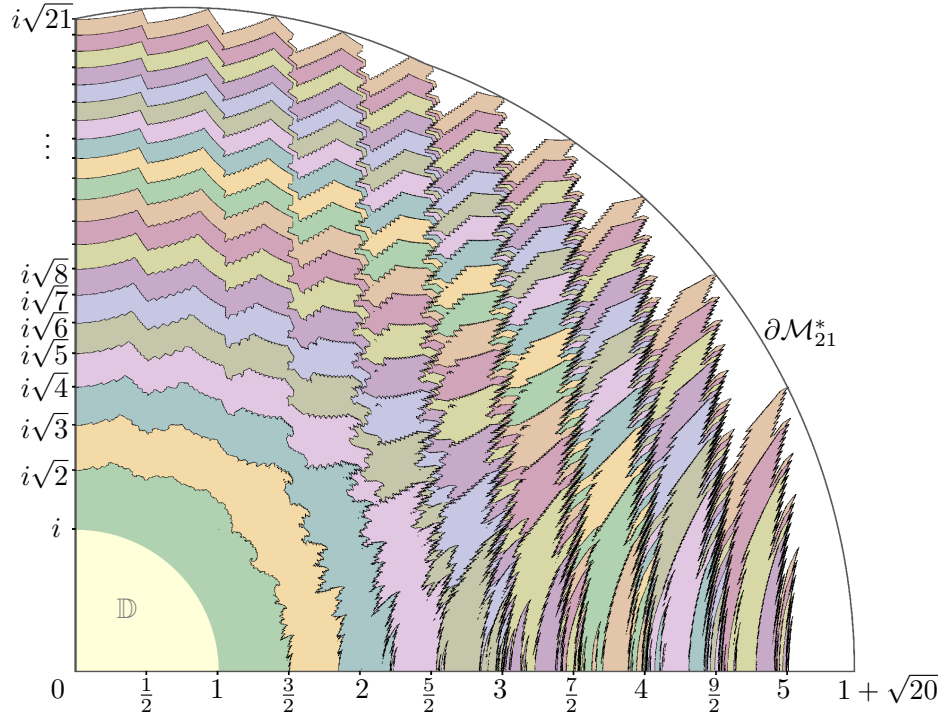
$$\mathcal{M}_n^* \setminus \mathbb{R} \subset B(0, 1 + \sqrt{n-1}).$$

Combining the estimates on and off the real axis gives

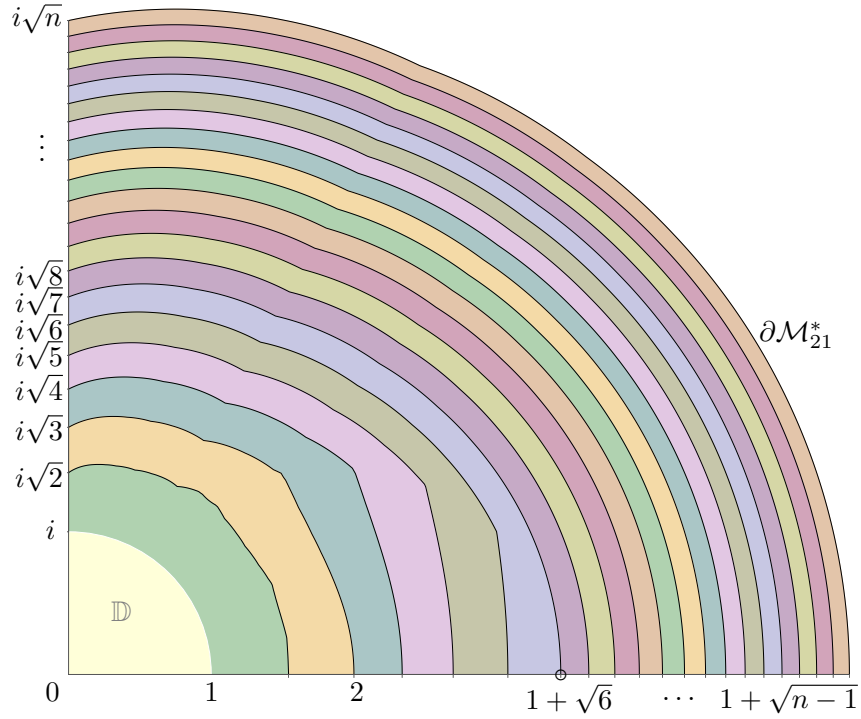
$$\mathcal{M}_n^* \subset B(0, 1 + \sqrt{n-1}) \cup [-n, n],$$

as required. □

Figure 3.3 displays this relaxation of the nested loci.



**Figure 3.2.** Nested arrangement of  $\mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_{21}$  in the upper-right parameter plane.



**Figure 3.3.** Convexity sets  $\mathcal{M}_2^*, \mathcal{M}_3^*, \dots, \mathcal{M}_{21}^*$ , with the disk bound from Proposition 3.9.

**Remark 3.10** (Convex hull and marked point). Let  $H(c, N)$  denote the convex hull of  $E(c, N)$ , with  $N = 2n - 1$ . Replacing the digit alphabet by its convex hull replaces the difference attractor by its convex hull. Thus both parameter sets test the same marked point  $2c$ :

$$\mathcal{M}_n = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : 2c \in E(c, N)\}, \quad \mathcal{M}_n^* = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : 2c \in H(c, N)\}.$$

Membership in  $\mathcal{M}_n^*$  is therefore a necessary condition for connectedness. The inner and outer tests in Theorem 6.15 will address membership in the difference attractor itself. Vass studies exact convex-hull reconstruction, finiteness of extremal sets, and constructive methods using periodic extremal addresses in specified classes [Vas13; Vas18]. The relaxation above requires only the hull-membership condition.

An inner bound comes from a different source. If the combined area scaling exceeds one, nested finite approximations must have overlapping pieces. The neighboring-piece argument then makes each approximation connected. On the real axis, an interval calculation gives the exact trace.

**Proposition 3.11** (Inner annulus and exact real trace). *For every  $n \geq 2$ ,*

$$\{c \in \mathbb{C} : 1 < |c| < \sqrt{n}\} \subset \mathcal{M}_n, \quad \mathcal{M}_n \cap \mathbb{R} = [-n, -1) \cup (1, n].$$

*Proof.* Fix  $1 < |c| < \sqrt{n}$ , put  $R = (n - 1)|c|/(|c| - 1)$ , and define

$$H_0 = \{z \in \mathbb{C} : |z| \leq R\}, \quad H_{k+1} = \bigcup_{t \in A_n} \left(t + \frac{H_k}{c}\right).$$

The choice of  $R$  gives  $H_1 \subset H_0$ . Applying the IFS operator repeatedly shows that the nonempty compact sets  $H_k$  are nested. Each has positive planar area, since it contains a similarity image of  $H_0$  obtained from a word of length  $k$ .

We prove by induction that each  $H_k$  is connected. This holds for  $H_0$ . If  $H_k$  is connected, then the sets  $t + c^{-1}H_k$ ,  $t \in A_n$ , are continua. They cannot be pairwise disjoint: otherwise similarity scaling and finite additivity of area would give

$$\text{area}(H_{k+1}) = \frac{n}{|c|^2} \text{area}(H_k) > \text{area}(H_k),$$

contrary to  $H_{k+1} \subset H_k$ . Among intersecting translates, choose a pair with the smallest index gap. The minimal-gap rectangle-crossing argument in the proof of Proposition 3.3 applies to the continuum  $c^{-1}H_k$ : its positive area ensures nonzero vertical extent, and any gap greater than one would yield a smaller gap. Thus adjacent translates intersect, and translation invariance gives an intersecting chain through all the pieces. Consequently  $H_{k+1}$  is connected. By Hutchinson's theorem,  $H_k$  converges to  $E(c, n)$  in the Hausdorff metric. Since the  $H_k$  are nested compact sets,

$$E(c, n) = \bigcap_{k \geq 0} H_k.$$

The nested-intersection theorem for continua proves the annular inclusion. The area obstruction also underlies the overlap bound in Lemma 2.5 (i) of [Sol98]. Here the crossing argument supplies the adjacent intersections needed to deduce connectedness.

For the real trace, let  $c \in (1, n]$ , and set

$$H = \left[ -\frac{(n-1)c}{c-1}, \frac{(n-1)c}{c-1} \right].$$

The IFS preserves the real line, and the address representation gives  $E(c, n) \subset H$ . Both endpoints of  $H$  occur through the constant addresses  $a_k \equiv -(n-1)$  and  $a_k \equiv n-1$ , respectively. Hence  $H$  is the convex hull of  $E(c, n)$ , and its contracted hull is

$$\frac{H}{c} = \left[ -\frac{n-1}{c-1}, \frac{n-1}{c-1} \right].$$

The convex hulls of two neighboring first-level pieces are

$$t + \frac{H}{c} \quad \text{and} \quad t + 2 + \frac{H}{c}.$$

They overlap whenever

$$\frac{n-1}{c-1} \geq 1,$$

which is equivalent to  $c \leq n$ . Thus, for  $1 < c \leq n$ , the intervals  $t + H/c$ ,  $t \in A_n$ , overlap consecutively and cover  $H$ , i.e.

$$H = \bigcup_{t \in A_n} \left( t + \frac{H}{c} \right).$$

By uniqueness of the compact invariant set for this real contracting IFS,  $E(c, n) = H$ . Hence  $E(c, n)$  is connected, and

$$(1, n] \subset \mathcal{M}_n \cap \mathbb{R}.$$

The symmetry  $c \mapsto -c$ , already visible from the reciprocal-series description, gives

$$[-n, -1) \subset \mathcal{M}_n \cap \mathbb{R}.$$

Together with the real upper bound in Proposition 3.9, and recalling that the parameter domain excludes  $[-1, 1]$ , this gives

$$\mathcal{M}_n \cap \mathbb{R} = [-n, -1) \cup (1, n].$$

□

**Remark 3.12** (Role of the annular inclusion). The annular inclusion in Proposition 3.11 verifies the annulus hypothesis in the application of Nakajima's general zero-set criterion [Nak24, Main Theorem B] in the proof of Theorem 3.13.

The inner annulus and Proposition 3.9 place the non-real boundary of  $\mathcal{M}_n$  in

$$\sqrt{n} \leq |c| < 1 + \sqrt{n-1},$$

while the exact real trace extends to the antenna tips  $\pm n$ .

The topology of the parameter locus now follows from its zero-set description. Bousch's Theorem 3, combined with his consecutive-digit criterion and annular inclusion, already gives connectedness [Bou92, Chapter 1, pp. 31, 37, 41]. Nakajima's general zero-set criterion gives connectedness and local connectedness together [Nak24, Main Theorem B].

**Theorem 3.13** (Connectedness and local connectedness). *For every  $n \geq 2$ , the locus  $\mathcal{M}_n$  is connected and locally connected.*

*Proof.* Put  $\lambda = 1/c$ , and let

$$Z_n = \left\{ \lambda \in \mathbb{D} : \exists (d_k)_{k \geq 1} \in D_n^{\mathbb{N}} \text{ such that } 1 + \sum_{k=1}^{\infty} d_k \lambda^k = 0 \right\}.$$

Since every series in the defining family has value 1 at  $\lambda = 0$ , we have  $0 \notin Z_n$ . By the reciprocal-series characterization Proposition 3.4,

$$c \in \mathcal{M}_n \iff \lambda = \frac{1}{c} \in Z_n.$$

Thus the inversion map  $\lambda \mapsto 1/\lambda$  is a homeomorphism from  $Z_n \subset \mathbb{D} \setminus \{0\}$  onto  $\mathcal{M}_n \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ . It is therefore enough to prove that  $Z_n$  is connected and locally connected.

We apply Nakajima's general criterion for zero sets of restricted power series [Nak24, Main Theorem B] to  $G = D_n$ . His Main Theorem A applies the same criterion to prove connectedness and local connectedness of the Mandelbrot sets of fractal  $n$ -gons [Nak24, Main Theorem A].

For a finite set  $G \subset \mathbb{C}$ , Nakajima defines

$$\Delta G = G - G, \quad R_G = \{\alpha \in \mathbb{C} : \alpha G \subset \Delta G\},$$

and puts an edge between  $a, b \in G$  when  $a - b \in R_G$ . This criterion, extending Bousch's binary zero-set argument [Bou88; Bou93], says that if  $G$  contains 1, if the graph  $(G, R_G)$  is connected, and if there exists  $L \in (0, 1)$  such that

$$\{\lambda \in \mathbb{C} : L < |\lambda| < 1\} \subset X^G, \quad X^G = \left\{ \lambda \in \mathbb{D} : \exists (a_k)_{k \geq 1} \in G^{\mathbb{N}}, 1 + \sum_{k \geq 1} a_k \lambda^k = 0 \right\},$$

then  $X^G$  is connected and locally connected.

Here  $Z_n = X^{D_n}$ , so it remains to check the hypotheses for  $G = D_n = \{-n+1, \dots, n-1\}$ . First,  $1 \in D_n$ , since  $n \geq 2$ . Moreover,

$$\Delta D_n = D_n - D_n = \{-2n+2, \dots, 2n-2\}.$$

Hence  $D_n \subset \Delta D_n$  and  $-D_n \subset \Delta D_n$ , so  $\pm 1 \in R_{D_n}$ . Consecutive digits  $j, j+1 \in D_n$  are therefore joined by an edge, because  $j - (j+1) = -1 \in R_{D_n}$ . Since  $D_n$  is a consecutive integer interval, the graph  $(D_n, R_{D_n})$  is connected.

It remains to check Nakajima's annulus hypothesis. Take

$$L = \frac{1}{\sqrt{n}}.$$

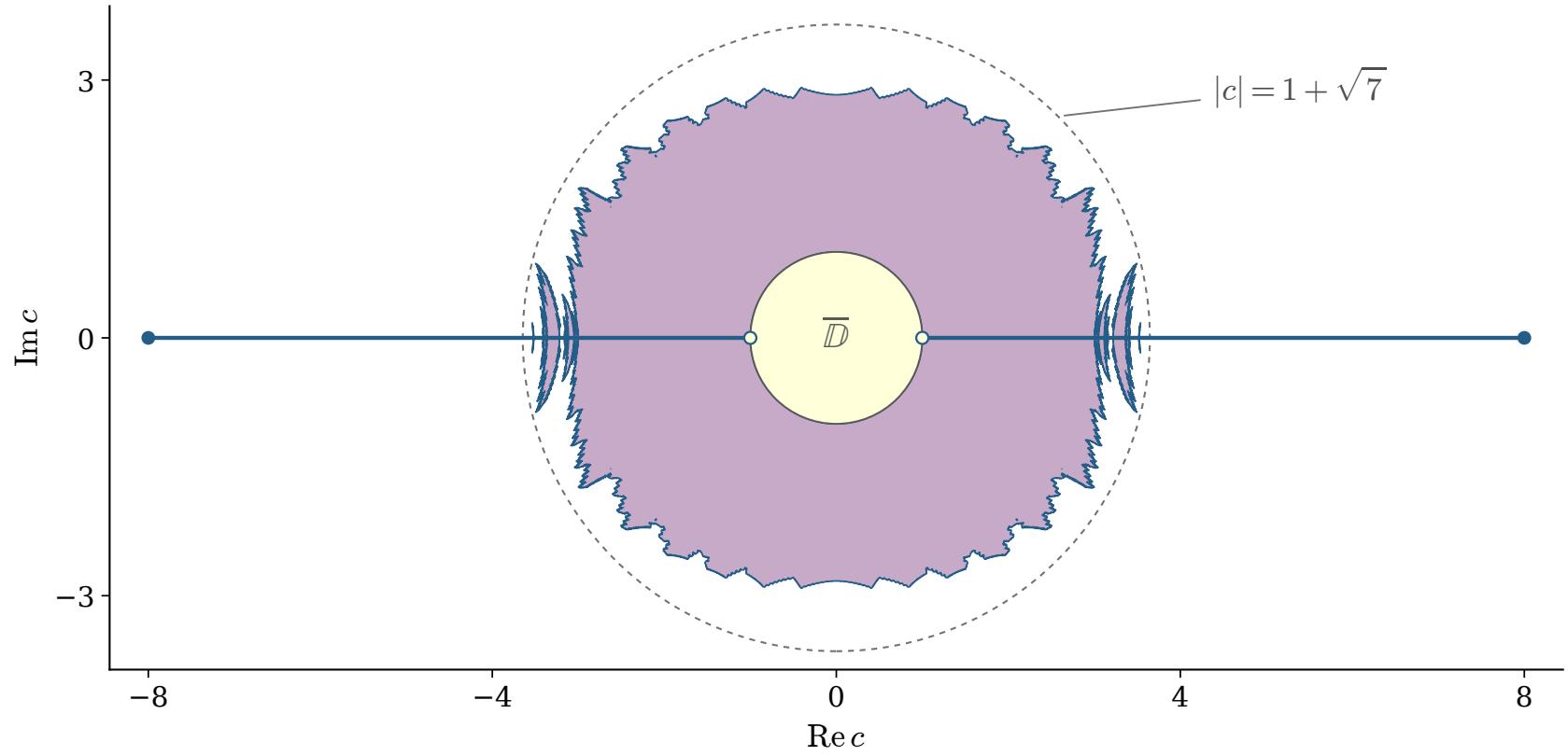
If  $L < |\lambda| < 1$  and  $c = 1/\lambda$ , then  $1 < |c| < \sqrt{n}$ . By the inner-annulus part of Proposition 3.11, this implies  $c \in \mathcal{M}_n$ . Using again the reciprocal-series characterization, we obtain  $\lambda \in Z_n$ . Thus

$$\left\{ \lambda \in \mathbb{C} : \frac{1}{\sqrt{n}} < |\lambda| < 1 \right\} \subset Z_n.$$

All hypotheses of Nakajima's Main Theorem B are satisfied with  $G = D_n$ , and hence  $Z_n = X^{D_n}$  is connected and locally connected. The inversion homeomorphism transfers both properties to  $\mathcal{M}_n$ .  $\square$

Figures 1.4 and 3.4 compare  $\mathcal{M}_7$  and  $\mathcal{M}_8$ . At the displayed depth and resolution,  $\mathcal{M}_7$  has a dominant shaded core, while the rendering of  $\mathcal{M}_8$  contains several apparently separated components near the real axis. The theorem establishes connectedness and local connectedness of both full loci. Approximation by interior parameters is a further property, addressed by the covering and finite-capture arguments below.

Four arithmetic subsequences are compared in Figures C.1 to C.4, beside the real-axis results and questions in Section C.4. Their nesting follows from Proposition 3.8; the finer near-real patterns are numerical observations.



**Figure 3.4.** The connectedness locus  $\mathcal{M}_8$  in the full parameter plane. The dashed circle is the bound  $|c| = 1 + \sqrt{7}$ . The horizontal blue segments show the exact real trace  $[-8, -1) \cup (1, 8]$ , including both antennae. The pale-yellow closed unit disk is excluded.

# From Covering Rectangles to Canonical Trap Geometry

We next ask whether connectedness parameters can be approximated by parameters for which connectedness persists under perturbation. For the non-real collinear locus, the interior-density statement corresponding to Bandt's binary conjecture is

$$\mathcal{M}_n \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

The proof combines two ingredients: restricted polynomial zeros are dense in the locus, and a self-covering rectangle makes every such non-real zero inside the parameter lens an interior point. This gives

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)} \quad (n \geq 2).$$

A comparison with the radial bound then gives

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n \quad (n \geq 21),$$

and hence regular-closedness away from the real axis for every  $n \geq 21$ ; see Theorem 4.5. The later finite-capture theory organizes this continuation mechanism around a canonical trap and enclosure. It also improves the lens-containment threshold to the sharp value  $n \geq 20$ ; see Chapter 6 and Theorem 6.44.

## 4.1. The Two-Disk Parameter Lens

By Definition 2.8, the two-disk parameter lens is

$$\mathcal{X}_n = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : |c - 1| < \sqrt{2n}, |c + 1| < \sqrt{2n}\}.$$

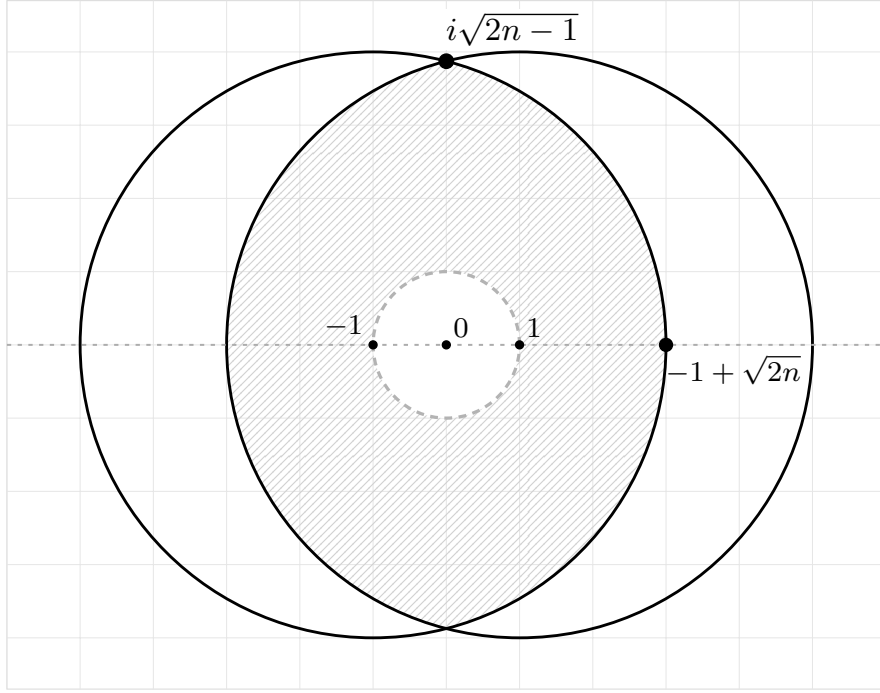
For  $c = x + iy \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , with  $\rho = |c|$  and  $N = 2n - 1$ , this is equivalently

$$c \in \mathcal{X}_n \iff \rho^2 + 2|x| < N.$$

This inequality will ensure overlap between horizontal sections of adjacent images of the covering rectangle. Figure 4.1 shows its two-disk geometry in the parameter plane.

## 4.2. Self-Covering and Finite Continuation

The self-covering method predates the rectangle construction developed here. Indlekofer, J  rai, and K  tai used a disk covered by its images under iterated maps of the difference system, together with finite entry into that disk, to certify connectedness in the binary family [IJK95, Lemma 7 and pp. 422–424]. Solomyak and Xu developed the rectangle method for  $\mathcal{M}_2$  [SX03]. In the present notation, they prove that every non-real connectedness parameter inside the binary lens is a limit of interior parameters. Related self-covering and zero-set



**Figure 4.1.** The two-disk parameter lens  $\mathcal{X}_n$  for  $n = 8$ : the intersection of the two disks centered at  $\pm 1$ , with the closed unit disk excluded.

techniques also appear in the study of restricted power series and polygonal fractals [SS06; Hun07].

The construction below gives a uniform family of rectangles  $R(c, N)$  for every collinearity, with  $N = 2n - 1$ . The vertices are given in closed form in  $c$  and  $N$ , and the covering proof uses all first-level images of the  $N$ -ary difference system. For every  $n \geq 2$ , the lens condition guarantees this covering throughout  $\mathcal{X}_n$ .

Self-covering supplies the infinite continuation after a finite address prefix. Continuous dependence of the rectangle on the parameter then turns strict entry into an interior certificate. Together with polynomial approximation, this proves

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}$$

for every  $n \geq 2$ . For  $n \geq 21$ , the radial bound from Proposition 3.9 places the entire non-real locus inside  $\mathcal{X}_n$ , yielding

$$\mathcal{M}_n \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

### 4.3. The Rectangle Mechanism

We first construct the rectangle for an alphabet of size  $m$ , then use  $m = N = 2n - 1$  for the difference system. For an integer  $m \geq 2$ , set

$$A_m = \{-m + 1, -m + 3, \dots, m - 1\}.$$

For  $c = x + iy \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , write  $\rho = |c|$  and define  $R(c, m)$  as the rectangle centered at the origin with vertex set

$$\{\psi(c, m), \overline{\psi(c, m)}, -\psi(c, m), -\overline{\psi(c, m)}\},$$

where

$$\psi(c, m) = \begin{cases} \frac{(m+1)c}{1+c}, & \operatorname{Re}(c) \geq 0, \\ \frac{(m+1)c}{1-c}, & \operatorname{Re}(c) < 0. \end{cases}$$

The half-widths of this rectangle are

$$|\operatorname{Re} \psi(c, m)|, \quad |\operatorname{Im} \psi(c, m)|.$$

For  $\operatorname{Re}(c) < 0$ , the representative  $\psi(c, m)$  has negative real part. The symmetric vertex set gives the same rectangle after replacing  $\psi$  by  $-\psi$ .

Equivalently, if  $\sigma = |x|$ , then

$$R(c, m) = \{u + iv : |u| \leq \alpha_R(c, m), |v| \leq \beta_R(c, m)\},$$

where

$$\alpha_R(c, m) = \frac{(m+1)(\rho^2 + \sigma)}{\rho^2 + 2\sigma + 1}, \quad \beta_R(c, m) = \frac{(m+1)|y|}{\rho^2 + 2\sigma + 1}.$$

These horizontal and vertical half-widths follow by taking real and imaginary parts of  $\psi(c, m)$ . They make the rectangle's continuous dependence on  $c$ , including across the imaginary axis, explicit.

The inequality

$$\rho^2 + 2|x| < m$$

will guarantee self-covering. For the  $n$ -ary connectedness problem we set

$$m = N = 2n - 1,$$

and this condition is precisely the two-disk lens condition  $c \in \mathcal{X}_n$ . Since  $m+1 = 2n$  in this application, we have

$$\psi(c, N) = \begin{cases} \frac{2nc}{1+c}, & \operatorname{Re}(c) \geq 0, \\ \frac{2nc}{1-c}, & \operatorname{Re}(c) < 0, \end{cases}$$

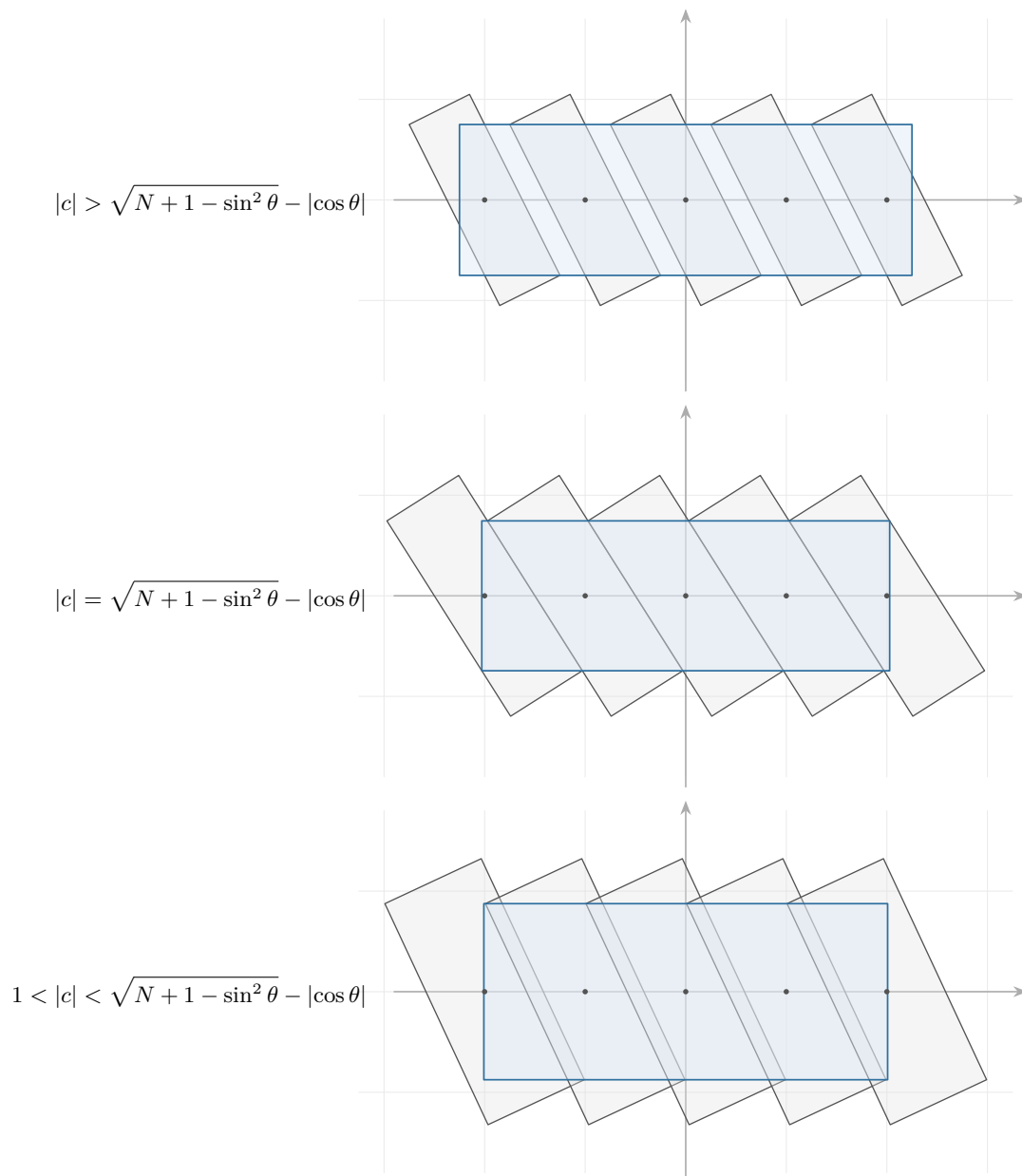
and

$$\alpha_R(c, N) = \frac{2n(\rho^2 + |x|)}{\rho^2 + 2|x| + 1}, \quad \beta_R(c, N) = \frac{2n|y|}{\rho^2 + 2|x| + 1}.$$

We will prove that the resulting rectangle is covered by its first-level images whenever  $c \in \mathcal{X}_n$ :

$$R(c, N) \subset \bigcup_{t \in A_N} \left( t + \frac{R(c, N)}{c} \right).$$

Figure 4.2 compares parameters outside, on the boundary of, and inside the lens. The critical covering configuration occurs when the adjacent side intervals meet at their endpoints.



**Figure 4.2.** Rectangle covering for  $N = 2n - 1 = 5$ , with  $c = |c|e^{i\theta}$ . From top to bottom: a non-covering configuration outside the lens, critical covering on its two-disk boundary, and covering inside the lens. Blue denotes  $R(c, N)$ ; gray denotes its images  $t + R(c, N)/c$ .

Before checking this geometry, we record its two consequences. A compact self-covering set lies in the attractor, and finite entry into such a set provides the tail of an overlap address. The canonical trap in Corollary 6.11 will use the same continuation principle.

**Lemma 4.1** (Compact self-covering sets lie in the attractor). *Let  $m \geq 2$ , let  $U \subset \mathbb{C}$  be compact, and let  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$ . If*

$$U \subset \bigcup_{t \in A_m} \left( t + \frac{U}{c} \right),$$

*then  $U \subset E(c, m)$ .*

*Proof.* The empty case is immediate. For nonempty  $U$ , the inclusion is exactly the hypothesis of the self-covering principle Proposition 2.9.  $\square$

**Lemma 4.2.** *If  $U \subset E(c, N)$  with  $N = 2n - 1$  and there exist  $d_1, \dots, d_q \in D_n$  such that*

$$1 + \sum_{k=1}^q d_k c^{-k} \in \frac{1}{2c^{q+1}} U,$$

*then  $c \in \mathcal{M}_n$ .*

*Proof.* The residual of the finite reciprocal sum has an address in the scaled difference attractor. Since  $U \subset E(c, N)$  and  $A_N = 2D_n$ , there are digits  $b_j \in D_n$  such that

$$1 + \sum_{k=1}^q d_k c^{-k} = \sum_{j=0}^{\infty} b_j c^{-(q+1+j)}.$$

Define a new digit sequence  $(e_k)_{k \geq 1} \in D_n^{\mathbb{N}}$  by

$$e_k = d_k \quad (1 \leq k \leq q), \quad e_{q+1+j} = -b_j \quad (j \geq 0).$$

This is allowed because  $D_n = -D_n$ . Then

$$1 + \sum_{k=1}^{\infty} e_k c^{-k} = 0.$$

By Proposition 3.4, this gives  $c \in \mathcal{M}_n$ .  $\square$

**Lemma 4.3** (Rectangle covering). *For every  $c \in \mathcal{X}_n$ , the rectangle  $R(c, N)$ , with  $N = 2n - 1$ , is covered by its first-level images:*

$$R(c, N) \subset \bigcup_{t \in A_N} \left( t + \frac{R(c, N)}{c} \right).$$

*Proof.* We prove covering for  $A_m$  under the slightly weaker condition  $\rho^2 + 2|x| \leq m$ , then take  $m = N = 2n - 1$ . Recall that

$$A_m = \{-m + 1, -m + 3, \dots, m - 1\}.$$

Complex conjugation and the symmetry  $c \mapsto -c$  reduce the non-real case to

$$c = x + iy, \quad x \geq 0, \quad y > 0.$$

Put

$$\rho = |c|, \quad \Delta_+ = \rho^2 + 2x + 1 = |1 + c|^2.$$

For  $x \geq 0$ , the defining vertex is

$$\psi(c, m) = \frac{c(m+1)}{1+c}.$$

Put

$$\alpha = \alpha_R(c, m), \quad \beta = \beta_R(c, m).$$

Since  $x \geq 0$  and  $y > 0$ , these half-widths are

$$\alpha = \frac{(m+1)(\rho^2 + x)}{\Delta_+}, \quad \beta = \frac{(m+1)y}{\Delta_+}.$$

Thus

$$R(c, m) = [-\alpha, \alpha] \times [-\beta, \beta].$$

Covering will follow by examining horizontal sections of the images

$$t + \frac{R(c, m)}{c}, \quad t \in A_m.$$

At the height  $v = \beta$  of the upper side of  $R(c, m)$ , substitution in  $z \mapsto t + z/c$  shows that the section of  $t + R(c, m)/c$  contains the interval

$$I_t^+ = \left[ t - \frac{(m+1)(1+x)}{\Delta_+}, t + \frac{(m+1)(1-x)}{\Delta_+} \right] \times \{\beta\}.$$

At the height  $v = -\beta$  of the lower side, it contains the reflected interval

$$I_t^- = \left[ t - \frac{(m+1)(1-x)}{\Delta_+}, t + \frac{(m+1)(1+x)}{\Delta_+} \right] \times \{-\beta\}.$$

Adjacent upper-side intervals  $I_t^+$  and  $I_{t+2}^+$  overlap whenever

$$t + \frac{(m+1)(1-x)}{\Delta_+} \geq t + 2 - \frac{(m+1)(1+x)}{\Delta_+},$$

which is equivalent to

$$\Delta_+ \leq m+1, \quad \text{that is,} \quad \rho^2 + 2x \leq m.$$

The same overlap criterion is obtained for the lower-side intervals  $I_t^-$ .

We must also cover the ends of each side. Under  $\rho^2 + 2x \leq m$ , the leftmost and rightmost intervals reach the corresponding endpoints  $-\alpha$  and  $\alpha$ , because the required inequalities reduce to

$$2(m - \rho^2 + x(m-1)) \geq 0 \quad \text{and} \quad 2(m - (\rho^2 + 2x)) \geq 0,$$

both of which hold under  $\rho^2 + 2x \leq m$ .

The upper and lower sides are therefore covered. To fill the rectangle between them, use the convexity of each image  $t + c^{-1}R(c, m)$ : at every intermediate height it contains the interval obtained by linearly interpolating the endpoints of  $I_t^-$  and  $I_t^+$ . Both the signed overlap of

adjacent interpolated intervals and their overhang at the ends of  $[-\alpha, \alpha]$  depend affinely on height. They are nonnegative at  $v = \pm\beta$ , hence throughout  $[-\beta, \beta]$ . Consequently,

$$R(c, m) \subset \bigcup_{t \in A_m} \left( t + \frac{R(c, m)}{c} \right)$$

whenever

$$\rho^2 + 2x \leq m.$$

For real parameters the rectangle degenerates to an interval. If  $c = x > 1$  and  $x^2 + 2x \leq m$ , then  $R(c, m)$  is the interval

$$\left[ -\frac{(m+1)x}{x+1}, \frac{(m+1)x}{x+1} \right].$$

Its first-level images are the intervals

$$\left[ t - \frac{m+1}{x+1}, t + \frac{m+1}{x+1} \right], \quad t \in A_m.$$

Since  $x^2 + 2x \leq m$  implies  $x < m$ , adjacent intervals overlap and the leftmost and rightmost intervals cover the two endpoints of  $R(c, m)$ . Hence the same covering inclusion holds for real  $c > 1$ . The case  $c < -1$  follows by the symmetry  $c \mapsto -c$ .

Now take  $m = N = 2n - 1$ . For  $x \geq 0$ , the parameter-lens condition is

$$\rho^2 + 2x < N.$$

Thus the covering condition above is satisfied, with strict inequality, for every  $c \in \mathcal{X}_n$  in the right half-plane. The cases  $y < 0$  and  $x < 0$  follow from the symmetries described above, replacing  $x$  by  $|x|$ . Therefore, for every  $c \in \mathcal{X}_n$ ,

$$R(c, N) \subset \bigcup_{t \in A_N} \left( t + \frac{R(c, N)}{c} \right).$$

□

## 4.4. The Rectangle Threshold

Inside the lens, a finite reciprocal sum gives an interior certificate when its residual lies strictly inside the corresponding scaled rectangle. We first show that the parameters where such a residual vanishes are dense in the locus. Let

$$\widehat{\mathcal{M}}_n = \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : \exists m \geq 1 \exists d_1, \dots, d_m \in D_n \text{ such that } 1 + \sum_{k=1}^m d_k c^{-k} = 0 \right\}.$$

The binary truncation argument appears in [Bou88, Sections 2–3]; the reciprocal root-closure identity is also obtained by Rouché approximation in [IJK95, Section 7]. The proof below applies this analytic mechanism to the full coefficient alphabet  $D_n$ .

**Lemma 4.4** (Rouché approximation by restricted polynomial zeros). *One has*

$$\widehat{\mathcal{M}}_n \subset \mathcal{M}_n \quad \text{and} \quad \mathcal{M}_n \subset \overline{\widehat{\mathcal{M}}_n}^{\mathbb{C} \setminus \overline{\mathbb{D}}}.$$

*Thus every parameter in  $\mathcal{M}_n$  is a limit, in the exterior parameter space, of parameters satisfying finite reciprocal identities.*

*Proof.* If  $c \in \widehat{\mathcal{M}}_n$ , then

$$1 + \sum_{k=1}^m d_k c^{-k} = 0$$

for some  $d_k \in D_n$ . Appending zero digits gives an infinite reciprocal identity, so Proposition 3.4 yields  $c \in \mathcal{M}_n$ .

Conversely, let  $c \in \mathcal{M}_n$ , put  $\lambda = 1/c$ , and choose  $(d_k)_{k \geq 1} \in D_n^{\mathbb{N}}$  such that

$$F(\lambda) = 1 + \sum_{k=1}^{\infty} d_k \lambda^k = 0.$$

Let

$$F_q(z) = 1 + \sum_{k=1}^q d_k z^k.$$

Choose radii  $r_j \downarrow 0$  such that  $\overline{B(\lambda, r_j)} \subset \mathbb{D} \setminus \{0\}$  and  $F$  has no zero on  $\partial B(\lambda, r_j)$ . Since  $F_q \rightarrow F$  uniformly on each such disk, Rouché's theorem gives, for all sufficiently large  $q$ , a zero of  $F_q$  in  $B(\lambda, r_j)$ . Choose one such zero  $\lambda_j$ , with truncation index  $q_j$ . Then

$$\lambda_j \rightarrow \lambda.$$

Setting  $c_j = 1/\lambda_j$ , we have  $c_j \rightarrow c$  and

$$1 + \sum_{k=1}^{q_j} d_k c_j^{-k} = 0.$$

Hence  $c_j \in \widehat{\mathcal{M}}_n$ , proving

$$\mathcal{M}_n \subset \overline{\widehat{\mathcal{M}}_n}^{\mathbb{C} \setminus \overline{\mathbb{D}}}.$$

□

At a non-real parameter inside the lens, the rectangle has positive width in both directions. A vanishing residual lies strictly inside it, and continuity preserves that inclusion for nearby parameters. Thus the finite identities in the preceding lemma yield interior neighborhoods.

**Theorem 4.5** (Approximation by interior parameters inside the lens). *For every  $n \geq 2$ ,*

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

Moreover, if  $n \geq 21$ , then

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n.$$

Consequently, for  $n \geq 21$ ,

$$\mathcal{M}_n \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

*Proof.* First consider the inclusion inside the lens:

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)}.$$

By Lemma 4.4, it suffices to show that every point of

$$\widehat{\mathcal{M}}_n \cap (\mathcal{X}_n \setminus \mathbb{R})$$

is interior to  $\mathcal{M}_n$ . Indeed, for  $c \in (\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R}$ , the lemma supplies  $c_j \in \widehat{\mathcal{M}}_n$  with  $c_j \rightarrow c$ . Since  $\mathcal{X}_n \setminus \mathbb{R}$  is open in the exterior parameter plane, discarding finitely many terms gives

$$c_j \in \widehat{\mathcal{M}}_n \cap (\mathcal{X}_n \setminus \mathbb{R}).$$

Once these  $c_j$  are interior points, it follows that  $c \in \overline{\text{int}(\mathcal{M}_n)}$ .

Let

$$c_0 \in \widehat{\mathcal{M}}_n \cap (\mathcal{X}_n \setminus \mathbb{R}).$$

Then there are digits  $d_1, \dots, d_m \in D_n$  such that

$$p(c_0) = 0, \quad p(c) = 1 + \sum_{k=1}^m d_k c^{-k}.$$

Put

$$q(c) := 2c^{m+1}p(c).$$

For  $c = x + iy$ ,  $\rho = |c|$ , and  $\sigma = |x|$ , the coordinate form of the rectangle is

$$R(c, N) = \{u + iv : |u| \leq \alpha_R(c, N), |v| \leq \beta_R(c, N)\},$$

where

$$\alpha_R(c, N) = \frac{(N+1)(\rho^2 + \sigma)}{\rho^2 + 2\sigma + 1}, \quad \beta_R(c, N) = \frac{(N+1)|y|}{\rho^2 + 2\sigma + 1}.$$

Both half-widths depend continuously on  $c$ , including across  $\text{Re}(c) = 0$ .

Since  $c_0 \notin \mathbb{R}$ , we have

$$\alpha_R(c_0, N) > 0, \quad \beta_R(c_0, N) > 0.$$

Moreover  $q(c_0) = 0$ . Hence

$$|\text{Re}(q(c_0))| < \alpha_R(c_0, N), \quad |\text{Im}(q(c_0))| < \beta_R(c_0, N).$$

By continuity of  $q$ ,  $\alpha_R$ , and  $\beta_R$ , and by openness of  $\mathcal{X}_n \setminus \mathbb{R}$ , there exists an open neighborhood  $U$  of  $c_0$  in the exterior parameter plane such that

$$U \subset \mathcal{X}_n \setminus \mathbb{R}$$

and

$$|\text{Re}(q(c))| < \alpha_R(c, N), \quad |\text{Im}(q(c))| < \beta_R(c, N) \quad (c \in U).$$

Equivalently,

$$q(c) \in \text{int } R(c, N) \quad (c \in U),$$

and therefore

$$p(c) \in \frac{1}{2c^{m+1}} \text{int } R(c, N) \subset \frac{1}{2c^{m+1}} R(c, N) \quad (c \in U).$$

By Lemma 4.3,

$$R(c, N) \subset \bigcup_{t \in A_N} \left( t + \frac{R(c, N)}{c} \right) \quad (c \in U).$$

Applying Lemma 4.1 with the difference alphabet  $A_N$ , we obtain

$$R(c, N) \subset E(c, N) \quad (c \in U).$$

Therefore the hypotheses of Lemma 4.2 hold for every  $c \in U$ , and hence

$$U \subset \mathcal{M}_n.$$

Thus  $c_0 \in \text{int}(\mathcal{M}_n)$ , and the density argument proves the inclusion inside the lens.

To extend the result to the full non-real locus for  $n \geq 21$ , compare the lens with the radial bound from Proposition 3.9:

$$\mathcal{M}_n \setminus \mathbb{R} \subset B(0, 1 + \sqrt{n-1}).$$

For  $n \geq 21$ , we have

$$1 + \sqrt{n-1} < \sqrt{2n} - 1.$$

Indeed, this is equivalent to

$$2 + \sqrt{n-1} < \sqrt{2n},$$

and, after squaring twice, to

$$n^2 - 22n + 25 > 0,$$

which holds for every  $n \geq 21$ . Therefore, if  $c \in \mathcal{M}_n \setminus \mathbb{R}$ , then

$$|c \pm 1| \leq |c| + 1 < 2 + \sqrt{n-1} < \sqrt{2n}.$$

Thus  $c \in \mathcal{X}_n$ , and

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n.$$

Combining this with the inclusion inside the parameter lens gives

$$\mathcal{M}_n \setminus \mathbb{R} \subseteq \overline{\text{int}(\mathcal{M}_n)} \quad (n \geq 21).$$

□

**Remark 4.6** (Rectangle threshold versus finite-capture threshold). The threshold  $n \geq 21$  in Theorem 4.5 comes from placing the disk bound inside  $\mathcal{X}_n$ :

$$\mathcal{M}_n \setminus \mathbb{R} \subset B(0, 1 + \sqrt{n-1}).$$

The canonical enclosure in Chapter 6 gives the finer estimate needed at  $n = 20$ . The certified off-lens roots in Proposition A.2 show that containment of the full non-real locus in  $\mathcal{X}_n$  fails for every  $2 \leq n \leq 19$ . These results make  $n = 20$  the sharp threshold for lens containment.

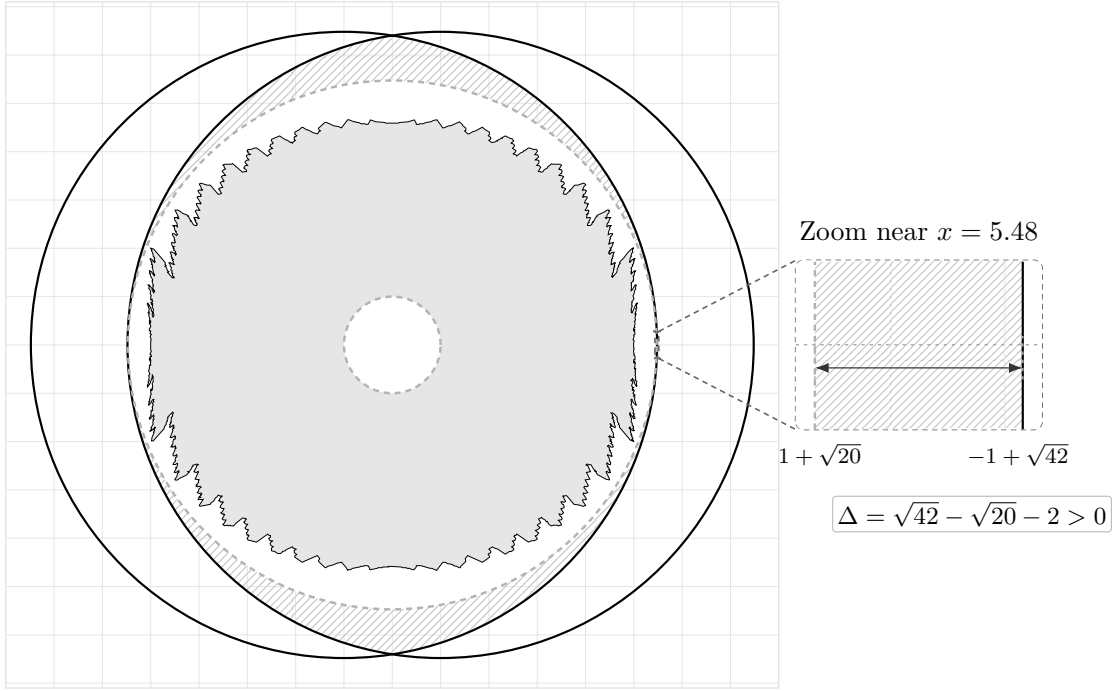
Figure 4.3 shows the positive gap between the bounding circle and the lens at  $n = 21$ .

## 4.5. From the Covering Rectangle to the Canonical Trap and Enclosure

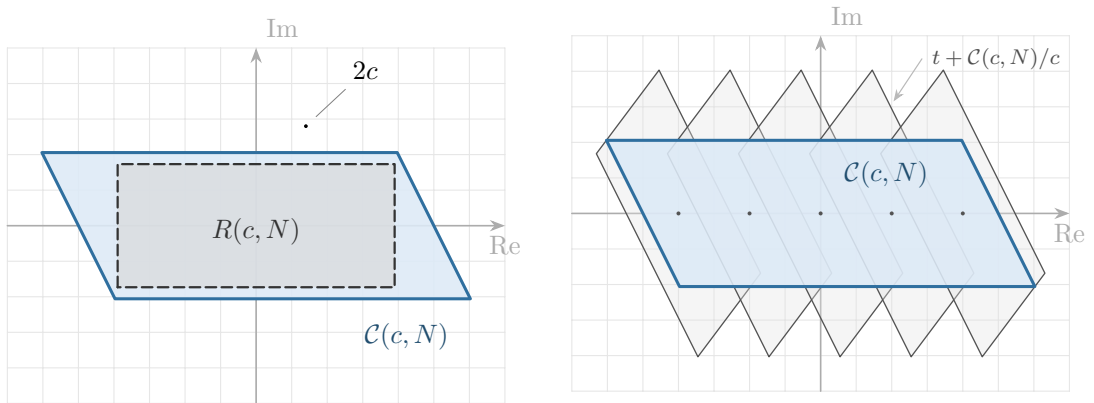
The rectangle proof converts finite reciprocal identities into open neighborhoods of connectedness parameters. Finite capture develops this mechanism into a filtration: a finite inverse address need only enter an open self-covering trap. A complementary enclosure controls the inverse addresses that have not yet entered the trap. For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , the canonical pair satisfies

$$\mathcal{C}(c, N) \subset \text{int } E(c, N) \subset E(c, N) \subseteq \mathcal{E}(c, N),$$

where  $N = 2n - 1$ . Entry into the trap certifies an interior parameter; extinction of all inverse branches in the enclosure certifies nonmembership in the connectedness locus. As the allowed depth increases, the captured region unfolds layer by layer. The two-step closure inclusion in Theorem 6.35 controls how these layers fit together. The enclosure estimates also settle lens containment at  $n = 20$  in Proposition 6.43. Figure 4.4 compares the covering rectangle and canonical trap at a representative parameter.



**Figure 4.3.** The disk bound for the non-real locus at  $n = 21$ . Every exterior parameter in the disk of radius  $1 + \sqrt{n-1}$  lies in  $\mathcal{X}_n$  when  $n \geq 21$ . The magnification shows the positive gap  $\Delta$  between the bounding circle and the two-disk boundary of  $\mathcal{X}_{21}$ .



**Figure 4.4.** The covering rectangle and canonical trap for  $n = 3$ ,  $N = 5$ , and  $c = 0.7 + 1.4i$ . Left: the rectangle lies inside the trap. Right: the trap is covered by its five first-level images. Dots mark the digits  $A_5 = \{-4, -2, 0, 2, 4\}$ .



PART III.

# Restricted Roots and Finite Capture

Chapters 5 and 6

*The mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite.*

— Hermann Weyl



# Restricted Polynomial Roots and Reciprocal Power Series

This chapter connects the finite reciprocal identities of Chapter 4 to polynomial roots and inverse dynamics. Rouché approximation and relative closedness identify  $\mathcal{M}_n$  with the exterior closure of  $\mathcal{R}_n$ . Reading the digits one at a time then turns polynomial evaluation into an inverse orbit: an exterior root is a parameter at which the orbit lands exactly at zero. This correspondence leads to the stable geometric tests of the next chapter.

## 5.1. Restricted Roots and the Marked Point

We retain  $N = 2n - 1$ ,  $D_n = \{-n + 1, \dots, n - 1\}$ , and the restricted-root set  $\mathcal{R}_n$  of Definition 2.7. Thus the difference alphabet is  $A_N = 2D_n$ .

**Remark 5.1** (Terminology for bounded coefficients). For an integer polynomial  $P(z) = \sum_{j=0}^m a_j z^j$ , its (naive) height is

$$H(P) = \max_{0 \leq j \leq m} |a_j|.$$

Hence the monic polynomials defining  $\mathcal{R}_n$  have integer coefficients and height at most  $n - 1$ . This is commonly described as a bounded-integer-coefficient or bounded-height condition; see [Fen16]. Here *restricted* specifies both the monic normalization and the finite coefficient alphabet  $D_n$ .

**Proposition 5.2** (Marked-point characterization). *For  $n \geq 2$ ,*

$$\mathcal{M}_n = \{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : 2c \in E(c, N)\}.$$

*Proof.* By Proposition 3.3, connectedness is equivalent to the existence of  $p, q \in E(c, n)$  with  $p/c = 2 + q/c$ , or  $p - q = 2c$ . The difference identity  $E(c, n) - E(c, n) = E(c, N)$  gives the claim.  $\square$

**Remark 5.3** (Disk and exterior coordinates). To compare with disk-coordinate formulations, write

$$E_\lambda(m) = \left\{ \sum_{k=0}^{\infty} a_k \lambda^k : a_k \in A_m \right\}, \quad 0 < |\lambda| < 1.$$

Then  $E(c, m) = E_{1/c}(m)$  for  $|c| > 1$ . The folded coordinate

$$\lambda_c = \begin{cases} c, & |c| < 1, \\ 1/\bar{c}, & |c| > 1, \end{cases} \quad c \in \mathbb{C}^* \setminus \partial\mathbb{D},$$

satisfies  $\lambda_{1/\bar{c}} = \lambda_c$ . In the disk chart, neighboring overlap and the difference identity give

$$E_\lambda(n) \text{ is connected} \iff 2 \in \lambda E_\lambda(N) \iff \frac{2}{\lambda} \in E_\lambda(N).$$

For an exterior parameter,  $\lambda_c = 1/\bar{c}$ . Since the digits are real, conjugation converts this to the marked-point condition:

$$\frac{2}{\lambda_c} \in E_{\lambda_c}(N) \iff 2\bar{c} \in E_{1/\bar{c}}(N) \iff 2c \in E(c, N).$$

The exterior chart gives the reciprocal-root equation and the inverse branches  $g_t(z) = c(z - t)$  their simplest form.

The real, symmetric coefficients also account for the symmetries used throughout the thesis. Conjugating a reciprocal-series identity proves invariance of  $\mathcal{M}_n$  under  $c \mapsto \bar{c}$ ; replacing its coefficients  $d_k$  by  $(-1)^k d_k \in D_n$  proves invariance under  $c \mapsto -c$ .

## 5.2. Exterior Closure of Restricted Roots

**Lemma 5.4** (Relative closedness of the marked-point locus). *The locus*

$$\{c \in \mathbb{C} \setminus \overline{\mathbb{D}} : 2c \in E(c, N)\}$$

*is closed relative to the exterior domain  $\mathbb{C} \setminus \overline{\mathbb{D}}$ .*

*Proof.* On each compact subset of  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , the maps  $z \mapsto t + c^{-1}z$ ,  $t \in A_N$ , have contraction ratios bounded above by a constant smaller than 1. Continuity of contractive IFS attractors therefore makes  $c \mapsto E(c, N)$  Hausdorff-continuous. The marked-point locus is the zero set of the continuous function

$$c \mapsto \text{dist}(2c, E(c, N)),$$

and is consequently relatively closed. □

Bousch's neighboring-overlap criterion (Proposition 3.3) provides the geometric antecedent of the root-closure identity. In the binary case, Indlekofer, J  rai, and K  tai explicitly establish the reciprocal identity by Rouch   approximation [IJK95, Section 7]. The uniform bounded-height formulation below is developed in [EJ26, Proposition 2.5]. Its proof combines approximation by finite reciprocal identities (Lemma 4.4) with relative closedness of the marked-point locus (Lemma 5.4).

**Theorem 5.5** (Exterior root closure as a connectedness locus). *For every  $n \geq 2$ ,*

$$\mathcal{M}_n = \overline{\mathcal{R}_n} \setminus \overline{\mathbb{D}}.$$

*Proof.* Multiplying a finite reciprocal identity by the appropriate power of  $c$  identifies the truncation-zero set from Lemma 4.4 as

$$\widehat{\mathcal{M}}_n = \mathcal{R}_n \cap (\mathbb{C} \setminus \overline{\mathbb{D}}).$$

That lemma gives  $\widehat{\mathcal{M}}_n \subset \mathcal{M}_n$  and density of  $\widehat{\mathcal{M}}_n$  in  $\mathcal{M}_n$ . The marked-point characterization and Lemma 5.4 show that  $\mathcal{M}_n$  is relatively closed. Hence

$$\mathcal{M}_n = \overline{\widehat{\mathcal{M}}_n}^{\mathbb{C} \setminus \overline{\mathbb{D}}} = \overline{\mathcal{R}_n}^{\mathbb{C}} \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) = \overline{\mathcal{R}_n} \setminus \overline{\mathbb{D}},$$

where the second equality uses that the exterior domain is open. □

### 5.3. Backward Dynamics

An inverse branch removes the first digit and rescales the remaining tail. With the composition convention of Definition 2.5, a word  $u = t_0 \cdots t_{k-1} \in A_N^k$  therefore reads the digits in their written order. The resulting recurrence is Horner's rule for a restricted polynomial.

**Proposition 5.6** (Backward iterates and polynomial evaluation). *Let  $u = t_0 \cdots t_{k-1} \in A_N^k$ , and write  $t_{j-1} = 2d_j$  with  $d_j \in D_n$  for  $1 \leq j \leq k$ . Then*

$$g_u(2c) = 2c P_u(c),$$

where

$$P_u(z) = z^k - d_1 z^{k-1} - d_2 z^{k-2} - \cdots - d_k$$

is a monic polynomial whose non-leading coefficients lie in  $D_n$ . Consequently,

$$g_u(2c) = 0 \iff P_u(c) = 0.$$

Conversely, every monic polynomial of degree  $k$  whose non-leading coefficients lie in  $D_n$  is  $P_u$  for a unique word  $u \in A_N^k$ .

*Proof.* Write  $u_j = t_0 \cdots t_{j-1}$ , with  $u_0 = \varepsilon$ , and start with  $P_{u_0}(z) = 1$ . If  $g_{u_{j-1}}(2c) = 2c P_{u_{j-1}}(c)$ , then

$$g_{u_j}(2c) = c(2c P_{u_{j-1}}(c) - 2d_j) = 2c(c P_{u_{j-1}}(c) - d_j).$$

Induction therefore gives  $g_{u_j}(2c) = 2c P_{u_j}(c)$ , with  $P_{u_j}(z) = z P_{u_{j-1}}(z) - d_j$ . This is Horner's recursion for the displayed polynomial. The equivalence follows because  $c \neq 0$ . Conversely, a coefficient  $a_j \in D_n$  determines the unique digit  $t_{j-1} = -2a_j \in A_N$ , giving the inverse correspondence.  $\square$

**Remark 5.7** (Expansion residual and exact landing). Let  $u = t_0 \cdots t_{m-1} \in A_N^m$ , put  $\varepsilon_k = t_{k-1}/2 \in D_n$ , and write  $z_m = g_u(2c)$ . Since  $f_u(z_m) = 2c$ ,

$$1 = \sum_{k=1}^m \varepsilon_k c^{-k} + \frac{z_m}{2c^{m+1}}.$$

We say that  $u$  gives an *exact landing at zero* when  $g_u(2c) = z_m = 0$ . Thus exact landing means that the displayed prefix is a terminating expansion of 1. More generally, if  $z_m \in E(c, N)$ , any address of  $z_m$  supplies a continuation of this prefix. Since  $0 \in E(c, N)$  by the constant zero address, exact landing gives  $2c = f_u(0) \in E(c, N)$ , and therefore  $c \in \mathcal{M}_n$ .

For example, if  $n = 3$ ,  $c = 1 + i$ , and  $u = (4, -4)$ , then  $g_4(2c) = -4$  and  $g_{-4}(-4) = 0$ . The associated polynomial  $P_u(z) = z^2 - 2z + 2$  therefore vanishes at  $c$ , and the corresponding terminating expansion is

$$1 = 2c^{-1} - 2c^{-2}.$$

### 5.4. Terminating Expansions and Finite Capture

For a fixed word, exact landing imposes the polynomial equation  $P_u(c) = 0$ . To obtain a neighborhood of parameters, the next chapter replaces this equality by strict entry into a continuously varying bounded open self-covering region. Entry supplies a continuation of the finite prefix and persists under small parameter perturbations. An outer enclosure controls the branches that have not yet entered, leading to a finite search with both interior and exterior certificates.



# Finite Capture: Inner and Outer Tests

The finite words of the preceding chapter now become geometric tests of the marked point  $2c$  (Definition 2.6) in  $E(c, N)$ , where  $N = 2n - 1$ . We construct two canonical parallelograms: an open inner target and a closed outer bound. A single trap hit certifies  $c \in \text{int}(\mathcal{M}_n)$ ; extinction of every branch retained by the outer bound certifies  $c \notin \mathcal{M}_n$ . The trap works on  $\mathcal{X}_n \setminus \mathbb{R}$ ; the enclosure is available for every non-real exterior parameter.

After constructing the pair in coordinates adapted to inverse iteration, we organize the tests by depth and turn them into a finite search (Sections 6.4 and 6.5). The main geometric question is how the captured regions fit together as depth increases. Within  $\mathcal{X}_n \setminus \mathbb{R}$ , every limit of depth- $k$  capture is captured by depth  $k+2$  (Section 6.6). The final section establishes the sharp range  $n \geq 20$  in which the lens contains the whole non-real connectedness locus (Section 6.7).

## 6.1. Coordinates for the Inverse Dynamics

Let  $c = x + iy$ ,  $\rho = |c|$ , and  $N = 2n - 1$ . For  $c \notin \mathbb{R}$ , pair the vertical coordinate  $v$  with a slanted coordinate  $s$  so that the new vertical coordinate under  $g_t$  is  $\rho s - yt$ . Choosing a nearby digit then controls the vertical coordinate; the relation between the two coordinates controls the remaining direction.

**Definition 6.1** (Canonical coordinates). Let

$$c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}), \quad \rho = |c|.$$

For  $z = u + iv \in \mathbb{C}$ , define

$$s = \ell_s(z; c) := \frac{\text{Im}(cz)}{\rho} = \frac{yu + xv}{\rho}, \quad v = \ell_v(z) := \text{Im } z.$$

We call

$$(s, v) = (\ell_s(z; c), \ell_v(z))$$

the *canonical coordinates of  $z$  relative to  $c$* . Since  $y \neq 0$ , this is a real-linear coordinate system, with inverse

$$u = \frac{\rho s - xv}{y}, \quad z = \frac{\rho s - xv}{y} + iv.$$

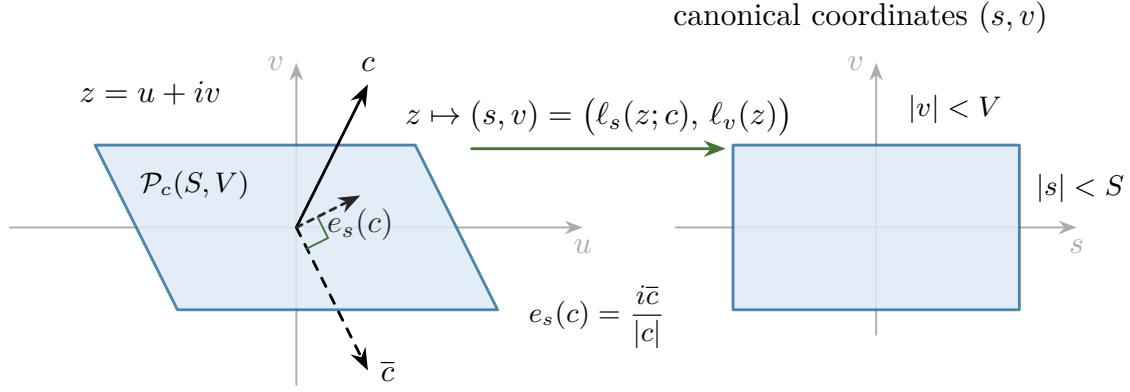
For  $S, V > 0$ , set

$$\mathcal{P}_c(S, V) = \{z \in \mathbb{C} : |\ell_s(z; c)| < S, |\ell_v(z)| < V\}.$$

Thus  $\mathcal{P}_c(S, V)$  is the rectangle  $(-S, S) \times (-V, V)$  in canonical coordinates and a parallelogram in the original complex plane.

Figure 6.1 illustrates this straightening. The canonical trap will be constructed on  $\mathcal{X}_n \setminus \mathbb{R}$ , where the lens condition from Definition 2.8 takes the form

$$\rho^2 + 2|x| < N.$$



**Figure 6.1.** Canonical coordinates straighten  $\mathcal{P}_c(S, V)$  into a rectangle.

**Lemma 6.2** (Inverse dynamics in canonical coordinates). *Let  $z = u + iv$ ,  $s = \ell_s(z; c)$ , and  $v = \ell_v(z)$ . If  $z' = g_t(z) = c(z - t)$ , then*

$$\ell_v(z') = \rho s - yt, \quad \ell_s(z'; c) = \frac{2x}{\rho} \ell_v(z') - \rho v.$$

*Proof.* Both coordinate functionals are real-linear. A direct calculation gives  $\ell_v(cz) = \text{Im}(cz) = \rho s$  and  $\ell_v(ct) = yt$ . This gives the first formula. Similarly,  $\ell_s(cz; c) = 2xs - \rho v$  and  $\ell_s(ct; c) = 2xyt/\rho$ . Substituting  $\ell_v(z') = \rho s - yt$  gives the second formula.  $\square$

## 6.2. The Inner Parallelogram

The self-covering principle of Section 2.7 already supplies the inclusion in the attractor. Here we need a bounded open set, so that entry can persist under parameter perturbation. This is the open version of the compact covering argument used in Lemma 4.1; Falconer's sub-self-similar terminology applies to the compact condition [Fal95].

**Remark 6.3** (Self-covering in inverse coordinates). For  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$  and a nonempty set  $U \subset \mathbb{C}$ , the self-covering condition for the difference IFS is

$$U \subset \bigcup_{t \in A_N} f_t(U).$$

In inverse form, every  $z \in U$  admits a digit  $t \in A_N$  such that  $g_t(z) \in U$ .

**Lemma 6.4** (Open self-covering sets lie in the attractor interior). *Let  $c \in \mathbb{C} \setminus \overline{\mathbb{D}}$ . If  $U$  is nonempty, bounded, open, and self-covering, then*

$$\overline{U} \subset E(c, N) \quad \text{and} \quad U \subset \text{int } E(c, N).$$

*Proof.* Apply Proposition 2.9 to the difference IFS.  $\square$

It remains to choose widths for which inverse continuation stays inside the parallelogram. The digit controls the new vertical coordinate first; the second formula in Lemma 6.2 then bounds the slanted coordinate.

**Definition 6.5** (Nearest admissible digit rule). Let  $c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , and set  $\rho = |c|$ . For  $r \in \mathbb{R}$ , let  $t^*(r) \in A_N$  be an admissible digit minimizing  $|r - t|$ ; if two digits are equally

close, choose the one with smaller absolute value, that is, the one nearer 0. This convention only makes the choice single-valued. For a point  $z$ , set

$$\alpha = \frac{\rho \ell_s(z; c)}{y},$$

and choose  $t = t^*(\alpha)$ .

**Lemma 6.6** (Nearest admissible digit). *If  $r \in [-N, N]$ , then there is  $t \in A_N$  with  $|r - t| \leq 1$ .*

*Proof.* The alphabet  $A_N = \{-N + 1, -N + 3, \dots, N - 1\}$  is an arithmetic progression of step 2. Every point of  $[-N, N]$  is therefore within distance 1 of an admissible digit.  $\square$

**Lemma 6.7** (Vertical control from the nearest digit). *Let  $c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , set  $\rho = |c|$ , and let  $z \in \mathbb{C}$ . Put  $s = \ell_s(z; c)$ , and choose*

$$t = t^*\left(\frac{\rho s}{y}\right).$$

*If  $|\rho s/y| \leq N$  and  $z' = g_t(z)$ , then*

$$|\ell_v(z')| \leq |y|.$$

*Proof.* The coordinate formula in Lemma 6.2 gives

$$\ell_v(z') = \rho s - yt = y \left( \frac{\rho s}{y} - t \right).$$

The defining property of the nearest admissible digit gives  $|\rho s/y - t| \leq 1$  whenever the argument lies in the admissible range, and therefore  $|\ell_v(z')| \leq |y|$ .  $\square$

**Proposition 6.8** (Nearest-digit self-covering criterion). *Let  $c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , set  $\rho = |c|$ , and let  $S, V > 0$ . Assume*

$$V > |y|, \quad \rho S \leq N|y|, \quad \rho V + \frac{2|x||y|}{\rho} \leq S.$$

*Then  $\mathcal{P}_c(S, V)$  is self-covering.*

*Proof.* Let  $z \in \mathcal{P}_c(S, V)$  and set  $s = \ell_s(z; c)$ . The digit-range condition gives

$$\left| \frac{\rho s}{y} \right| \leq \frac{\rho S}{|y|} \leq N.$$

By Lemma 6.6, the nearest admissible digit  $t$  satisfies  $|\rho s/y - t| \leq 1$ . Hence the vertical coordinate of  $g_t(z)$  satisfies  $|\ell_v(g_t(z))| \leq |y| < V$ . The second formula of Lemma 6.2, together with  $|\ell_v(z)| < V$ , gives

$$|\ell_s(g_t(z); c)| \leq \frac{2|x|}{\rho} |\ell_v(g_t(z))| + \rho |\ell_v(z)| < \frac{2|x|}{\rho} |y| + \rho V \leq S.$$

Thus  $g_t(z) \in \mathcal{P}_c(S, V)$ , proving self-covering.  $\square$

**Proposition 6.9** (Feasibility of the nearest-digit criterion). *Let  $c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , set  $\rho = |c|$ , and let  $N = 2n - 1$ . There exist  $S, V > 0$  such that  $\mathcal{P}_c(S, V)$  is a nonempty open self-covering set certified by Proposition 6.8 if and only if  $c \in \mathcal{X}_n \setminus \mathbb{R}$ .*

*Proof.* If  $S, V$  satisfy the hypotheses of Proposition 6.8, then

$$\rho^2 V + 2|x||y| \leq \rho S \leq N|y|.$$

Since  $V > |y|$ , division by  $|y|$  gives  $\rho^2 + 2|x| < N$ .

Conversely, assume  $\rho^2 + 2|x| < N$ . Choose  $V > |y|$  sufficiently close to  $|y|$  that  $\rho^2 V + 2|x||y| < N|y|$ , and set

$$S = \rho V + \frac{2|x||y|}{\rho}.$$

Then the three inequalities in Proposition 6.8 hold, so  $\mathcal{P}_c(S, V)$  is self-covering.  $\square$

To fix one parallelogram for each parameter, take the largest  $S$  allowed by the digit range and the largest corresponding  $V$  allowed by the shear inequality. Both half-widths are then explicit functions of  $c$ .

**Definition 6.10** (Canonical trap). For  $c = x + iy \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , set  $\rho = |c|$  and define

$$S(c, N) := \frac{N|y|}{\rho}, \quad V(c, N) := \frac{(N - 2|x|)|y|}{\rho^2}.$$

The canonical trap is

$$\mathcal{C}(c, N) := \begin{cases} \mathcal{P}_c(S(c, N), V(c, N)), & V(c, N) > 0, \\ \emptyset, & V(c, N) \leq 0. \end{cases}$$

For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , the half-widths  $S(c, N)$  and  $V(c, N)$  are positive. Hence

$$z \in \mathcal{C}(c, N) \iff \begin{cases} |\ell_s(z; c)| < S(c, N), \\ |\ell_v(z)| < V(c, N), \end{cases}$$

whereas

$$z \in \overline{\mathcal{C}(c, N)} \iff \begin{cases} |\ell_s(z; c)| \leq S(c, N), \\ |\ell_v(z)| \leq V(c, N). \end{cases}$$

Strict inequalities make trap entry stable under parameter perturbation. At a limit of capture witnesses, only the weak inequalities are initially available; Section 6.6 will recover strict entry after two further steps.

**Corollary 6.11** (Sharp validity of the canonical trap). *Let  $c = x + iy$ ,  $\rho = |c| > 1$ ,  $y \neq 0$ , and  $N = 2n - 1$ . With the half-widths in Definition 6.10, the trap is nonempty if and only if  $2|x| < N$ .*

*Moreover,  $\mathcal{C}(c, N)$  is a nonempty open self-covering set if and only if  $c \in \mathcal{X}_n \setminus \mathbb{R}$ ; on the two-disk boundary  $\rho^2 + 2|x| = N$ , it is not self-covering.*

*Proof.* Nonemptiness is equivalent to  $V(c, N) > 0$ , i.e.  $2|x| < N$ . If  $\rho^2 + 2|x| < N$ , then  $V(c, N) > |y|$  and the identities

$$\rho S(c, N) = N|y|, \quad S(c, N) = \rho V(c, N) + \frac{2|x||y|}{\rho}$$

show that Proposition 6.8 applies.

Conversely, if  $\rho^2 + 2|x| \geq N$  and the trap is nonempty, then  $V(c, N) \leq |y|$ . The point 1 belongs to the trap, but for every  $t \in A_N$  the integer  $1 - t$  is odd, hence  $|1 - t| \geq 1$ , and

$$|\ell_v(g_t(1))| = |y| |1 - t| \geq |y| \geq V(c, N).$$

Thus no inverse iterate  $g_t(1)$  stays in the trap, so self-covering fails.  $\square$

**Corollary 6.12** (Validity of the canonical trap). *For every  $c \in \mathcal{X}_n \setminus \mathbb{R}$ ,*

$$\mathcal{C}(c, N) \subset \text{int } E(c, N).$$

*Proof.* By Corollary 6.11, the trap is nonempty, bounded, open, and self-covering. Apply Lemma 6.4.  $\square$

### 6.3. The Outer Parallelogram

An outer bound complements the inner trap. If an inverse branch leaves a set containing  $E(c, N)$ , it cannot continue an address of the marked point. The smallest parallelogram in the canonical directions is obtained by maximizing the two coordinate functionals separately.

**Definition 6.13** (Canonical enclosure). Let  $c = x + iy \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$ , and write  $c = \rho e^{i\theta}$ . Define

$$\mathcal{V}_{\mathcal{E}}(c, N) = (N-1) \sum_{k=1}^{\infty} |\ell_v(c^{-k})| = (N-1) \sum_{k=1}^{\infty} \rho^{-k} |\sin(k\theta)|$$

and

$$\mathcal{S}_{\mathcal{E}}(c, N) = (N-1) \sum_{k=0}^{\infty} |\ell_s(c^{-k}; c)| = (N-1) \frac{|y|}{\rho} + \frac{1}{\rho} \mathcal{V}_{\mathcal{E}}(c, N).$$

The canonical enclosure is the closed canonical parallelogram

$$\mathcal{E}(c, N) = \overline{\mathcal{P}_c(\mathcal{S}_{\mathcal{E}}(c, N), \mathcal{V}_{\mathcal{E}}(c, N))}.$$

**Proposition 6.14** (Exact canonical enclosure). *For  $c = x + iy$ ,  $|c| > 1$ , and  $y \neq 0$ ,*

$$\sup_{z \in E(c, N)} |\ell_v(z)| = \mathcal{V}_{\mathcal{E}}(c, N), \quad \sup_{z \in E(c, N)} |\ell_s(z; c)| = \mathcal{S}_{\mathcal{E}}(c, N).$$

Consequently,

$$E(c, N) \subseteq \mathcal{E}(c, N).$$

Moreover, among all sets of the form  $\overline{\mathcal{P}_c(S, V)}$ , with the fixed coordinate directions of Definition 6.1, the set  $\mathcal{E}(c, N)$  is the unique smallest one under inclusion that contains  $E(c, N)$ .

*Proof.* Let  $\varphi : \mathbb{C} \rightarrow \mathbb{R}$  be real-linear. If  $z = \sum_{k \geq 0} a_k c^{-k} \in E(c, N)$ , then

$$|\varphi(z)| \leq (N-1) \sum_{k=0}^{\infty} |\varphi(c^{-k})|.$$

This bound is sharp: choose

$$a_k = (N-1) \text{sgn}(\varphi(c^{-k})) \in A_N, \quad \text{sgn}(0) = 0.$$

Applying this support-function computation to  $\varphi = \ell_v$  gives the first displayed formula, since  $\ell_v(1) = 0$ . Applying it to  $\varphi = \ell_s(\cdot; c)$ , and using  $\ell_s(1; c) = y/\rho$  and  $\ell_s(c^{-k}; c) = \rho^{-1} \ell_v(c^{-(k-1)})$  for  $k \geq 1$ , gives the second. The two sharp bounds are exactly the inequalities defining  $\mathcal{E}(c, N)$ . Minimality follows because any closed canonical parallelogram containing  $E(c, N)$  must contain the support extrema in both canonical coordinates.  $\square$

**Theorem 6.15** (Canonical trap–enclosure framework). *For every  $c \in \mathcal{X}_n \setminus \mathbb{R}$ ,*

$$\mathcal{C}(c, N) \subset \text{int } E(c, N) \subset E(c, N) \subseteq \mathcal{E}(c, N).$$

*Proof.* Combine Corollary 6.12 and Proposition 6.14. □

**Remark 6.16** (Relation with exact convex-hull methods). Vass studies exact convex hulls through the symbolic addresses of extremal points [Vas13; Vas18]. For the inverse search here, the two support directions  $\ell_v$  and  $\ell_s$  suffice: Proposition 6.14 computes their extrema and hence the smallest enclosing parallelogram in those directions. Exactness refers to this fixed class of parallelograms; the full convex hull can be smaller.

## 6.4. Finite Entry and Finite Escape

We now organize trap entry and enclosure survival by word length. This gives increasing inner approximations and decreasing outer approximations in parameter space.

### 6.4.1. From Polynomial Roots to Finite Entry

A restricted root gives a finite word  $u$  with  $g_u(2c) = 0$  (Proposition 5.6). For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , the origin lies in the open trap, so this word already reaches its interior. Allowing the final remainder to lie anywhere in the trap replaces exact equality by strict inequalities. The example after the definition explains why these give a certificate on a parameter neighborhood.

**Definition 6.17** (Finite-capture sets and capture time). For  $c \in \mathcal{X}_n \setminus \mathbb{R}$  and a finite word  $u \in A_N^m$ , call  $g_u(2c)$  the *inverse iterate of the marked point associated with  $u$* . The word  $u$  is a *capture word* for  $c$  if

$$g_u(2c) \in \mathcal{C}(c, N).$$

We say that  $c$  is *captured by depth  $k$*  if it has a capture word of length at most  $k$ .

For  $k \geq 0$ , define

$$\Theta_k(n) = \left\{ c \in \mathcal{X}_n \setminus \mathbb{R} : \exists u \in A_N^{\leq k} \text{ such that } g_u(2c) \in \mathcal{C}(c, N) \right\}.$$

The capture time is

$$k(c) = \min\{m \geq 0 : \exists u \in A_N^m \text{ such that } g_u(2c) \in \mathcal{C}(c, N)\},$$

with  $k(c) = \infty$  if no such word exists. The finite-capture locus is

$$\Theta_{\text{fin}}(n) = \bigcup_{k \geq 0} \Theta_k(n).$$

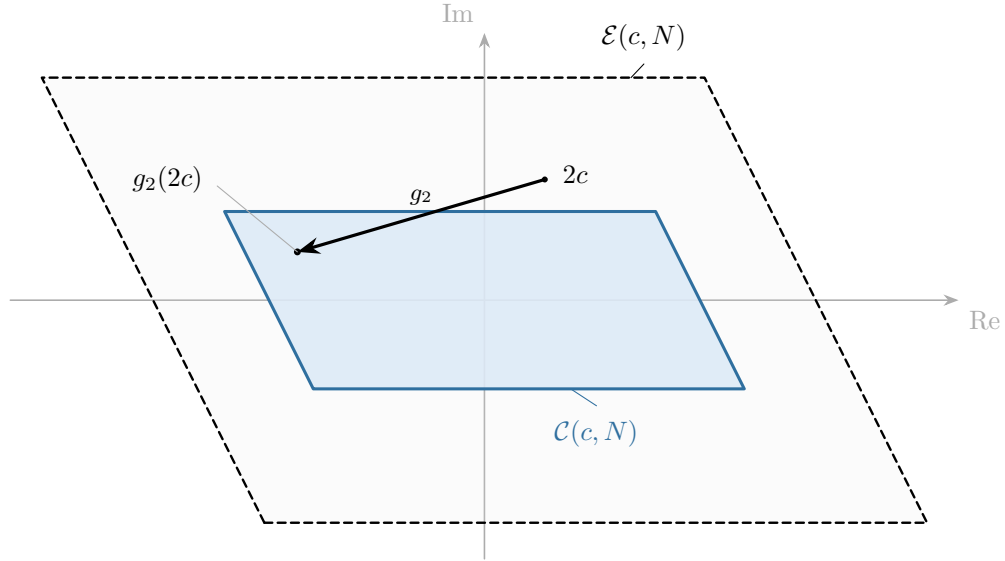
The parameters first captured at depth  $k$  form the layer  $\Theta_k(n) \setminus \Theta_{k-1}(n)$ , with  $\Theta_{-1}(n) = \emptyset$ . These layers record when parameters become visible to the search. Their behavior under limits is the subject of Theorem 6.35.

**Example 6.18** (Reading a capture word). Let  $c \in \mathcal{X}_n \setminus \mathbb{R}$  and  $u = t_0 t_1 t_2 \in A_N^3$ . The condition

$$g_u(2c) = g_{t_2}(g_{t_1}(g_{t_0}(2c))) \in \mathcal{C}(c, N)$$

means that the marked point has been pulled into the canonical trap after three inverse steps. Since  $\mathcal{C}(c, N) \subset \text{int } E(c, N)$ , applying the forward branch  $f_u = f_{t_0} \circ f_{t_1} \circ f_{t_2}$  sends an open neighborhood of  $g_u(2c)$  back to an open neighborhood of  $2c$  inside  $E(c, N)$ . Because the trap inequalities depend continuously on the parameter and are strict, the same word remains a witness on a neighborhood of  $c$ . Thus  $u$  proves that  $c \in \text{int}(\mathcal{M}_n)$ .

Figure 6.2 shows a one-step capture witness.



**Figure 6.2.** Trap-enclosure certification for  $c = 0.7 + 1.4i$  and  $N = 5$ . The marked point  $2c$  lies outside the canonical trap, while  $g_2(2c) = -4.34 + 1.12i \in \mathcal{C}(c, 5)$ . Branch pruning and trap entry are illustrated in Figure 2.1.

### 6.4.2. Enclosure Survival and Finite Escape

Every address of  $2c$  has all its inverse remainders in  $E(c, N)$ , hence in the enclosure. We retain only words that pass this necessary test at every step; an exterior certificate requires all such words to disappear.

**Definition 6.19** (Enclosure-admissible words). A word  $u \in A_N^m$  is  $\mathcal{E}$ -admissible if every prefix iterate remains inside the canonical enclosure:

$$g_{u'}(2c) \in \mathcal{E}(c, N) \quad \text{for every prefix } u' \preceq u.$$

For  $m = 0$ , this means that  $u = \varepsilon$  and  $2c \in \mathcal{E}(c, N)$ .

**Definition 6.20** (Survival sets). For  $k \geq 0$ , define

$$\Xi_k(n) = \{c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}) : \exists u \in A_N^k \text{ that is } \mathcal{E}\text{-admissible}\}.$$

The survival sets satisfy  $\Xi_{k+1}(n) \subseteq \Xi_k(n)$ , since every prefix of an admissible word is admissible. Thus each additional step can remove parameters from the outer approximation.

**Proposition 6.21** (Finite-depth extinction gives an exterior certificate). *If  $c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$  satisfies  $c \notin \Xi_k(n)$  for some  $k \geq 0$ , then  $c \notin \mathcal{M}_n$ .*

*Proof.* If  $c \in \mathcal{M}_n \setminus \mathbb{R}$ , then Proposition 5.2 gives  $2c \in E(c, N)$ . Choose an address  $(t_j)_{j \geq 0} \in A_N^{\mathbb{N}_0}$  for  $2c$ . Every prefix iterate belongs to  $E(c, N)$ , hence to  $\mathcal{E}(c, N)$  by Proposition 6.14. Thus the length- $k$  prefix is  $\mathcal{E}$ -admissible, so  $c \in \Xi_k(n)$ .  $\square$

**Corollary 6.22** (Survival characterizes  $\mathcal{M}_n$  off the real axis). *For  $c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ ,*

$$c \in \mathcal{M}_n \iff c \in \bigcap_{k \geq 0} \Xi_k(n).$$

*Proof.* The forward implication is Proposition 6.21. Conversely, if  $c \in \bigcap_k \Xi_k(n)$ , the tree of  $\mathcal{E}$ -admissible words is finitely branching and has a vertex at every depth. König's lemma gives an infinite branch  $(t_j)_{j \geq 0} \in A_N^{\mathbb{N}_0}$ . For  $m \geq 1$ , the prefix iterates  $z_m = g_{t_0 \dots t_{m-1}}(2c)$  remain in the bounded enclosure, and unwinding the inverse dynamics gives

$$2c = t_0 + \frac{t_1}{c} + \dots + \frac{t_{m-1}}{c^{m-1}} + \frac{z_m}{c^m}.$$

The last term tends to zero, so  $2c \in E(c, N)$ , and Proposition 5.2 gives  $c \in \mathcal{M}_n$ .  $\square$

**Proposition 6.23** (Inner and outer bounds at depth  $k$ ). *On  $\mathcal{X}_n \setminus \mathbb{R}$ , for every  $k \geq 0$ ,*

$$\Theta_k(n) \subset (\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} \subset \Xi_k(n).$$

*Proof.* If  $c \in \Theta_k(n)$ , then some word  $u$  satisfies  $g_u(2c) \in \mathcal{C}(c, N) \subset \text{int } E(c, N)$ . Applying the forward branch  $f_u$  gives  $2c \in E(c, N)$ , hence  $c \in \mathcal{M}_n$ . The second inclusion is Proposition 6.21.  $\square$

### 6.4.3. The First Capture Layers

Figure 6.3 compares the parameter and dynamical pictures: capture of the marked point  $2c$  in  $E(c, 5)$  certifies connectedness of the original attractor  $E(c, 3)$ . Matching depth colours show how the two finite-capture filtrations develop near  $c$  and  $2c$ .

**Proposition 6.24** (The depth-zero lens). *Let  $n \geq 2$ , set  $N = 2n - 1$ , and write  $c = x + iy$  and  $\rho = |c|$ . Then*

$$\Theta_0(n) = \{c = x + iy \in \mathcal{X}_n \setminus \mathbb{R} : \rho^2 + |x| < N/2\}.$$

*Equivalently,*

$$\Theta_0(n) = \left\{ c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}) : |c - \tfrac{1}{2}| < \sqrt{n - \tfrac{1}{4}}, \quad |c + \tfrac{1}{2}| < \sqrt{n - \tfrac{1}{4}} \right\}.$$

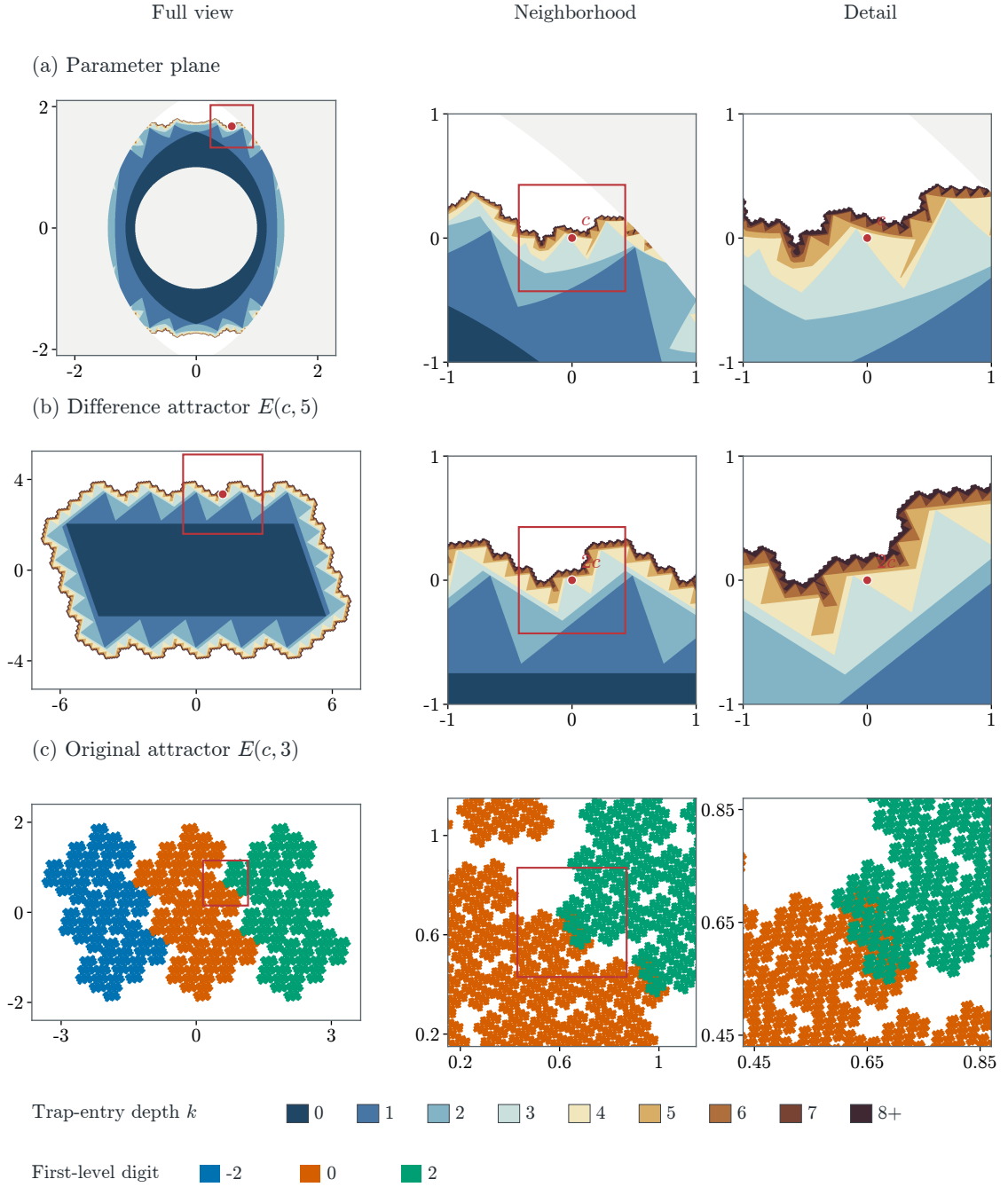
*Proof.* By definition of  $\Theta_0(n)$ , a parameter  $c \in \mathcal{X}_n \setminus \mathbb{R}$  belongs to  $\Theta_0(n)$  if and only if  $2c \in \mathcal{C}(c, N)$ . Since  $\mathcal{C}(c, N) = \mathcal{P}_c(S(c, N), V(c, N))$ , this condition is

$$|\ell_v(2c)| < V(c, N), \quad |\ell_s(2c; c)| < S(c, N).$$

Here

$$|\ell_v(2c)| = 2|y|, \quad |\ell_s(2c; c)| = \frac{|\text{Im}(2c^2)|}{\rho} = \frac{4|x||y|}{\rho}.$$

$$n = 3, \quad c = 0.585 + 1.675i$$



**Figure 6.3.** Numerical finite-capture filtration for  $n = 3$  at  $c = 0.585 + 1.675i$ . The rows show the parameter field, the difference attractor  $E(c, 5)$ , and the original attractor  $E(c, 3)$ , whose three first-level pieces have distinct colours. Red boxes locate successive magnifications. In the first two rows, the local coordinates are  $(c' - c)/s$  and  $(z - 2c)/(5s)$ , with  $s = 0.35$  and  $0.15$ . The shared key gives first trap-entry depths up to the search limit 20, grouping depths 8–20 as 8+. White denotes numerical exclusion, dark gray unresolved pixel centres, and light gray the region outside the canonical parameter lens. The marked parameter belongs to  $\Theta_4(3)$ .

Since  $c \notin \mathbb{R}$ , we have  $|y| > 0$ . Therefore the vertical inequality is

$$2|y| < \frac{(N - 2|x|)|y|}{\rho^2} \iff \rho^2 + |x| < N/2.$$

The slanted inequality is

$$\frac{4|x||y|}{\rho} < \frac{N|y|}{\rho} \iff 4|x| < N.$$

In the exterior parameter plane, the vertical inequality already implies the slanted one.

If  $N = 3$ , equivalently  $n = 2$ , the exterior condition gives  $\rho > 1$ . Hence the vertical inequality implies

$$|x| < \frac{3}{2} - \rho^2 < \frac{1}{2} < \frac{3}{4} = \frac{N}{4},$$

and therefore  $4|x| < N$ .

If  $N \geq 5$ , then  $|x| \leq \rho$ , so the vertical inequality gives

$$|x|^2 + |x| < N/2.$$

Hence

$$|x| < \frac{-1 + \sqrt{1 + 2N}}{2} \leq \frac{N}{4},$$

where the last inequality holds for  $N \geq 5$ . Therefore  $4|x| < N$  also in this case. Thus, on the exterior parameter plane, the single inequality  $\rho^2 + |x| < N/2$  is equivalent to  $2c \in \mathcal{C}(c, N)$ .

Finally,

$$\rho^2 + |x| < N/2 \iff \rho^2 + x < N/2 \text{ and } \rho^2 - x < N/2.$$

Completing the square and using  $N = 2n - 1$  gives

$$|c + \frac{1}{2}|^2 < n - \frac{1}{4}, \quad |c - \frac{1}{2}|^2 < n - \frac{1}{4}.$$

Conversely, any  $c \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$  satisfying these two-disk inequalities satisfies  $\rho^2 + |x| < N/2$ . The redundancy argument above gives  $|x| < N/4$ , and hence

$$\rho^2 + 2|x| = (\rho^2 + |x|) + |x| < \frac{N}{2} + \frac{N}{4} < N.$$

Thus such a point lies in  $\mathcal{X}_n \setminus \mathbb{R}$ , so the disk formulation is exactly equivalent to the first one.  $\square$

Figure 6.4 summarizes the depth-zero capture geometry of  $\Theta_0(n)$ .

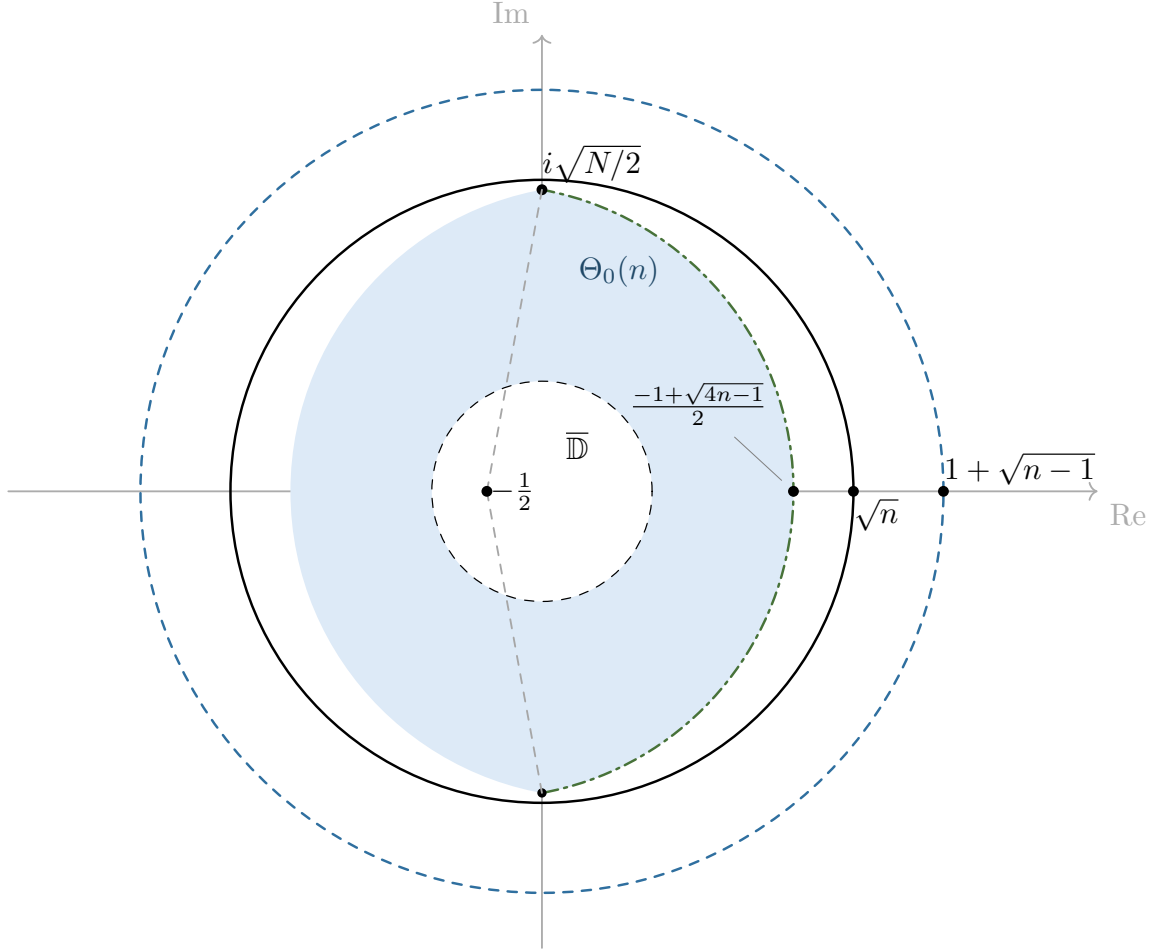
For comparison, the basic inner annulus gives an interior region without an inverse search.

**Proposition 6.25** (Inner annulus). *For every  $n \geq 2$ ,*

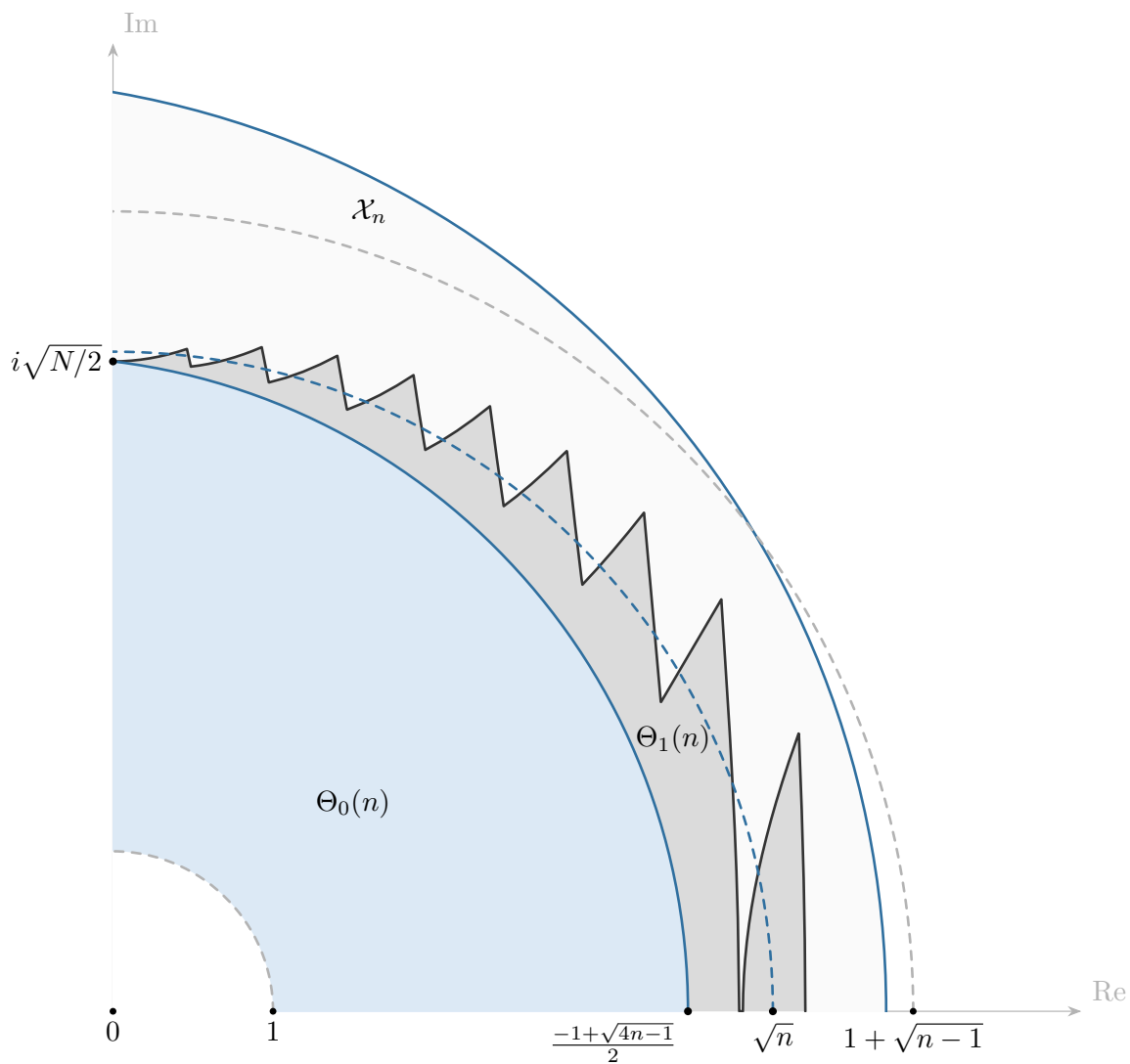
$$\{c \in \mathbb{C} : 1 < |c| < \sqrt{n}\} \subset \text{int}(\mathcal{M}_n).$$

*Proof.* The annulus lies in  $\mathcal{M}_n$  by the basic collinear attractor estimate of Proposition 3.11. Since it is open in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , it is contained in  $\text{int}(\mathcal{M}_n)$ .  $\square$

Figure 6.5 shows  $\Theta_1(n)$ , the set of parameters captured by depth one, in the first quadrant together with the base layer and radial reference curves.



**Figure 6.4.** Depth-zero capture geometry for  $\Theta_0(n)$ . The shaded region is  $\{c = x + iy \in \mathcal{X}_n \setminus \mathbb{R} : \rho^2 + |x| < N/2\}$ , equivalently the intersection of the two disks  $|c \pm \frac{1}{2}| < \sqrt{n - \frac{1}{4}}$ , with  $\overline{\mathbb{D}} \cup \mathbb{R}$  removed. The guides mark  $|c| = \sqrt{n}$ ,  $|c| = 1 + \sqrt{n-1}$ , the imaginary-axis endpoint  $i\sqrt{N/2}$ , and the real endpoint  $(-1 + \sqrt{4n-1})/2$ .



**Figure 6.5.** First-quadrant view of  $\Theta_0(n) \subset \Theta_1(n)$  for  $n = 17$ , with solid blue arcs marking the outer boundaries of the base capture set and ambient parameter lens. The dashed blue and outer dashed gray arcs mark  $|c| = \sqrt{n}$  and  $|c| = 1 + \sqrt{n-1}$ , respectively.

## 6.5. A Finite Inverse Search

Bandt's inverse-iteration method for the binary locus [Ban02] rests on the expansion of distance from the attractor. For  $E = E(c, N)$  and every digit  $t \in A_N$ ,

$$\text{dist}(g_t(z), E) = |c| \text{dist}(z, f_t(E)) \geq |c| \text{dist}(z, E),$$

because  $f_t(E) \subset E$ . A point outside  $E$  therefore moves farther away under every inverse branch. The canonical enclosure provides an explicit escape bound, and the trap adds the complementary entry test.

The search uses the full alphabet  $A_N$ , with  $N = 2n - 1$ . A trap hit returns **Interior**; exhaustion of all surviving branches returns **Exterior**. A depth or node limit reached before either event returns **Undetermined**. To implement the exact bounds of Definition 6.13 and Proposition 6.14, let

$$\tilde{S} \geq \mathcal{S}_{\mathcal{E}}(c, N), \quad \tilde{V} \geq \mathcal{V}_{\mathcal{E}}(c, N)$$

be verified upper bounds, for example bounds obtained by outward-rounded interval arithmetic. Define the corresponding *working enclosure* by

$$\tilde{\mathcal{E}}(c, N) = \overline{\mathcal{P}_c(\tilde{S}, \tilde{V})}.$$

Then

$$E(c, N) \subseteq \mathcal{E}(c, N) \subseteq \tilde{\mathcal{E}}(c, N).$$

Equality in both bounds,

$$\tilde{S} = \mathcal{S}_{\mathcal{E}}(c, N), \quad \tilde{V} = \mathcal{V}_{\mathcal{E}}(c, N),$$

recovers the exact canonical enclosure; larger bounds retain a larger search region.

The algorithm is stated with exact comparisons. An interval implementation must certify lens membership and both strict trap inequalities before accepting a capture word, and may prune a branch only when it verifies that the corresponding iterate lies outside the working enclosure. Unresolved comparisons require higher precision or retention of the branch.

**Proposition 6.26** (Correctness of the inverse search). *If Algorithm 1 returns Interior, then  $c \in \text{int}(\mathcal{M}_n)$ . If it returns Exterior, then  $c \notin \mathcal{M}_n$ .*

*Proof.* Set

$$z_0 = 2c, \quad \tilde{\mathcal{E}} = \tilde{\mathcal{E}}(c, N).$$

**Interior.** If the algorithm returns **Interior**, it has generated a word  $u \in A_N^{\leq k_{\max}}$ , possibly  $u = \varepsilon$ , such that

$$g_u(z_0) \in \mathcal{C}(c, N).$$

This can occur only when  $c \in \mathcal{X}_n \setminus \mathbb{R}$ . Then Corollary 6.12 gives  $\mathcal{C}(c, N) \subset \text{int } E(c, N)$ , so  $g_u(z_0) \in E(c, N)$ . Applying the forward branch gives

$$z_0 = f_u(g_u(z_0)) \in f_u(E(c, N)) \subset E(c, N).$$

By Proposition 5.2,  $c \in \mathcal{M}_n$ .

To see that  $c$  is an interior point, define

$$\mathcal{U}_u = \{c' \in \mathcal{X}_n \setminus \mathbb{R} : g_u(2c') \in \mathcal{C}(c', N)\}.$$

**Algorithm 1** Inverse-iteration test for  $c \in \mathcal{M}_n$ 


---

**Require:**  $c = x + iy \in \mathbb{C}$  with  $|c| > 1$  and  $y \neq 0$ ;  $n \geq 2$ ; set  $N = 2n - 1$   
max depth  $k_{\max}$ ; max nodes per level  $L_{\max}$   
certified bounds  $\tilde{S} \geq \mathcal{S}_{\mathcal{E}}(c, N)$ ,  $\tilde{V} \geq \mathcal{V}_{\mathcal{E}}(c, N)$

**Ensure:** Verdict: Interior, Exterior, or Undetermined

- 1:  $\rho \leftarrow |c|$ ,  $(s_0, v_0) \leftarrow (\ell_s(2c; c), \ell_v(2c))$
- 2:  $\text{inLens} \leftarrow (c \in \mathcal{X}_n \setminus \mathbb{R})$
- 3: **if**  $\text{inLens}$  **then**
- 4:   Compute trap half-widths  $\mathcal{S} \leftarrow S(c, N)$  and  $\mathcal{V} \leftarrow V(c, N)$   
*Step 1: initial node checks*
- 5: **if**  $|s_0| > \tilde{S}$  **or**  $|v_0| > \tilde{V}$  **then**
- 6:   **return** Exterior
- 7: **if**  $\text{inLens}$  **and**  $|s_0| < \mathcal{S}$  **and**  $|v_0| < \mathcal{V}$  **then**
- 8:   **return** Interior  
*Step 2: level-by-level traversal of the backward-orbit tree*
- 9:  $W \leftarrow \{(s_0, v_0, \varepsilon)\}$   $\triangleright$  Track coordinates  $(s, v)$  and address word  $u$
- 10: **for**  $k = 1$  **to**  $k_{\max}$  **do**
- 11:    $W' \leftarrow \emptyset$
- 12:   **for all**  $(s, v, u) \in W$  **do**
- 13:     **for all**  $t \in A_N$  **such that**  $|\rho s - yt| \leq \tilde{V}$  **do**  $\triangleright$  Vertical branch truncation
- 14:        $v' \leftarrow \rho s - yt$
- 15:        $s' \leftarrow \frac{2x}{\rho} v' - \rho v$
- 16:       **if**  $|s'| \leq \tilde{S}$  **then**  $\triangleright$  Slanted branch truncation
- 17:         **if**  $\text{inLens}$  **and**  $|s'| < \mathcal{S}$  **and**  $|v'| < \mathcal{V}$  **then**
- 18:         **return** Interior
- 19:          $W' \leftarrow W' \cup \{(s', v', ut)\}$
- 20:         **if**  $|W'| \geq L_{\max}$  **then**
- 21:         **return** Undetermined
- 22:   **if**  $W' = \emptyset$  **then**
- 23:     **return** Exterior  $\triangleright$  Enclosure exhausted
- 24:    $W \leftarrow W'$
- 25: **return** Undetermined

---

Membership in  $\mathcal{C}(c', N)$  is determined by strict inequalities in continuous functions of  $c'$ . Hence  $\mathcal{U}_u$  is open, contains  $c$ , and the same witness  $u$  shows that every  $c' \in \mathcal{U}_u$  belongs to  $\mathcal{M}_n$ . Thus  $\mathcal{U}_u \subset \mathcal{M}_n$ , so  $c \in \text{int}(\mathcal{M}_n)$ .

Exterior. If the algorithm returns Exterior at the initial check, then  $z_0 \notin \tilde{\mathcal{E}}$ . Since

$$E(c, N) \subseteq \mathcal{E}(c, N) \subseteq \tilde{\mathcal{E}},$$

it follows that  $z_0 \notin E(c, N)$ , and hence  $c \notin \mathcal{M}_n$  by Proposition 5.2.

Suppose instead that the algorithm returns Exterior because the working set becomes empty at some depth. If  $z_0 \in E(c, N)$ , choose an address  $(t_j)_{j \geq 0} \in A_N^{\mathbb{N}_0}$  for  $z_0$ , and for  $m \geq 1$  set

$$z_m = g_{t_0 \dots t_{m-1}}(z_0).$$

Then

$$z_m \in E(c, N) \subseteq \tilde{\mathcal{E}} \quad (m \geq 0).$$

Each prefix of this address survives until the search stops. Indeed, the empty prefix passes the initial check. If the length- $m$  prefix has survived, its child corresponding to  $t_m$  is  $z_{m+1}$ , whose canonical coordinates satisfy

$$|v_{m+1}| \leq \tilde{V}, \quad |s_{m+1}| \leq \tilde{S}.$$

This child is retained at depth  $m + 1$ . Induction contradicts extinction of the working set. Therefore  $z_0 \notin E(c, N)$ , and Proposition 5.2 gives  $c \notin \mathcal{M}_n$ .  $\square$

Figure 2.1 illustrates branch pruning and trap entry.

The companion software provides a floating-point implementation of the canonical-coordinate inverse search in Algorithm 1, with export of inverse words and search metadata. Its numerical output can guide the search for certificates; certification requires verification of the inequalities as described above.

**Proposition 6.27** (Vertical branch truncation). *Let  $c = x + iy \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$ . For a backward-search node with canonical coordinate  $s$ , define*

$$\mathbf{A}(s) = \{t \in A_N : |\rho s - yt| \leq \mathcal{V}_{\mathcal{E}}(c, N)\}.$$

*Every child that satisfies the vertical enclosure test is generated by a digit in  $\mathbf{A}(s)$ , and*

$$\#\mathbf{A}(s) \leq \left\lfloor \frac{\mathcal{V}_{\mathcal{E}}(c, N)}{|y|} \right\rfloor + 1.$$

*Proof.* The vertical coordinate of the child  $g_t(z)$  is  $\rho s - yt$ . Thus the vertical enclosure test is exactly the condition  $t \in \mathbf{A}(s)$ . The allowed digits are the intersection of the arithmetic progression  $A_N$ , with step 2, and an interval of length  $2\mathcal{V}_{\mathcal{E}}(c, N)/|y|$ . Such an interval contains at most  $\lfloor \mathcal{V}_{\mathcal{E}}(c, N)/|y| \rfloor + 1$  admissible digits.  $\square$

**Theorem 6.28** (Uniform branching bound for  $|c| \geq \sqrt{n}$ ). *Let  $c = x + iy \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{R})$ , and assume  $\rho = |c| \geq \sqrt{n}$ . Then every node in the exact backward-search tree has at most*

$$b_n = \left\lceil \frac{2(\sqrt{n} + 1)}{\sqrt{n} - 1} \right\rceil = \left\lceil \frac{4}{\sqrt{n} - 1} \right\rceil + 2$$

*children that can pass the vertical enclosure test. In particular the tree has at most  $b_n^k$  nodes at depth  $k$ , and  $b_n = 3$  for all  $n \geq 25$ .*

*Proof.* By Proposition 6.27,

$$\#\mathbf{A}(s) \leq \left\lfloor \frac{\mathcal{V}_{\mathcal{E}}(c, N)}{|y|} \right\rfloor + 1.$$

Write  $c = \rho e^{i\theta}$ . The estimate  $|\sin(k\theta)| \leq k|\sin \theta|$ , strict for some term because  $c \notin \mathbb{R}$ , gives

$$\frac{\mathcal{V}_{\mathcal{E}}(c, N)}{|y|} < \frac{N - 1}{(\rho - 1)^2} \leq \frac{2(n - 1)}{(\sqrt{n} - 1)^2} = \frac{2(\sqrt{n} + 1)}{\sqrt{n} - 1}.$$

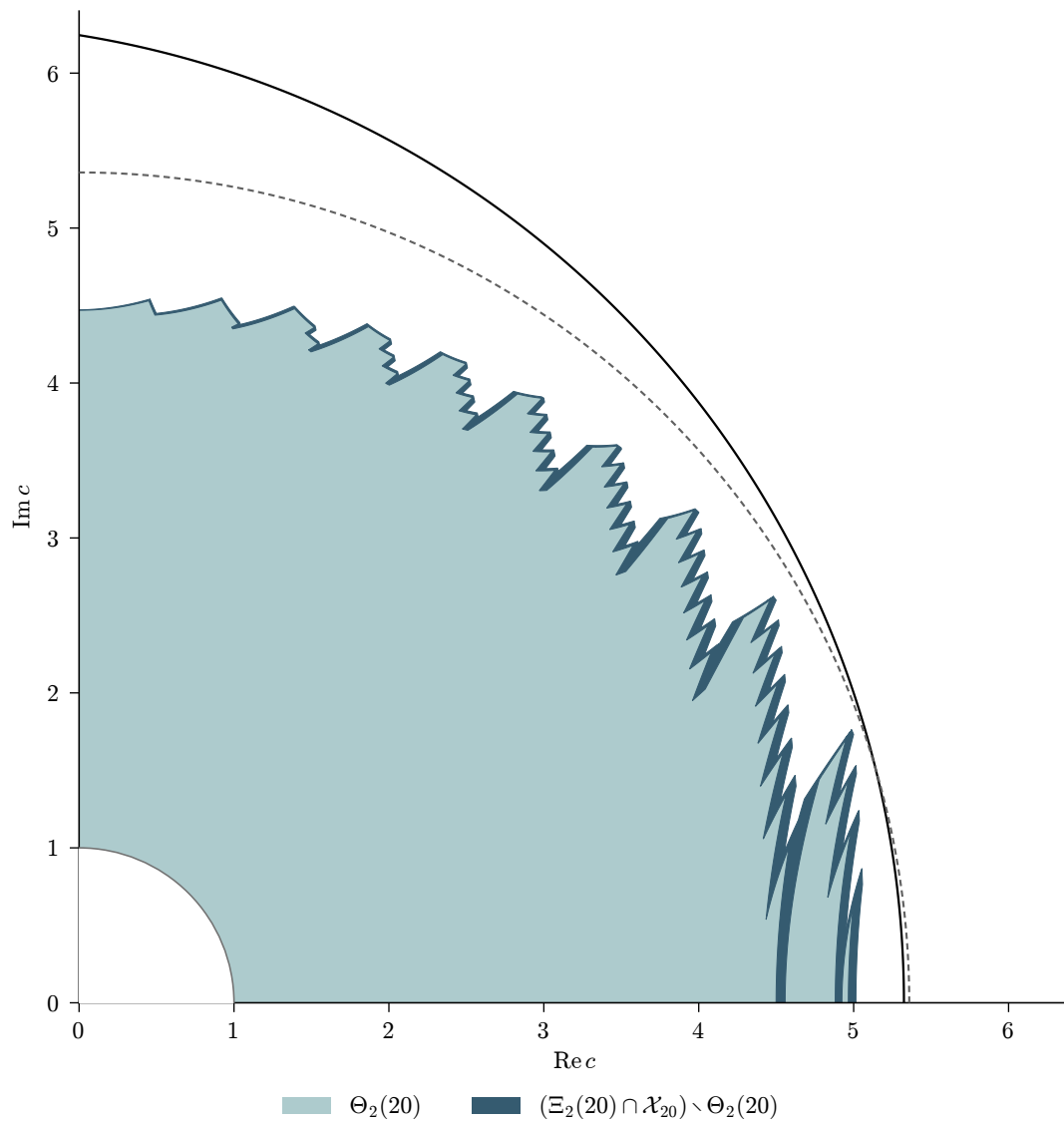
Taking the integer bound yields the formula. The last assertion follows from  $\frac{2(\sqrt{n}+1)}{\sqrt{n}-1} \leq 3$ , equivalently  $n \geq 25$ .  $\square$

For an enlarged working enclosure

$$\tilde{\mathcal{E}}(c, N) \supseteq \mathcal{E}(c, N),$$

the vertical count uses  $\tilde{V}$  in place of  $\mathcal{V}_{\mathcal{E}}(c, N)$ . Additional children may survive, so Theorem 6.28 bounds the exact admissible tree, not necessarily the working tree of an implementation.

Figure 6.6 gives a first-quadrant view of the depth-2 inner and outer tests when  $n = 20$ .



**Figure 6.6.** Numerical first-quadrant view of  $\Theta_2(n)$  (pale teal) and  $(\Xi_2(n) \cap \mathcal{X}_n) \setminus \Theta_2(n)$  (deep blue) for  $n = 20$ , the threshold case in which  $\mathcal{M}_n \setminus \mathbb{R}$  is contained in the parameter lens  $\mathcal{X}_n$ . The solid arc is the lens boundary  $|c + 1| = \sqrt{40}$ ; the dashed arc is the radial bound  $|c| = 1 + \sqrt{19}$ .

## 6.6. Stability under Limits

As the parameter varies, the endpoint of a fixed capture word may reach the boundary of the canonical trap, so the strict capture inequalities need not survive a limit. Inside the non-real lens, however, at most two additional inverse steps restore strict entry. We first express this fact through the forward iterates of the trap and then transfer it to the finite-capture levels.

**Definition 6.29** (Forward trap iterates). For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , define

$$\mathcal{C}_0(c) = \mathcal{C}(c, N), \quad \mathcal{C}_{k+1}(c) = \mathcal{H}_{c,N}(\mathcal{C}_k(c)).$$

**Lemma 6.30** (Forward invariance of the interior). Let  $E = E(c, N)$ . For every  $t \in A_N$ ,

$$f_t(\text{int } E) \subset \text{int } E.$$

Consequently, if  $U \subset \text{int } E(c, N)$ , then

$$\mathcal{H}_{c,N}^k(U) \subset \text{int } E(c, N) \quad (k \geq 0).$$

*Proof.* If  $z \in \text{int } E$ , choose  $r > 0$  with  $B(z, r) \subset E$ . The affine homeomorphism  $f_t$  takes this ball to an open neighborhood of  $f_t(z)$  contained in  $f_t(E) \subset E$ . Thus  $f_t(z) \in \text{int } E$ . Taking unions over the digits and inducting on  $k$  proves the iterated statement.  $\square$

Self-covering gives  $\mathcal{C}_0(c) \subset \mathcal{C}_1(c)$ . Monotonicity of the Hutchinson operator then yields  $\mathcal{C}_k(c) \subset \mathcal{C}_{k+1}(c)$  for all  $k \geq 0$ . These increasing open sets remain inside  $\text{int } E(c, N)$ . Figure 6.7 shows the trap and its first two forward iterates.

**Proposition 6.31** (Forward–backward duality). For a word  $u$ , the condition  $g_u(2c) \in \mathcal{C}(c, N)$  is equivalent to

$$2c \in f_u(\mathcal{C}(c, N)).$$

*Proof.* Apply  $f_u$  to the capture condition and use  $g_u = f_u^{-1}$ .  $\square$

Together with the nesting of the forward iterates, this gives

$$c \in \Theta_k(n) \iff 2c \in \mathcal{C}_k(c).$$

Thus the same search can be read by moving the marked point backward or expanding the trap forward. The next proposition shows how this expansion absorbs the trap boundary.

**Proposition 6.32** (Two-step inclusion for the closed trap). For  $c \in \mathcal{X}_n \setminus \mathbb{R}$ ,

$$\overline{\mathcal{C}_0(c)} \subset \mathcal{C}_2(c).$$

Consequently, for every  $k \geq 0$ ,

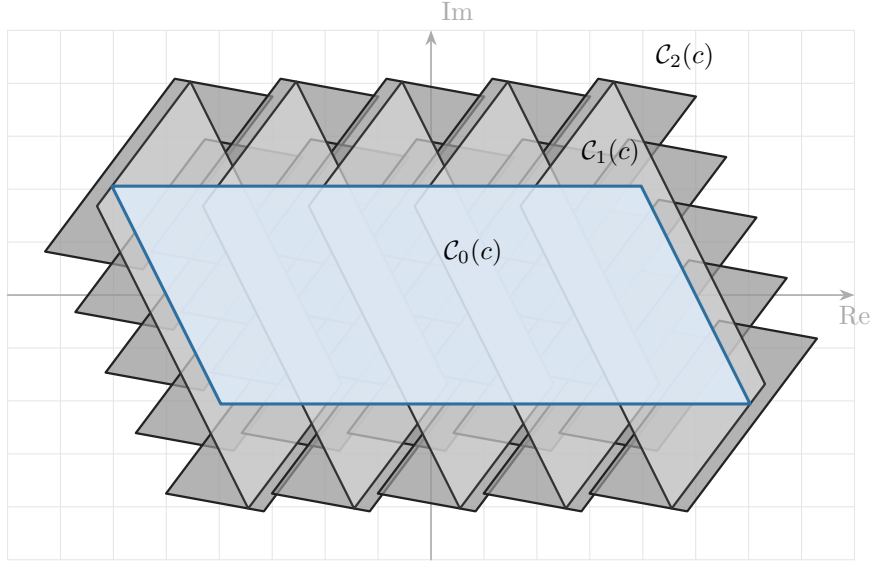
$$\overline{\mathcal{C}_k(c)} \subset \mathcal{C}_{k+2}(c).$$

*Proof.* The nearest admissible digit rule makes the vertical inequality strict after one inverse step. A second step transfers this strictness to the slanted inequality. To make this precise, write

$$c = x + iy, \quad \rho = |c|, \quad N = 2n - 1,$$

and let

$$S = S(c, N) = \frac{N|y|}{\rho}, \quad V = V(c, N) = \frac{(N - 2|x|)|y|}{\rho^2}$$



**Figure 6.7.** The canonical trap  $\mathcal{C}_0(c)$  and its first two forward iterates for  $N = 5$  and  $c = 0.7 + 1.4i$ .

be the canonical trap half-widths.

Since  $c \in \mathcal{X}_n \setminus \mathbb{R}$ , we have  $y \neq 0$  and

$$\rho^2 + 2|x| < N.$$

Hence

$$N - 2|x| > \rho^2,$$

and therefore

$$\frac{V}{|y|} = \frac{N - 2|x|}{\rho^2} > 1,$$

so  $|y| < V$ . We shall also use the identity

$$\frac{\rho S}{|y|} = N.$$

Let  $z_0 \in \overline{\mathcal{C}_0(c)}$ , and write

$$s_j = \ell_s(z_j; c), \quad v_j = \ell_v(z_j).$$

Then

$$|s_0| \leq S, \quad |v_0| \leq V.$$

In particular,

$$\left| \frac{\rho s_0}{y} \right| \leq \frac{\rho S}{|y|} = N,$$

so the nearest admissible digit rule applies. Choose the corresponding digit  $t_0 \in A_N$ , and set  $z_1 = g_{t_0}(z_0)$ . By Lemma 6.7,

$$|v_1| \leq |y| < V.$$

Using the second identity in Lemma 6.2, we also obtain

$$|s_1| \leq \frac{2|x|}{\rho} |v_1| + \rho |v_0| \leq \frac{2|x||y|}{\rho} + \rho V = S.$$

If  $|s_1| < S$ , then

$$z_1 \in \mathcal{C}_0(c),$$

and hence

$$z_0 \in \mathcal{C}_1(c) \subset \mathcal{C}_2(c).$$

It remains to handle the boundary case

$$|s_1| = S.$$

Then

$$\left| \frac{\rho s_1}{y} \right| = \frac{\rho S}{|y|} = N,$$

so the nearest admissible digit rule applies once more. Choose the corresponding digit  $t_1 \in A_N$ , and set  $z_2 = g_{t_1}(z_1)$ . Again,

$$|v_2| \leq |y| < V.$$

Moreover,

$$\begin{aligned} |s_2| &\leq \frac{2|x|}{\rho}|v_2| + \rho|v_1| \\ &\leq \frac{2|x||y|}{\rho} + \rho|v_1| \\ &< \frac{2|x||y|}{\rho} + \rho V \\ &= S. \end{aligned}$$

The strict inequality follows from  $|v_1| < V$ ; in particular, the argument also covers the case  $x = 0$ . Thus

$$z_2 \in \mathcal{C}_0(c),$$

and consequently

$$z_0 \in \mathcal{C}_2(c).$$

This proves

$$\overline{\mathcal{C}_0(c)} \subset \mathcal{C}_2(c).$$

For the general case, observe that

$$\mathcal{C}_k(c) = \bigcup_{|u|=k} f_u(\mathcal{C}_0(c)).$$

The union is finite and each  $f_u$  is a homeomorphism, so

$$\overline{\mathcal{C}_k(c)} = \bigcup_{|u|=k} f_u(\overline{\mathcal{C}_0(c)}).$$

Using the first part,

$$\overline{\mathcal{C}_k(c)} \subset \bigcup_{|u|=k} f_u(\mathcal{C}_2(c)) = \mathcal{C}_{k+2}(c).$$

□

**Lemma 6.33** (Closed graph for fixed trap words). *Fix a word  $u \in A_N^k$ . Suppose  $c_j \rightarrow c$  in  $\mathcal{X}_n \setminus \mathbb{R}$  and*

$$g_u(2c_j) \in \overline{\mathcal{C}(c_j, N)} \quad \text{for all } j.$$

*Then*

$$g_u(2c) \in \overline{\mathcal{C}(c, N)}.$$

*Proof.* The half-widths  $S(c, N)$  and  $V(c, N)$  are continuous on  $\mathcal{X}_n \setminus \mathbb{R}$ , and the slanted functional  $\ell_s(z; c)$  is jointly continuous in  $z$  and  $c$  there; see Definition 6.10. The inverse iterate  $g_u(2c)$  is polynomial in  $c$ , hence continuous. Since  $\overline{\mathcal{C}(c, N)}$  is described by the two non-strict inequalities

$$|\ell_s(z; c)| \leq S(c, N), \quad |\ell_v(z)| \leq V(c, N),$$

the assumed inequalities for  $z_j = g_u(2c_j)$  pass to the limit. Thus the limit point  $z = g_u(2c)$  belongs to  $\overline{\mathcal{C}(c, N)}$ .  $\square$

The *capture chart* of a finite word  $u$  is

$$\mathcal{U}_u = \{c \in \mathcal{X}_n \setminus \mathbb{R} : g_u(2c) \in \mathcal{C}_0(c)\}.$$

**Lemma 6.34** (Openness of capture sets). *For each  $k \geq 0$ , the set  $\Theta_k(n)$  is open in  $\mathcal{X}_n \setminus \mathbb{R}$ ; consequently  $\Theta_{\text{fin}}(n)$  is open in  $\mathcal{X}_n \setminus \mathbb{R}$ .*

*Proof.* Fix a word  $u$ . The map  $c \mapsto g_u(2c)$ , the trap half-widths, and the canonical coordinate functions are continuous. Moreover,

$$\mathcal{U}_u = \left\{ c \in \mathcal{X}_n \setminus \mathbb{R} : \begin{array}{l} |\ell_s(g_u(2c); c)| < S(c, N), \\ |\ell_v(g_u(2c))| < V(c, N) \end{array} \right\}.$$

Thus  $\mathcal{U}_u$  is defined by two strict continuous inequalities and is open. Consequently,

$$\Theta_k(n) = \bigcup_{|u| \leq k} \mathcal{U}_u$$

is a finite union of open sets. Taking the union over  $k$  proves openness of  $\Theta_{\text{fin}}(n)$ .  $\square$

### 6.6.1. Two-Step Stability of the Capture Levels

For a fixed word, closed-trap membership passes to parameter limits. The two-step inclusion for the closed trap then converts this weak limiting membership into strict capture at depth at most two levels greater.

**Theorem 6.35** (Two-step stability of finite capture). *For every  $n \geq 2$  and  $k \geq 0$ ,*

$$\overline{\Theta_k(n)} \cap (\mathcal{X}_n \setminus \mathbb{R}) \subset \Theta_{k+2}(n).$$

*Proof.* Let  $c_j \in \Theta_k(n)$  converge to  $c \in \mathcal{X}_n \setminus \mathbb{R}$ . For each  $j$  choose a word  $u_j$  of length at most  $k$  witnessing capture. There are finitely many such words, so one word  $u$  occurs along a subsequence. Then

$$g_u(2c_j) \in \mathcal{C}(c_j, N) \subset \overline{\mathcal{C}(c_j, N)}$$

for all  $j$  in the subsequence, and Lemma 6.33 gives

$$g_u(2c) \in \overline{\mathcal{C}(c, N)}.$$

By Proposition 6.32,

$$\overline{\mathcal{C}(c, N)} \subset \mathcal{C}_2(c).$$

Hence, there is a word  $w \in A_N^{\leq 2}$  such that

$$g_w(g_u(2c)) \in \mathcal{C}(c, N).$$

The concatenated word  $uw$  has length at most  $k + 2$ , so it witnesses  $c \in \Theta_{k+2}(n)$ .  $\square$

**Corollary 6.36** (Finite-depth boundaries are captured two levels later). *For every  $n \geq 2$  and every  $k \geq 0$ ,*

$$\partial_{\mathcal{X}_n \setminus \mathbb{R}} \Theta_k(n) \subset \Theta_{k+2}(n) \setminus \Theta_k(n).$$

*Thus the boundary of each finite-capture level within the non-real lens is absorbed after at most two additional inverse steps.*

*Proof.* By Lemma 6.34,  $\Theta_k(n)$  is open in  $\mathcal{X}_n \setminus \mathbb{R}$ , so

$$\partial_{\mathcal{X}_n \setminus \mathbb{R}} \Theta_k(n) = (\overline{\Theta_k(n)} \cap (\mathcal{X}_n \setminus \mathbb{R})) \setminus \Theta_k(n).$$

Apply Theorem 6.35 to the term in parentheses.  $\square$

The filtration unfolds layer by layer: the boundary of every level already lies inside the level two steps later. This controls each finite-depth frontier. Recovering the connectedness locus from the closure of the entire union requires the separate approximation argument below.

### 6.6.2. Capture Charts and Closure of the Union

Each capture chart is determined by two strict trap inequalities. Their algebraic form gives finite-depth regularity, while exact landing places restricted polynomial roots inside the charts.

**Lemma 6.37** (Semialgebraicity at fixed depth). *Fix a word  $u$  and an open quadrant  $Q$ . Then  $\mathcal{U}_u \cap Q$  is an open semialgebraic subset of  $Q$ . Consequently, for each  $k \geq 0$ ,  $\Theta_k(n) \cap Q$  is a finite union of open semialgebraic sets.*

*Proof.* By Proposition 5.6, the real and imaginary parts of  $g_u(2c)$  are polynomials in  $(x, y)$ . In a fixed quadrant, the signs of  $x$  and  $y$  resolve the absolute values in the trap half-widths. Squaring and clearing positive denominators then expresses the two strict trap inequalities as polynomial inequalities in  $(x, y)$ . The lens condition is also semialgebraic, and the openness was proved in Lemma 6.34.  $\square$

**Lemma 6.38** (Restricted polynomials give trap witnesses). *Let  $c \in \mathcal{R}_n \cap (\mathcal{X}_n \setminus \mathbb{R})$ . Then  $c \in \Theta_{\text{fin}}(n)$ . More precisely, if*

$$P(z) = z^m + d_1 z^{m-1} + \cdots + d_m, \quad d_j \in D_n, \quad P(c) = 0,$$

*then there is a word  $u \in A_N^m$  such that  $g_u(2c) = 0 \in \mathcal{C}_0(c)$ .*

*Proof.* Set  $t_{j-1} = 2(-d_j) \in A_N$  for  $1 \leq j \leq m$ , and let  $u = t_0 \cdots t_{m-1}$ . The polynomial-evaluation identity in Proposition 5.6 gives  $g_u(2c) = 2cP(c) = 0$ . The origin belongs to the open canonical trap, so  $u$  witnesses capture.  $\square$

**Capture charts and polynomial roots.** Figure 6.8 shows how these charts overlap at depth three. Each root of  $P_u$  in the non-real lens belongs to  $\mathcal{U}_u$  by exact landing, but need not lie at its geometric center. For the highlighted word  $u = (10, -40, -40)$ ,

$$P_u(z) = z^3 - 5z^2 + 20z + 20.$$

Its first-quadrant root  $c_*$  lies in  $\mathcal{X}_{21}$  and satisfies

$$g_{(10, -40)}(2c_*) = -40 \in \mathcal{C}(c_*, 41), \quad g_u(2c_*) = 0.$$

Indeed, the vertical coordinate of  $-40$  is zero and its slanted coordinate has magnitude  $40|\operatorname{Im} c_*|/|c_*|$ , strictly less than the trap half-width  $41|\operatorname{Im} c_*|/|c_*|$ . Thus  $c_*$  is already captured by depth two. Its length-three chart nevertheless contributes points to the next layer shown in the figure: the length of the chart-defining word can exceed the first-capture depth of its associated root.

Combining Lemma 6.38 with the root-closure theorem now identifies the closure of the capture union.

**Theorem 6.39** (Non-real closure in the parameter lens). *For every  $n \geq 2$ ,*

$$(\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R} = \overline{\Theta_{\text{fin}}(n)} \cap (\mathcal{X}_n \setminus \mathbb{R}).$$

*Proof.* By Proposition 6.23,  $\Theta_{\text{fin}}(n) \subset \mathcal{M}_n$ . Since  $\mathcal{M}_n$  is relatively closed in  $\mathbb{C} \setminus \overline{\mathbb{D}}$  by Theorem 5.5, taking closures gives

$$\overline{\Theta_{\text{fin}}(n)} \cap (\mathcal{X}_n \setminus \mathbb{R}) \subseteq (\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R}.$$

Conversely, let  $c \in (\mathcal{M}_n \cap \mathcal{X}_n) \setminus \mathbb{R}$ . The root-closure identity of Theorem 5.5 gives a sequence  $(c_m)_{m \geq 1} \in \mathcal{R}_n^{\mathbb{N}}$  with  $c_m \rightarrow c$ . The set  $\mathcal{X}_n \setminus \mathbb{R}$  is open, so  $c_m \in \mathcal{X}_n \setminus \mathbb{R}$  for all sufficiently large  $m$ . For those indices, Lemma 6.38 gives  $c_m \in \Theta_{\text{fin}}(n)$ . Hence

$$c \in \overline{\Theta_{\text{fin}}(n)} \cap (\mathcal{X}_n \setminus \mathbb{R}),$$

which proves the reverse inclusion.  $\square$

The closure is essential here: the approximating capture depths may increase without bound. The theorem recovers the locus by approximation; the capture filtration need not contain every interior parameter (Section C.2).

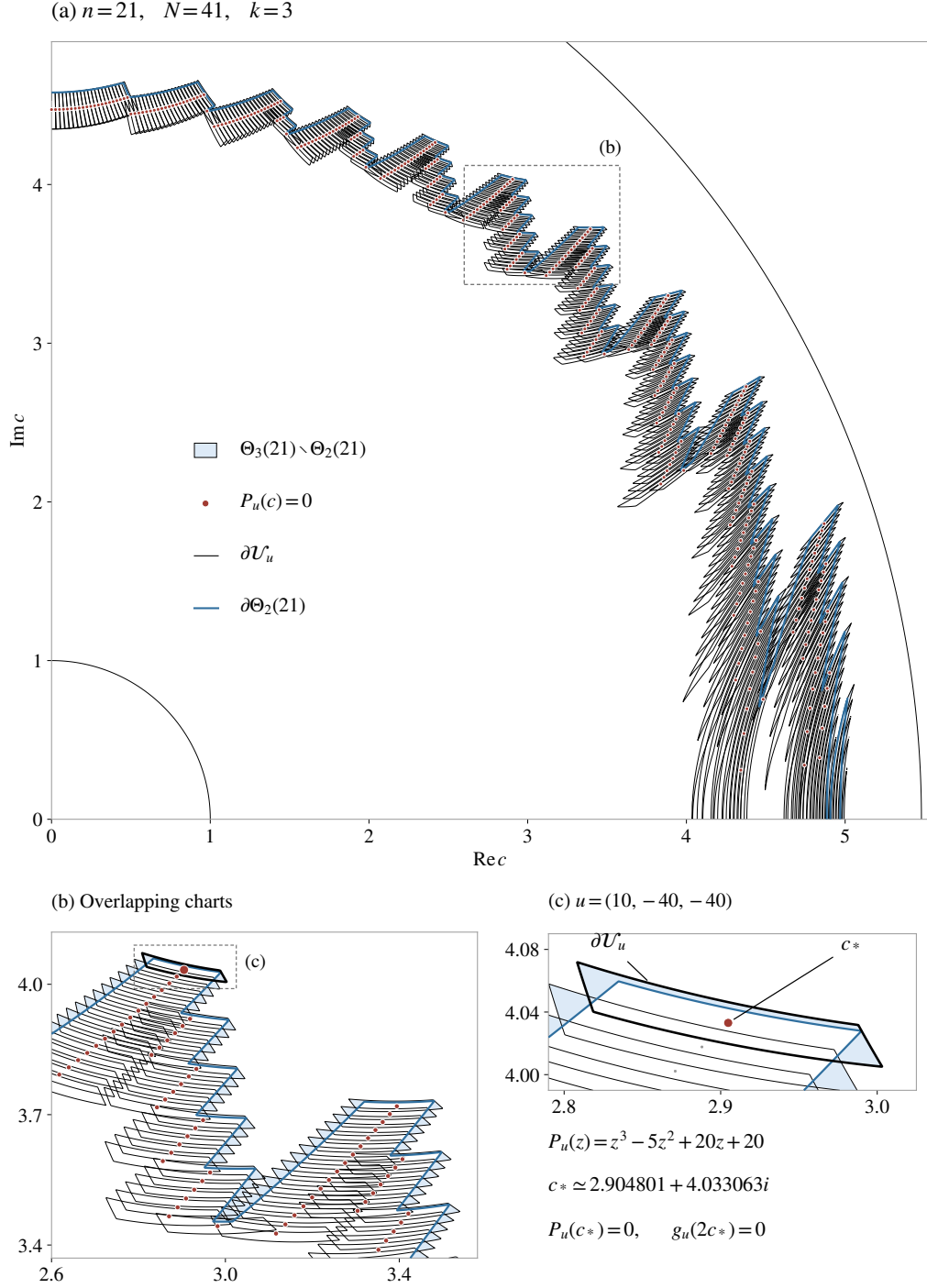
**Corollary 6.40** (Completion by the real trace in the parameter lens). *For every  $n \geq 2$ ,*

$$\mathcal{M}_n \cap \mathcal{X}_n = (\overline{\Theta_{\text{fin}}(n)} \cup (\partial \mathcal{M}_n \cap \mathbb{R})) \cap \mathcal{X}_n.$$

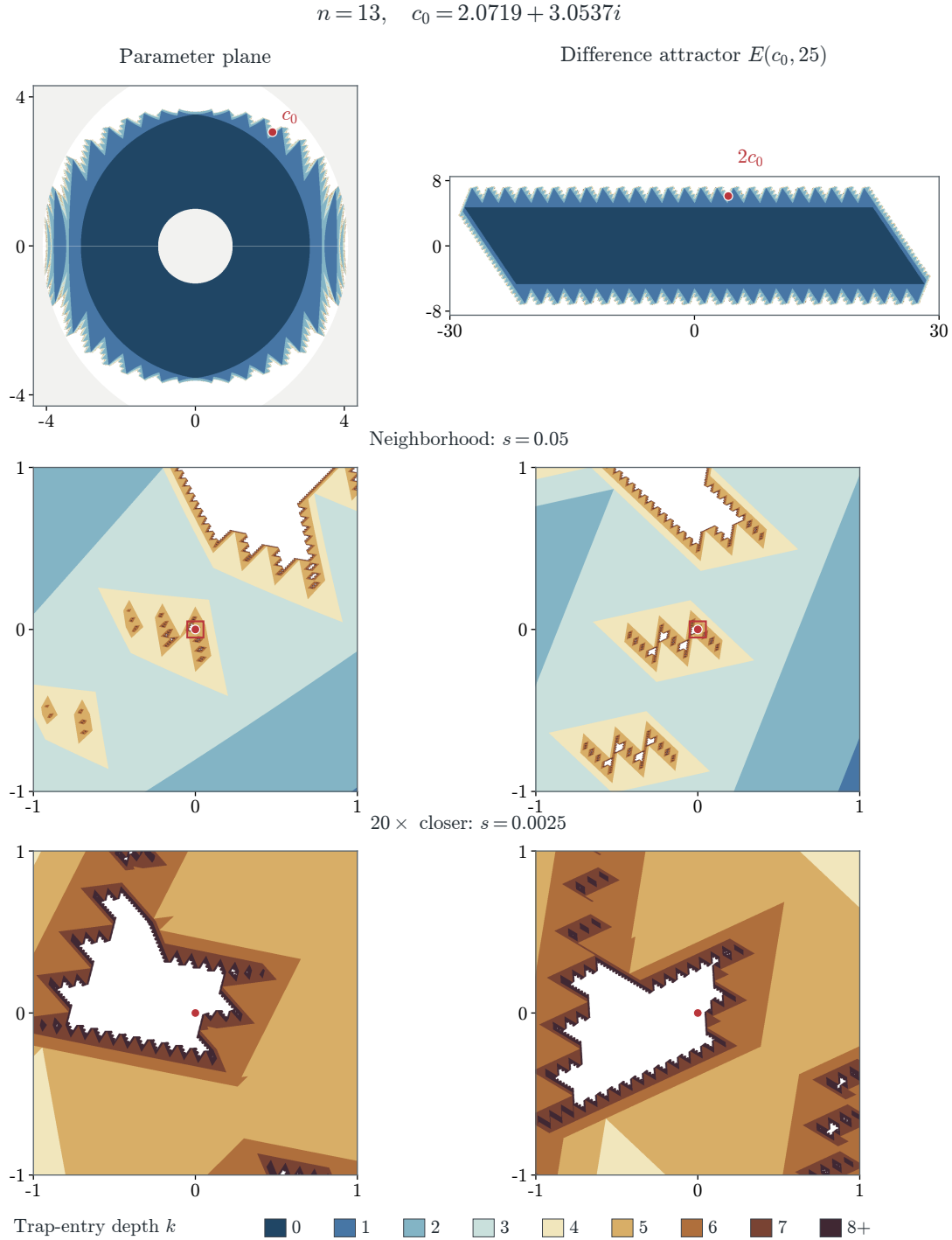
*Proof.* The non-real part follows from Theorem 6.39. A real point of  $\mathcal{M}_n \cap \mathcal{X}_n$  either belongs to  $\partial \mathcal{M}_n \cap \mathbb{R}$  or lies in  $\operatorname{int}(\mathcal{M}_n) \cap \mathcal{X}_n$ . In the latter case, approximate it by non-real points of this open set. The theorem places those points in  $\overline{\Theta_{\text{fin}}(n)}$ , and closedness places their limit there as well.

Conversely,  $\Theta_{\text{fin}}(n) \subset \mathcal{M}_n$  by Proposition 6.23, and  $\mathcal{M}_n$  is relatively closed in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . Hence  $\overline{\Theta_{\text{fin}}(n)} \subset \mathcal{M}_n$ . Moreover,  $\partial \mathcal{M}_n \cap \mathbb{R} \subset \mathcal{M}_n$ . Intersecting with  $\mathcal{X}_n$  proves the reverse inclusion.  $\square$

For  $n = 13$ , Figure 6.9 compares successive magnifications of parameter and dynamical space near an apparent inner loop.



**Figure 6.8.** Length-three capture charts for  $n = 21, N = 41$ . (a) Numerically resolved chart components meeting  $\Theta_3(21) \setminus \Theta_2(21)$ ; (b), (c) successive enlargements. Light blue marks the resolved new-layer regions, dark blue the boundary of  $\Theta_2(21)$  relative to the non-real lens, and black the individual chart boundaries. Dots mark associated polynomial roots. Components sampled as lying entirely in  $\Theta_2(21)$  are omitted. The thin outer arc is part of  $\partial \mathcal{X}_{21}$ ; the lower-left arc is  $|c| = 1$ . In (c), the bold contour selects the root-containing component of  $\mathcal{U}_u$ , with  $u = (10, -40, -40)$ ; its root  $c_*$  is already captured by depth two. Curves, root positions, and component selection are numerical, not a certified exhaustive classification.



**Figure 6.9.** Numerical finite-capture fields for  $n = 13$ , with  $c_0 = 2.0719 + 3.0537i$ . The full views above locate the paired magnifications of the parameter plane and  $E(c_0, 25)$ . Local coordinates are  $(c - c_0)/s$  on the left and  $(z - 2c_0)/(4s)$  on the right, for the indicated values of  $s$ ; red boxes locate the final row. The colours give first trap-entry depths up to 20, with 8+ grouping later entry. White denotes numerical exclusion and light gray the region outside the canonical parameter lens.

## 6.7. Sharp Lens Containment

Lens containment makes the closure description of Theorem 6.39 global away from the real axis. We prove this containment for every  $n \geq 20$ , and exhibit off-lens roots for every  $2 \leq n \leq 19$ . The starting estimate

$$\mathcal{M}_n \setminus \mathbb{R} \subset B(0, 1 + \sqrt{n-1})$$

follows from the convexity bound in Proposition 3.9. Here we derive it from the exact enclosure of Proposition 6.14 in the canonical coordinates of Definition 6.1. The disk estimate gives lens containment for  $n \geq 21$ . At  $n = 20$ , the enclosure also restricts the first inverse digit, excluding the small off-lens region left by the disk bound.

**Proposition 6.41** (Enclosure proof of the non-real disk bound). *For every  $n \geq 2$ ,*

$$\mathcal{M}_n \setminus \mathbb{R} \subset \{c \in \mathbb{C} : |c| < 1 + \sqrt{n-1}\}.$$

*Proof.* Let  $c = \rho e^{i\theta} = x + iy \notin \mathbb{R}$ , and suppose

$$\rho \geq 1 + \sqrt{n-1}.$$

We show that  $2c \notin \mathcal{E}(c, N)$ . Since  $|\ell_v(2c)| = 2|y|$ , it is enough to prove  $\mathcal{V}_{\mathcal{E}}(c, N) < 2|y|$ . Using  $|\sin(k\theta)| \leq k|\sin \theta|$ , with strict inequality for at least one term because  $c \notin \mathbb{R}$ , we get

$$\mathcal{V}_{\mathcal{E}}(c, N) = (N-1) \sum_{k=1}^{\infty} \rho^{-k} |\sin(k\theta)| < (N-1) |\sin \theta| \sum_{k=1}^{\infty} k \rho^{-k}.$$

Since

$$\sum_{k=1}^{\infty} k \rho^{-k} = \frac{\rho}{(\rho-1)^2} \quad \text{and} \quad |y| = \rho |\sin \theta|,$$

this gives

$$\mathcal{V}_{\mathcal{E}}(c, N) < \frac{(N-1)|y|}{(\rho-1)^2} = \frac{2(n-1)|y|}{(\rho-1)^2}.$$

The assumption  $\rho \geq 1 + \sqrt{n-1}$  implies

$$(\rho-1)^2 \geq n-1,$$

and hence

$$\mathcal{V}_{\mathcal{E}}(c, N) < 2|y|.$$

Therefore  $2c \notin \mathcal{E}(c, N)$ , so  $2c \notin E(c, N)$ . By Proposition 5.2,  $c \notin \mathcal{M}_n$ . Thus every non-real point of  $\mathcal{M}_n$  satisfies

$$|c| < 1 + \sqrt{n-1}.$$

□

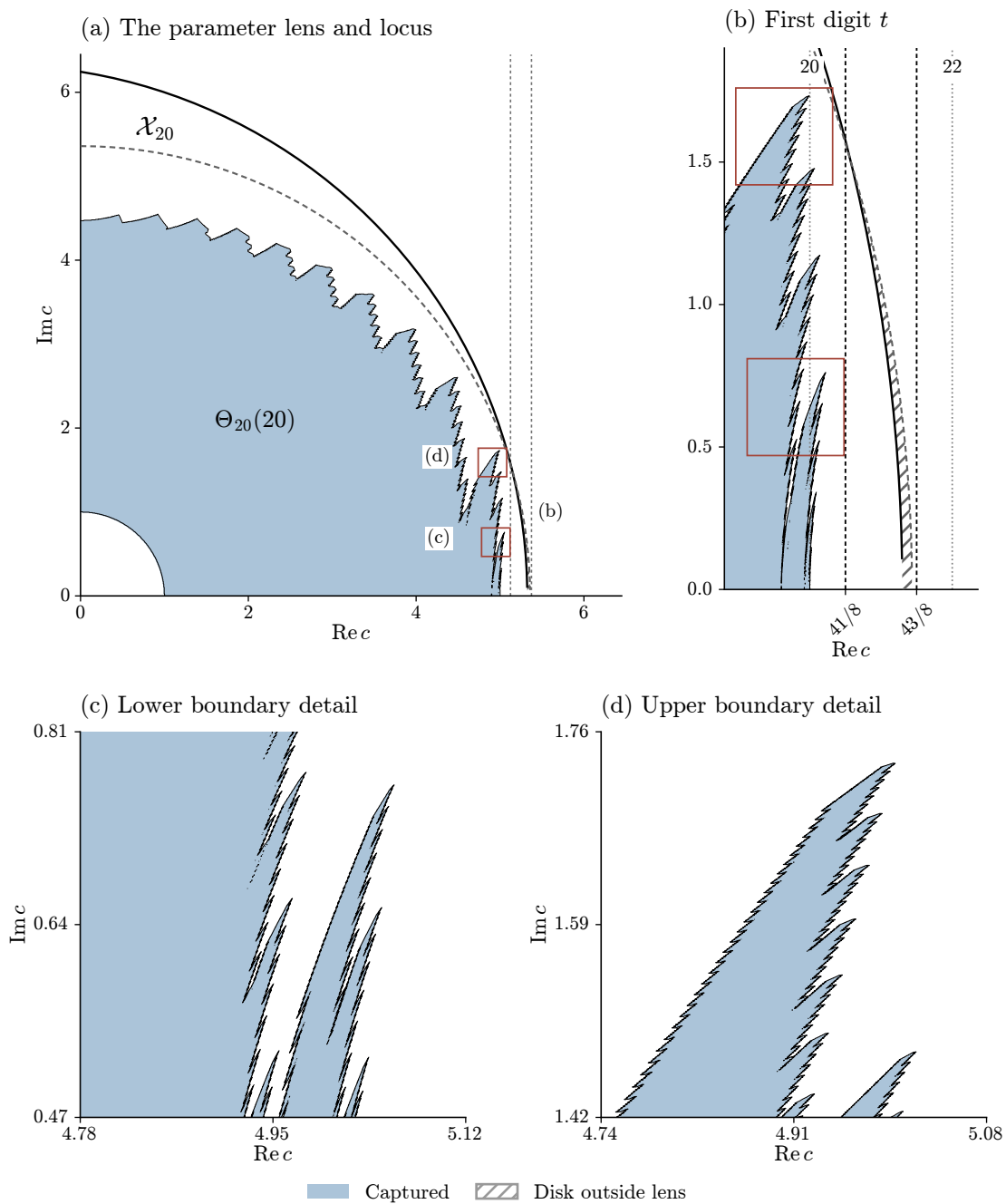
**Corollary 6.42.** *For every  $n \geq 21$ ,  $\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n$ .*

*Proof.* By Proposition 6.41,  $|c| < 1 + \sqrt{n-1}$  for  $c \in \mathcal{M}_n \setminus \mathbb{R}$ . For  $n \geq 21$ ,

$$1 + \sqrt{n-1} < \sqrt{2n} - 1,$$

so  $|c \pm 1| \leq |c| + 1 < \sqrt{2n}$ . These are the inequalities defining  $\mathcal{X}_n$  in the exterior parameter plane. □

For  $n = 20$ , part of  $B(0, 1 + \sqrt{19})$  lies outside  $\mathcal{X}_{20}$ , see Figure 6.10.



**Figure 6.10.** Off-lens geometry for  $n = 20$ . (a) The solid and dashed curves mark  $|c + 1| = \sqrt{40}$  and  $|c| = 1 + \sqrt{19}$ . (b) Hatching marks the disk outside the lens. In  $41/8 < x < 43/8$ , the vertical enclosure bound permits only first digits 20, 22.

**Proposition 6.43** (The borderline case  $n = 20$ ).

$$\mathcal{M}_{20} \setminus \mathbb{R} \subset \mathcal{X}_{20}.$$

*Proof.* Set  $n = 20$ , so  $N = 39$ . We restrict a possible off-lens parameter and its first inverse digit until the slanted enclosure bound fails. Suppose, for contradiction, that

$$c = x + iy \in \mathcal{M}_{20} \setminus \mathcal{X}_{20} \quad \text{and} \quad y \neq 0.$$

Using the symmetries  $c \mapsto \bar{c}$  and  $c \mapsto -c$ , we may assume  $x \geq 0$  and  $y > 0$ . Write  $\rho = |c|$ .

By Proposition 6.41,

$$\rho < 1 + \sqrt{19}.$$

Since  $x \geq 0$ , one has  $|c + 1| \geq |c - 1|$ . Thus  $c \notin \mathcal{X}_{20}$  implies

$$|c + 1| \geq \sqrt{40}.$$

Equivalently,

$$\rho^2 + 2x + 1 \geq 40, \quad \text{so} \quad \rho^2 + 2x \geq 39.$$

Also,

$$\rho + 1 \geq |c + 1| \geq \sqrt{40},$$

hence

$$\rho \geq \sqrt{40} - 1 > 5.$$

Set

$$\beta = \frac{\mathcal{V}_{\mathcal{E}}(c, 39)}{y}.$$

The estimate used in Proposition 6.41 gives

$$\beta < \frac{38}{(\rho - 1)^2} < \frac{38}{16} = \frac{19}{8}.$$

Connectedness supplies the inverse step to which these bounds apply. By Proposition 5.2,  $2c \in E(c, 39)$ , so some first inverse digit  $t \in A_{39}$  satisfies

$$z_1 = g_t(2c) \in E(c, 39) \subset \mathcal{E}(c, 39).$$

At the root node  $z_0 = 2c$ , the canonical coordinates are

$$\ell_v(2c) = 2y, \quad \ell_s(2c; c) = \frac{\operatorname{Im}(2c^2)}{\rho} = \frac{4xy}{\rho}.$$

Using Lemma 6.2, we obtain

$$\ell_v(z_1) = y(4x - t)$$

and

$$\ell_s(z_1; c) = -\frac{2y}{\rho}(\rho^2 + x(t - 4x)).$$

Since  $z_1 \in \mathcal{E}(c, 39)$ , the vertical enclosure inequality gives

$$|4x - t| \leq \beta < \frac{19}{8}.$$

The slanted enclosure inequality gives

$$\frac{2y}{\rho} |\rho^2 + x(t - 4x)| \leq \mathcal{S}_{\mathcal{E}}(c, 39).$$

Using

$$\mathcal{S}_{\mathcal{E}}(c, 39) = 38\frac{y}{\rho} + \frac{1}{\rho}\mathcal{V}_{\mathcal{E}}(c, 39) = \frac{y}{\rho}(38 + \beta),$$

we get

$$|\rho^2 + x(t - 4x)| \leq \frac{38 + \beta}{2} < \frac{38 + 19/8}{2} = \frac{323}{16}.$$

The off-lens condition and disk bound also force  $x$  into a narrow strip. From

$$\rho^2 + 2x \geq 39 \quad \text{and} \quad \rho < 1 + \sqrt{19},$$

we obtain

$$2x > 39 - (1 + \sqrt{19})^2 = 19 - 2\sqrt{19}.$$

Since

$$\frac{19 - 2\sqrt{19}}{2} > \frac{41}{8},$$

we have

$$x > \frac{41}{8}.$$

On the other hand,

$$x < \rho < 1 + \sqrt{19} < \frac{43}{8}.$$

Thus

$$\frac{41}{8} < x < \frac{43}{8}.$$

This strip leaves only two possible digits. Since  $t \in A_{39}$ , the digit  $t$  is even, and the inequalities

$$\frac{41}{8} < x < \frac{43}{8}, \quad |4x - t| < \frac{19}{8}$$

force

$$t \in \{20, 22\}.$$

Both violate the slanted bound. Indeed,  $t \geq 20$  and  $\rho^2 + 2x \geq 39$  give

$$\rho^2 + x(t - 4x) \geq 39 - 2x + 20x - 4x^2 = 39 + 18x - 4x^2.$$

The function

$$h(x) = 39 + 18x - 4x^2$$

is decreasing on  $(41/8, 43/8)$ , because  $h'(x) = 18 - 8x < 0$  there. Hence

$$h(x) > h(43/8) = \frac{323}{16}.$$

Therefore

$$\rho^2 + x(t - 4x) > \frac{323}{16},$$

contradicting the slanted enclosure bound

$$|\rho^2 + x(t - 4x)| < \frac{323}{16}.$$

This contradiction proves

$$\mathcal{M}_{20} \setminus \mathbb{R} \subset \mathcal{X}_{20}.$$

□

**Theorem 6.44** (The sharp  $n \geq 20$  lens-containment threshold). *For every  $n \geq 20$ ,*

$$\mathcal{M}_n \setminus \mathbb{R} \subset \mathcal{X}_n.$$

*For every  $2 \leq n \leq 19$ , there exists a non-real parameter in  $\mathcal{M}_n \setminus \mathcal{X}_n$ .*

*Proof.* For  $n \geq 21$ , use Corollary 6.42. The borderline case  $n = 20$  is Proposition 6.43. For each  $2 \leq n \leq 19$ , Proposition A.2 provides a Rouché-certified non-real root in  $\mathcal{M}_n \setminus \mathcal{X}_n$ . The certificate data are listed in Table A.1 in Appendix A.  $\square$

**Corollary 6.45** (Closure of the finite-capture locus). *If  $n \geq 20$ , then*

$$\mathcal{M}_n = \overline{\Theta_{\text{fin}}(n)} \cup (\partial\mathcal{M}_n \cap \mathbb{R}).$$

*Proof.* By Theorems 6.39 and 6.44,

$$\mathcal{M}_n \setminus \mathbb{R} \subset \overline{\Theta_{\text{fin}}(n)}.$$

If  $c \in \text{int}(\mathcal{M}_n) \cap \mathbb{R}$ , choose non-real points of  $\text{int}(\mathcal{M}_n)$  converging to  $c$ . The inclusion just proved places these points in  $\overline{\Theta_{\text{fin}}(n)}$ , and closedness places their limit there as well. The remaining real points of  $\mathcal{M}_n$  belong to  $\partial\mathcal{M}_n \cap \mathbb{R}$ . This proves

$$\mathcal{M}_n \subset \overline{\Theta_{\text{fin}}(n)} \cup (\partial\mathcal{M}_n \cap \mathbb{R}).$$

The reverse inclusion follows from  $\Theta_{\text{fin}}(n) \subset \mathcal{M}_n$ , the relative closedness of  $\mathcal{M}_n$ , and  $\partial\mathcal{M}_n \cap \mathbb{R} \subset \mathcal{M}_n$ .  $\square$

The two canonical coordinates become dependent on the real axis, where the trap degenerates. The term  $\partial\mathcal{M}_n \cap \mathbb{R}$  records the real boundary trace needed to complete the closure description.

**Corollary 6.46** (Density of finite capture). *For every  $n \geq 2$ , the finite-capture locus  $\Theta_{\text{fin}}(n)$  is open and dense in*

$$(\text{int}(\mathcal{M}_n) \cap \mathcal{X}_n) \setminus \mathbb{R}.$$

*If  $n \geq 20$ , then  $\Theta_{\text{fin}}(n)$  is open and dense in*

$$\text{int}(\mathcal{M}_n) \setminus \mathbb{R}.$$

*Consequently, for  $n \geq 20$ ,*

$$\overline{\Theta_{\text{fin}}(n)} = \overline{\text{int}(\mathcal{M}_n)}.$$

*Proof.* By Lemma 6.34,  $\Theta_{\text{fin}}(n)$  is open in  $\mathcal{X}_n \setminus \mathbb{R}$ , hence in the exterior parameter plane. Together with  $\Theta_{\text{fin}}(n) \subset \mathcal{M}_n$  from Proposition 6.23, this gives

$$\Theta_{\text{fin}}(n) \subset (\text{int}(\mathcal{M}_n) \cap \mathcal{X}_n) \setminus \mathbb{R}.$$

Every point of the set on the right belongs to  $\overline{\Theta_{\text{fin}}(n)}$  by Theorem 6.39, proving the first density statement. For  $n \geq 20$ , Theorem 6.44 places all non-real points of  $\mathcal{M}_n$  in  $\mathcal{X}_n$ , so the same conclusion holds throughout  $\text{int}(\mathcal{M}_n) \setminus \mathbb{R}$ .

The non-real part of an open subset of the plane is dense in that subset. Thus, for  $n \geq 20$ , the density just proved and  $\Theta_{\text{fin}}(n) \subset \text{int}(\mathcal{M}_n)$  give

$$\overline{\Theta_{\text{fin}}(n)} = \overline{\text{int}(\mathcal{M}_n)}.$$

$\square$



PART IV.

# Synthesis

Chapters 7 and 8

*Mathematics is not about numbers, equations, computations,  
or algorithms: it is about understanding.*

— William Thurston



# Synthesis and Discussion

The preceding chapters distinguish three ways to obtain information from an address: an exact algebraic relation, a stable interior certificate, and approximation by parameters carrying finite witnesses. Comparing these uses explains both the strength of finite capture and the points its canonical trap misses. The dependence on search depth gives a second comparison: how the captured regions fit together and how many inverse branches can survive.

## 7.1. What a Finite Address Certifies

The address of the marked point links geometric overlap to a signed expansion of 1. A terminating address gives a restricted monic polynomial root; taking the closure of these roots recovers  $\mathcal{M}_n$  outside the closed unit disk (Theorem 5.5). This closure statement includes parameters whose addresses do not terminate.

Finite capture asks for a different property of a finite word. Its inverse orbit need only enter the trap, where self-covering supplies an infinite continuation. Strict entry persists under small parameter changes and therefore certifies interior membership. Exact landing records an algebraic identity; trap entry records stable continuation.

The enclosure supplies the exterior test. Off the real axis, extinction of every enclosure-admissible branch at a finite depth rules out every address. For every  $n \geq 3$ , even the interior within the lens contains points never captured by the canonical trap (Section C.2).

## 7.2. From Covering to Finite Certificates

The rectangle and canonical trap use self-covering geometry at different stages of the argument. The rectangle in Chapter 4 supplies a family of subsets of the difference attractor throughout the two-disk lens  $\mathcal{X}_n$ . Near a non-real finite reciprocal zero in the lens, the residual

$$p(c) = 1 + \sum_{k=1}^m d_k c^{-k}$$

enters the scaled rectangle  $(2c^{m+1})^{-1}R(c, N)$ . This dynamical-plane inclusion places a neighborhood of the parameter in  $\mathcal{M}_n$ . Together with approximation by finite reciprocal zeros, it proves density of interior parameters in the non-real part of the locus inside the lens.

The canonical trap in Chapter 6 makes this continuation mechanism an explicit inverse test for  $2c$ . Once a backward branch enters the trap, the corresponding forward composition places  $2c$  in the interior of  $E(c, N)$ . An outer enclosure, expressed in the same coordinates, prunes the search: a branch that leaves it cannot begin an address of  $2c$  in  $E(c, N)$ . Finite capture therefore records both a certificate and the depth at which it first becomes available.

The closure of the captured parameter set recovers the non-real locus inside the lens by Theorem 6.39. The sharp lens-containment result (Theorem 6.44) makes this recovery global

off the real axis for  $n \geq 20$ . Table 7.1 compares the certificates; Section 7.4 explains the threshold.

| Construction        | Finite witness                                           | Immediate conclusion and domain                                          | Further use                                                                        |
|---------------------|----------------------------------------------------------|--------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| Rectangle           | Reciprocal residual strictly inside the scaled rectangle | Interior membership in the non-real lens, every $n \geq 2$               | Interior approximation; disk comparison gives the global result for $n \geq 21$    |
| Canonical trap      | A finite inverse word enters the trap                    | Interior membership in the non-real lens, every $n \geq 2$               | Capture layers and closure; lens containment gives global recovery for $n \geq 20$ |
| Canonical enclosure | Every admissible branch is extinct at a finite depth     | Nonmembership in $\mathcal{M}_n$ , for $c \notin \mathbb{R}$ , $ c  > 1$ | Exterior certification                                                             |
| Root disks          | Strict Rouché inequality and off-lens disk bounds        | A non-real restricted root outside the lens                              | Witnesses for $2 \leq n \leq 19$ establish sharpness of lens containment           |

**Table 7.1.** Finite certificates, their domains, and the further conclusions obtained by combining them with approximation or containment results.

### 7.3. Layered Interior and Bounded Boundary Branching

Openness and semialgebraicity describe an individual capture stage: in each open quadrant,  $\Theta_k(n)$  is a finite union of open semialgebraic sets (Lemmas 6.34 and 6.37). The two-step closure theorem (Theorem 6.35) describes how the stages fit together:

$$\partial_{\mathcal{X}_n \setminus \mathbb{R}} \Theta_k(n) \subset \Theta_{k+2}(n) \setminus \Theta_k(n).$$

Each finite-depth frontier is therefore absorbed by depth  $k + 2$ , as long as the point remains in  $\mathcal{X}_n \setminus \mathbb{R}$ . The certified interior unfolds layer by layer with a uniform bound on this extra depth. The required total depth can still grow without bound from one parameter to another.

The enclosure also limits the branching of the inverse search. For non-real parameters with  $|c| \geq \sqrt{n}$ , the bound in Theorem 6.28 reduces a depth- $k$  enumeration from  $(2n - 1)^k$  words to at most  $3^k$  admissible nodes when  $n \geq 25$ . Thus increasing the alphabet no longer increases the exponential base of this bound. Since every non-real boundary parameter lies in  $|c| \geq \sqrt{n}$ , it applies throughout a region containing the non-real boundary. The distinction between the exact tree and a numerically enlarged working tree is discussed in Section 7.5.

### 7.4. Why the Two Thresholds Differ

The thresholds  $n \geq 21$  and  $n \geq 20$  come from two ways of comparing the same non-real bound with the parameter lens. The inclusion

$$\mathcal{M}_n \setminus \mathbb{R} \subset B(0, 1 + \sqrt{n - 1})$$

is first obtained from the convexity-set estimate and later reproved in canonical coordinates. By the triangle inequality, the exterior part

$$B(0, 1 + \sqrt{n-1}) \setminus \overline{\mathbb{D}}$$

of this disk is contained in the two-disk parameter lens  $\mathcal{X}_n$  only for  $n \geq 21$ . This is the threshold for the earlier rectangle-covering proof of regular-closedness away from the real axis.

At  $n = 20$ , the disk  $B(0, 1 + \sqrt{n-1})$  still leaves a residual off-lens region, and this region is excluded directly from  $\mathcal{M}_n$  by the canonical-enclosure inequalities in Proposition 6.43. The certified off-lens roots of Proposition A.2 for  $2 \leq n \leq 19$  show that the resulting lens-containment threshold is sharp. Thus the improvement comes from excluding the residual region, while retaining the same disk radius. Together with Theorem 6.39, this sharper containment lets the closure of finite capture recover the whole non-real locus for  $n \geq 20$ .

## 7.5. Scope and Limitations

**The off-lens range.** For each  $2 \leq n \leq 19$ , Theorem 6.44 supplies an explicit non-real witness  $c_n \in \mathcal{M}_n \setminus \mathcal{X}_n$ . The canonical trap is both nonempty and self-covering precisely on  $\mathcal{X}_n \setminus \mathbb{R}$  (Corollary 6.11), so its interior certificate is unavailable at these parameters. An off-lens finite-capture description requires other traps. Whether their finite-depth frontiers satisfy a uniform absorption rule is a further question, developed in Section C.1.

**The real trace.** The closure identities retain  $\partial\mathcal{M}_n \cap \mathbb{R}$  as an additional set. The two-dimensional coordinates of the trap and enclosure degenerate on  $\mathbb{R}$ , although the difference IFS remains contracting for  $|c| > 1$ . Classifying the real trace as planar interior or boundary therefore requires separate information, see Section C.4.

**The residual interior set.** The finite-capture locus  $\Theta_{\text{fin}}(n)$  is open and dense in  $\text{int}(\mathcal{M}_n) \cap (\mathcal{X}_n \setminus \mathbb{R})$  by Corollary 6.46. The parameters omitted by the canonical test form the residual interior set

$$\Phi_n := \text{int}(\mathcal{M}_n) \setminus \Theta_{\text{fin}}(n).$$

Its non-real lens part is relatively closed and nowhere dense in  $\text{int}(\mathcal{M}_n) \cap (\mathcal{X}_n \setminus \mathbb{R})$ , but it is nonempty for every  $n \geq 3$ . The quadratic parameters in Section C.2 are isolated points of this residual set [Esp26f, Theorem 1.1]. These examples separate recovery by closure from exhaustive interior detection.

**Exact versus working search trees.** Theorem 6.28 bounds the exact  $\mathcal{E}(c, N)$ -admissible tree. An outward-rounded working enclosure may retain additional branches, and a search can continue indefinitely when neither trap entry nor enclosure extinction occurs. The bound of three admissible continuations per node for  $n \geq 25$  therefore controls exact branching without guaranteeing an implementation's running time. The off-lens certificates also illustrate the need for careful arithmetic near the threshold: the witnesses for  $n = 17$  and  $n = 19$  in Table A.1 use Rouché radii of order  $10^{-6}$  and  $10^{-9}$ , respectively.

## 7.6. Directions Arising from the Results

### Parameter and dynamical space near boundary points

At a finite reciprocal zero, the same finite address relation governs both the parameter equation and the position of  $2c$  in  $E(c, N)$ . This suggests comparing their local geometry after suitable rescaling. An asymptotic self-similarity theorem for  $\mathcal{M}_n$  would provide an  $n$ -ary counterpart to Solomyak’s binary result [Sol05]. The boundary-accessibility criteria of Silvestri and Pérez [SP23] suggest a complementary question: under which address conditions can such a comparison produce a path in the complement approaching the parameter? Both questions require control of local geometry beyond the closure statements proved here.

Bandt’s magnification-flow algorithm for self-similar sets [Ban25a] provides a related computational approach. Comparing its descriptions of local scenery with finite-capture data may help relate parameter-space magnifications to dynamical-plane magnifications near the marked point  $2c$ .

### Constrained addresses and measures

Restricting the admissible infinite words produces the base loci introduced in Section 1.3. The finite search suggests corresponding languages of exact landing, trap entry, and enclosure survival. Section C.3 asks when these languages admit finite-state descriptions and how their automata relate to neighbor graphs.

The same address differences also enter the theory of self-similar measures. Solomyak and Xu connect the binary locus with complex Bernoulli convolutions [SX03], and the general absolute-continuity theorem of Shmerkin and Solomyak applies directly to every collinear alphabet [SS16]. The remaining effective questions concern specified parameters, exceptional sets, and quantitative regularity. Kleptsyn, Pollicott, and Vytnova’s diffusion-operator method gives rigorous dimension bounds for Bernoulli convolutions and related affine IFS measures [KPV22]; developing comparable computational bounds for the complex collinear family is a further direction.

# Conclusions

This thesis identifies the collinear connectedness loci with exterior closures of restricted monic polynomial roots:

$$\mathcal{M}_n = \overline{\mathcal{R}_n} \setminus \overline{\mathbb{D}}.$$

The common witness is an address of the marked point in the difference attractor: terminating addresses give polynomial roots, and approximation by finite words recovers the full locus. The resulting geometric setting gives nesting, connectedness, local connectedness, the exact real trace, and radial bounds on the non-real boundary.

The explicit parametric rectangle construction turns these algebraic relations into stable interior information. It proves that every non-real point of  $\mathcal{M}_n \cap \mathcal{X}_n$  is approximated by interior parameters; the disk comparison extends this conclusion to the whole non-real locus for  $n \geq 21$  (Theorem 4.5).

The canonical trap and enclosure make finite words into verifiable certificates. Trap entry certifies interior membership within the non-real lens; extinction of every enclosure-admissible branch certifies nonmembership at a non-real parameter with  $|c| > 1$ . These two outcomes support a finite search that reports its unresolved cases explicitly.

The capture filtration describes how such certificates accumulate. Its fixed-depth stages are open and semialgebraic in each open quadrant, and every finite-depth frontier within  $\mathcal{X}_n \setminus \mathbb{R}$  is absorbed after at most two further inverse steps. The closure of their union recovers the non-real locus inside the lens. The refined enclosure estimate at  $n = 20$  makes this recovery global off the real axis for  $n \geq 20$ . Certified off-lens roots for  $2 \leq n \leq 19$  establish the sharpness of this lens-containment threshold and locate parameters requiring other trap geometries.

Finally, the enclosure controls the branching of the exact search. For  $n \geq 25$  and non-real  $c$  with  $|c| \geq \sqrt{n}$ , each node has at most three admissible continuations, independently of the difference alphabet's size. This region contains every non-real boundary parameter. The companion software makes the geometry explorable and exports finite words and search data for verification. Finite capture thus supplies both individual certificates and a description of their depth and branching across parameter space.



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# Finite Certificates

This appendix supplies the off-lens witnesses that make the threshold in Theorem 6.44 sharp. Each row of Table A.1 consists of a polynomial, a parameter disk, and two strict margins, recorded as the certificate tuple

$$(P_n, \tilde{c}_n, r_n, \Delta_n, \Lambda_n).$$

Here  $P_n$  is monic with non-leading coefficients in  $D_n$ , and the closed disk has center  $\tilde{c}_n$  and radius  $r_n$ . The first margin is

$$\Delta_n = |P'_n(\tilde{c}_n)|r_n - |P_n(\tilde{c}_n)| - \mathcal{R}_{P_n}(\tilde{c}_n, r_n),$$

so  $\Delta_n > 0$  verifies the Rouché hypothesis in Lemma A.1. The second margin is

$$\Lambda_n = (|\tilde{c}_n + 1| - r_n)^2 - 2n.$$

For every listed disk,  $\text{Im}(\tilde{c}_n) > r_n$ , which also gives  $|\tilde{c}_n + 1| - r_n > 0$ . Hence  $\Lambda_n > 0$  places the entire disk outside  $\mathcal{X}_n$ , since  $|\tilde{c}_n + 1| - r_n > \sqrt{2n}$ . The two margins isolate a root and certify its required location.

**Lemma A.1** (Rouché disk test). *Let  $P(z) = \sum_{k=0}^d a_k z^k \in \mathbb{C}[z]$ ,  $z_0 \in \mathbb{C}$ , and  $r > 0$ . Define*

$$\mathcal{R}_P(z_0, r) = \sum_{k=2}^d |a_k|((|z_0| + r)^k - |z_0|^k - k|z_0|^{k-1}r).$$

*If*

$$|P(z_0)| + \mathcal{R}_P(z_0, r) < |P'(z_0)|r,$$

*then  $P$  has exactly one zero in  $\overline{B(z_0, r)}$ , counted with multiplicity.*

*Proof.* Let

$$L(z) = P(z_0) + P'(z_0)(z - z_0).$$

For  $|z - z_0| = r$ , Taylor's formula gives

$$|P(z) - L(z)| \leq \mathcal{R}_P(z_0, r),$$

while

$$|L(z)| \geq |P'(z_0)|r - |P(z_0)|.$$

The stated inequality gives

$$|P(z) - L(z)| < |L(z)| \quad (|z - z_0| = r).$$

By Rouché's theorem,  $P$  and  $L$  have the same number of zeros in  $B(z_0, r)$ , counted with multiplicity. Since  $L$  is affine, its unique zero is

$$z_0 - \frac{P(z_0)}{P'(z_0)}.$$

| $n$ | $P_n(c)$                                    | $\tilde{c}_n$                       | $r_n$       | $\Delta_n >$ | $\Lambda_n >$            |
|-----|---------------------------------------------|-------------------------------------|-------------|--------------|--------------------------|
| 2   | $c^3 - c^2 + 1$                             | $0.877438833123 + 0.744861766620 i$ | 0.001       | 0.00266845   | 0.0755570                |
| 3   | $c^3 - 2c^2 + 2$                            | $1.419643377607 + 0.606290729207 i$ | 0.001       | 0.00282944   | 0.217274                 |
| 4   | $c^3 - 3c^2 + c + 3$                        | $1.884646177119 + 0.589742805022 i$ | 0.001       | 0.00319771   | 0.663092                 |
| 5   | $c^3 - 4c^2 + 3c + 3$                       | $2.273409138442 + 0.563821092829 i$ | 0.001       | 0.00323211   | 1.02645                  |
| 6   | $c^3 - 5c^2 + 5c + 4$                       | $2.755773570847 + 0.474476778007 i$ | 0.001       | 0.00311966   | 2.32339                  |
| 7   | $c^3 - 5c^2 + 5c + 4$                       | $2.755773570847 + 0.474476778007 i$ | 0.001       | 0.00311966   | 0.323393                 |
| 8   | $c^3 - 6c^2 + 7c + 7$                       | $3.313682542356 + 0.421052806988 i$ | 0.001       | 0.00332164   | 2.77647                  |
| 9   | $c^3 - 6c^2 + 7c + 7$                       | $3.313682542356 + 0.421052806988 i$ | 0.001       | 0.00332164   | 0.776475                 |
| 10  | $c^3 - 6c^2 + 8c + 9$                       | $3.353263977249 + 1.222280727312 i$ | 0.001       | 0.0103477    | 0.435835                 |
| 11  | $c^3 - 7c^2 + 10c + 9$                      | $3.806735133791 + 0.423562585034 i$ | 0.001       | 0.00374312   | 1.27445                  |
| 12  | $c^3 - 7c^2 + 10c + 11$                     | $3.855311981586 + 0.784808055439 i$ | 0.001       | 0.00725305   | 0.180142                 |
| 13  | $c^3 - 7c^2 + 12c + 12$                     | $3.846271905939 + 1.591733658655 i$ | 0.001       | 0.0152924    | 0.00976647               |
| 14  | $c^3 - 8c^2 + 13c + 13$                     | $4.342889763045 + 0.309577572123 i$ | 0.001       | 0.00309835   | 0.631606                 |
| 15  | $c^4 - 8c^3 + 14 \sum_{j=0}^2 c^j$          | $4.459402595334 + 1.287852995168 i$ | 0.001       | 0.0673229    | 1.45242                  |
| 16  | $c^4 - 8c^3 + 15 \sum_{j=0}^2 c^j$          | $4.462920327187 + 1.637845297937 i$ | 0.001       | 0.0891868    | 0.514630                 |
| 17  | $c^8 - 8c^7 + 16 \sum_{j=0}^6 c^j$          | $4.499982156746 + 1.936550004261 i$ | 0.000001    | 0.0655749    | $1.79816 \times 10^{-5}$ |
| 18  | $c^4 - 9c^3 + 17 \sum_{j=0}^2 c^j$          | $4.962660230936 + 0.943348614313 i$ | 0.001       | 0.0577094    | 0.431150                 |
| 19  | $c^{12} - 9c^{11} + 18 \sum_{j=0}^{10} c^j$ | $4.999999990671 + 1.414213647216 i$ | 0.000000001 | 0.0496780    | $1.15695 \times 10^{-7}$ |

**Table A.1.** Certified data for off-lens witnesses  $c_n \in \mathcal{M}_n \setminus \mathcal{X}_n$ ,  $2 \leq n \leq 19$ . Each row records a certificate tuple  $(P_n, \tilde{c}_n, r_n, \Delta_n, \Lambda_n)$ ; the displayed  $\Delta_n$  and  $\Lambda_n$  are certified strict lower bounds.

The same strict inequality gives

$$\left| \frac{P(z_0)}{P'(z_0)} \right| < r,$$

so this zero lies in  $B(z_0, r)$ . Therefore  $P$  has exactly one zero in the open disk.

Moreover, the strict inequality on  $\partial B(z_0, r)$  implies that neither  $L$  nor  $P$  vanishes on the boundary. Hence the same unique zero is the only zero in the closed disk  $\overline{B(z_0, r)}$ .  $\square$

**Proposition A.2** (Certified off-lens witnesses). *For each  $2 \leq n \leq 19$ , there exists a non-real parameter*

$$c_n \in \mathcal{M}_n \setminus \mathcal{X}_n.$$

*Proof.* Fix  $n \in \{2, 3, \dots, 19\}$ , and use the corresponding row of Table A.1. The inequality  $\Delta_n > 0$  is exactly the hypothesis of Lemma A.1, so  $P_n$  has a unique zero  $c_n$  in  $\overline{B(\tilde{c}_n, r_n)}$ .

The inequality

$$\operatorname{Im}(\tilde{c}_n) > r_n$$

makes this disk disjoint from  $\mathbb{R}$ , so  $c_n \notin \mathbb{R}$ . Moreover,  $\operatorname{Im}(\tilde{c}_n) > r_n$  implies

$$|\tilde{c}_n + 1| - r_n > 0.$$

The inequality  $\Lambda_n > 0$  therefore gives

$$|\tilde{c}_n + 1| - r_n > \sqrt{2n},$$

and therefore every  $c \in \overline{B(\tilde{c}_n, r_n)}$  satisfies

$$|c + 1| \geq |\tilde{c}_n + 1| - r_n > \sqrt{2n}.$$

In particular,

$$c_n \notin \mathcal{X}_n.$$

We also need  $c_n$  to lie outside the closed unit disk. Since

$$|c_n| \geq |c_n + 1| - 1 > \sqrt{2n} - 1 \geq 1,$$

and the inequality is strict, we have  $|c_n| > 1$ .

Finally,  $P_n$  is monic with non-leading coefficients in  $D_n$ , so  $c_n \in \mathcal{R}_n$ . Since  $|c_n| > 1$ , Theorem 5.5 gives  $c_n \in \mathcal{M}_n$ . Thus

$$c_n \in \mathcal{M}_n \setminus \mathcal{X}_n$$

and  $c_n$  is non-real. □



# Provenance of the Results

Table B.1 identifies the joint papers from which the principal thesis formulations are drawn and records their coauthorship. Their antecedents include Bousch’s neighboring-overlap criterion Proposition 3.3 [Bou92, Chapter 1, p. 41]. The series and marked-point descriptions in Propositions 5.2 and 3.4 express this criterion in the present normalization. The exterior root-closure correspondence in Theorem 5.5 combines this description with approximation by polynomial truncations; its binary case appears explicitly in Indlekofer, J  rai, and K  tai [IJK95, Section 7]. The local-connectedness argument applies Nakajima’s general zero-set criterion [Nak24, Main Theorem B].

| Thesis result                 | Source formulation      | Content in the thesis                                                             | Coauthorship                              |
|-------------------------------|-------------------------|-----------------------------------------------------------------------------------|-------------------------------------------|
| Theorem 3.13                  | [EJS24], Theorem 1      | Local connectedness via the Nakajima transfer argument                            | Joint work with D. Juher and J. Salda  a. |
| Theorem 4.5                   | [EJS24], Theorem 2      | Rectangle covering and regular-closedness away from the real axis for $n \geq 21$ | Joint work with D. Juher and J. Salda  a. |
| Proposition 5.2               | [EJ26], Proposition 2.1 | Marked-point form of the neighboring-overlap criterion                            | Joint work with D. Juher.                 |
| Theorem 5.5                   | [EJ26], Proposition 2.5 | Explicit exterior monic root-closure correspondence                               | Joint work with D. Juher.                 |
| Theorem 6.35                  | [EJ26], Theorem A       | Finite-capture closure inclusion via the fixed-word closed-graph lemma            | Joint work with D. Juher.                 |
| Theorem 6.39                  | [EJ26], Theorem B       | Finite-capture closure inside the non-real parameter lens                         | Joint work with D. Juher.                 |
| Theorem 6.44                  | [EJ26], Theorem D       | Sharp lens-containment threshold $n \geq 20$                                      | Joint work with D. Juher.                 |
| Proposition A.2 and Table A.1 | [EJ26], Appendix A      | Explicit certificates for non-real off-lens roots                                 | Joint work with D. Juher.                 |
| Corollary 6.45                | [EJ26], Corollary E     | Closure identity for the connectedness locus and finite capture                   | Joint work with D. Juher.                 |

**Table B.1.** Sources and coauthorship of the principal thesis results.

The finite certificates used for sharpness are given in Appendix A.



# Outlook and Related Questions

The limits of canonical capture lead to questions about other traps, constrained address languages, and the real boundary. This appendix develops those questions alongside companion results on connectedness loci, attractor topology, and measures. The cited papers and manuscripts are separate from the proofs of the main chapters.

## C.1. Regular-closedness and off-lens finite capture

Regular-closedness away from the real axis is proved for every  $n \geq 2$  in a separate unpublished manuscript, using a stable linked-translate construction, a short-chain argument, and a Rouché perturbation lemma [Esp26g, Theorem 1.1]. For  $n \geq 20$ , the main chapters give the more specific description by closure of canonical capture (Corollary 6.45).

For  $2 \leq n \leq 19$ , the witnesses of Proposition A.2 lie beyond the canonical trap's self-covering range. The question is how to extend finite capture there and organize its layers. Within  $\mathcal{X}_n \setminus \mathbb{R}$ , every finite-depth boundary enters the filtration after at most two additional inverse steps, for every  $n \geq 2$  (Corollary 6.36).

Finite unions of geometric pieces, or covering relations involving higher iterates, are possible sources of off-lens self-covering traps. Suppose that a family of open traps  $T(c)$  is available on an open off-lens region  $U$ , with trap entry stable under parameter perturbations. Let  $\Theta_j^T(n) \subset U$  denote the parameters for which an inverse word of length at most  $j$  sends the marked point  $2c$  into  $T(c)$ . One can then ask whether a finite integer  $r \geq 1$  exists such that

$$\partial_U \Theta_j^T(n) \subseteq \Theta_{j+r}^T(n) \setminus \Theta_j^T(n), \quad j \geq 0.$$

Here  $r$  is independent of  $j$ , but may depend on  $n$ , the region, and the trap family. Determining where such a bound exists, and how it changes with the trap geometry, would describe the extent of layered regularity beyond the lens. A related question is whether

$$\mathcal{M}_n \cap U = \overline{\bigcup_{j \geq 0} \Theta_j^T(n)}^U,$$

where the closure is relative to  $U$ . All depths count inverse steps in the original difference system, even when the trap is constructed using compositions.

Regular-closedness is compatible with complicated complementary topology. A separate preprint proves that  $\mathcal{M}_n$  has infinitely many holes for every  $n \geq 2$ , using exact finite-capture loop certificates [Esp26e].

## C.2. Limits of canonical capture

For every  $n \geq 3$ , the manuscript [Esp26f, Theorem 1.1] identifies

$$c_n = \frac{1 + i\sqrt{4n-1}}{2}$$

as an interior parameter with no canonical capture word. It lies strictly inside the non-real lens, since

$$|c_n|^2 + 2|\operatorname{Re} c_n| = n + 1 < 2n - 1.$$

More precisely, there is a neighborhood  $U_n \subset \mathcal{X}_n \setminus \mathbb{R}$  of  $c_n$  such that

$$U_n \setminus \{c_n\} \subset \Theta_{\text{fin}}(n), \quad c_n \notin \Theta_{\text{fin}}(n).$$

Thus  $c_n$  is an isolated point of the residual interior set  $\Phi_n$  defined in Section 7.5.

Captured parameters are dense near  $c_n$  (Corollary 6.46), but every sequence of them approaching  $c_n$  has minimal capture depths tending to infinity. Otherwise, a bounded-depth subsequence would put  $c_n$  in  $\overline{\Theta_k(n)}$  for some  $k$ , hence in  $\Theta_{k+2}(n)$  by Theorem 6.35, contradicting noncapture. The companion manuscript determines the logarithmic rate of this divergence.

These isolated examples lead to two further problems: characterize the full residual set and find additional finite geometric tests that detect its points.

## C.3. Base loci and finite certificate languages

For a finite digit set  $D \subset \mathbb{C}$  and a fixed value  $x \in \mathbb{C}$ , define the base locus

$$\mathcal{B}(D, x) = \left\{ c \in \mathbb{C} \setminus \overline{\mathbb{D}} : x = \sum_{k=1}^{\infty} e_k c^{-k} \text{ for some } (e_k)_{k \geq 1} \in D^{\mathbb{N}} \right\}.$$

The thesis correspondence is  $\mathcal{M}_n = \mathcal{B}(D_n, 1)$ : the represented value and the digit alphabet are fixed, while the base varies.

For a language  $L \subset D^{\mathbb{N}}$ , write  $\mathcal{B}(L, x)$  for the bases admitting such a representation with  $(e_k)_{k \geq 1} \in L$ . The Fibonacci–Mandelbrot construction discussed in Section 1.3 motivates asking how  $\omega$ -regular, sofic, substitutive, or other constrained infinite-word languages select subloci of  $\mathcal{B}(D_n, 1)$ .

Finite-type theory describes the relative positions of overlapping pieces at a fixed parameter. Bandt and Mesing develop an algorithmic framework [BM09]; Bandt’s complex-Pisot theorem applies to the collinear systems whenever  $c$  is complex Pisot [Ban25b, Theorem 1.3]. Here this means that  $c$  is a non-real algebraic integer with  $|c| > 1$  whose conjugates other than  $c$  and  $\bar{c}$  lie in  $\mathbb{D}$ . Indeed,  $f_t(z) = c^{-1}(z + ct)$  has  $s_t = 1$  and  $v_t = ct \in \mathbb{Z}[c]$ , so the theorem’s algebraic and finite-rotation hypotheses hold. These finite-type results do not by themselves establish finite-state recognition of the particular trap-entry or enclosure-survival languages used here.

The inverse search produces its own finite-word languages: exact-landing words, first trap-entry words, and enclosure-survival words. At specified algebraic parameters, it is natural to seek automata recognizing these languages, or transducers recording capture depth, and to compare their state graphs with neighbor graphs of self-affine tiles [Fro92; FP18]. An eventually periodic expansion yields a finite cyclic address relation, but does not by itself imply entry into the canonical trap.

## C.4. The real axis

The real-trace formula settles membership on the real axis. The finer classification asks which of those points lie in the planar interior and which are approached by non-real parameters. The companion results below identify exact thresholds at specific alphabet sizes.

For  $n \geq 2$ , set

$$\mathcal{N}_n^+ := \overline{\mathcal{M}_n \setminus \mathbb{R}} \cap (1, \infty), \quad T_n := \sup \mathcal{N}_n^+,$$

so  $T_n$  is the last positive real point accumulated by non-real parameters of  $\mathcal{M}_n$ . Also define

$$\Omega_2(n) := \left\{ \lambda \in (0, 1) : f(\lambda) = f'(\lambda) = 0 \text{ for some } f(z) = 1 + \sum_{k \geq 1} a_k z^k, \quad a_k \in D_n \right\}.$$

**The ternary case.** At the first sixth-cyclotomic order  $H_2 = \Phi_6(2) = 3$ , one has  $D_3 = \{-2, -1, 0, 1, 2\}$ . The five-symbol manuscript proves

$$\Omega_2(3) = \left\{ \frac{1}{2} \right\} \cup [\rho^{-1}, 1), \quad \rho^3 - \rho^2 - \rho - 1 = 0, \quad \rho > 1.$$

In the notation of Shmerkin and Solomyak this is the set  $\Omega'_2$ . The identity settles the numerical observation in their Remark 6.4 and identifies exactly the interval endpoint previously known only within a short numerical interval [SS06]; see [Esp26h, Theorem 1.1]. The isolated point  $1/2$  does not determine the geometric antenna endpoint: its reciprocal 2 is a purely real boundary point, whereas  $\rho$  is approached by non-real parameters. Hence  $T_3 = \rho$  [Esp26h, Theorem 1.2].

**Higher sixth-cyclotomic orders.** The higher-order companion manuscript treats the arithmetic family

$$H_q = \Phi_6(q) = q^2 - q + 1 \quad (q \geq 3).$$

Together, the two manuscripts give the endpoint formula

$$T_{H_q} = \begin{cases} \rho, & q = 2, \\ q, & q \geq 3. \end{cases}$$

For the two cases, see [Esp26h; Esp26a].

At  $H_3 = 7$ , the endpoint and terminal tail are exact:

$$T_7 = 3, \quad [3, 7] \subset \partial \mathcal{M}_7, \quad \min \Omega_2(7) = \frac{1}{3},$$

as stated in [Esp26a, Theorem C]; the local classification immediately below 3 remains a separate problem. For every  $q \geq 4$ , the stronger complete classification is

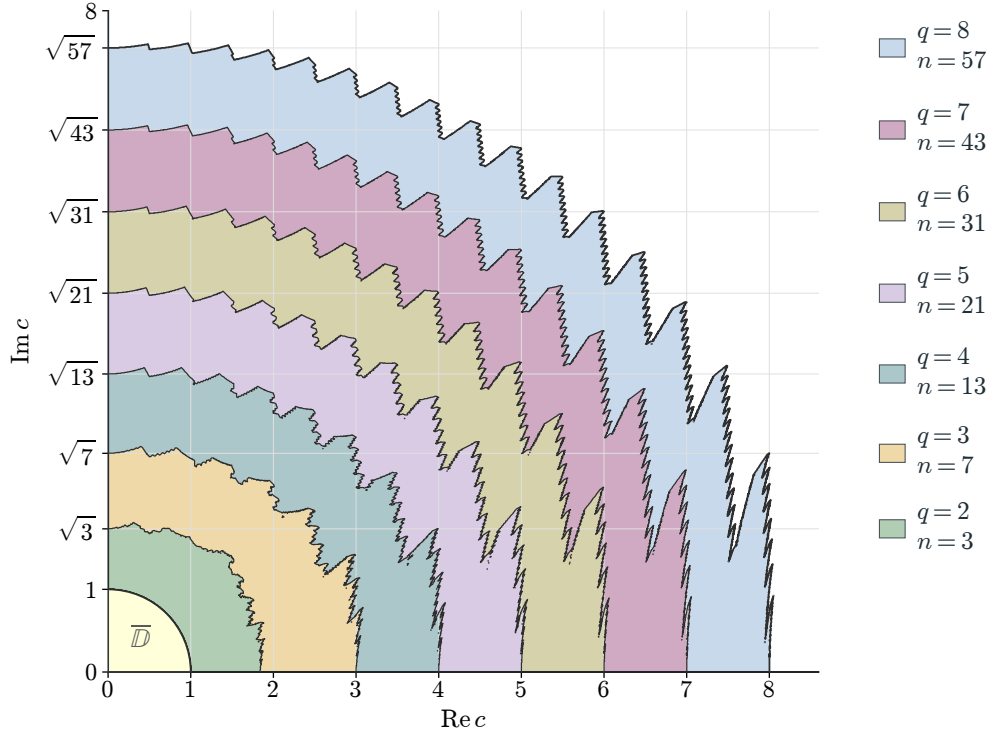
$$\text{int}(\mathcal{M}_{H_q}) \cap (1, \infty) = (1, q), \quad \partial \mathcal{M}_{H_q} \cap (1, \infty) = [q, H_q],$$

$$\overline{\mathcal{M}_{H_q} \setminus \mathbb{R}} \cap (1, \infty) = (1, q], \quad \Omega_2(H_q) = [1/q, 1).$$

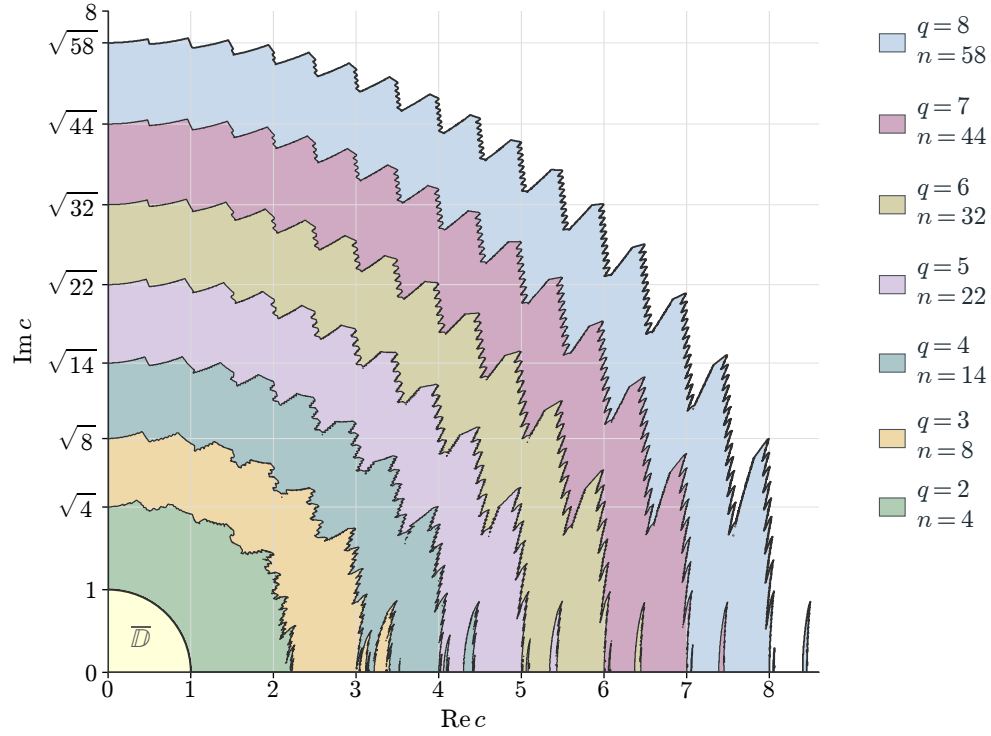
The endpoint  $T_{H_q}$  is approached by non-real parameters; the interval  $(T_{H_q}, H_q]$  is the purely real terminal antenna [Esp26a, Theorems A and B]. Figure C.1 shows this subsequence for  $q = 2, \dots, 8$ . For a full-plane comparison of the order  $H_3 = 7$  with the neighboring order  $n = 8$ , see Figures 1.4 and 3.4.

**Arithmetic subsequences of the nested loci.** The following figures compare the sixth-cyclotomic orders with their successors and with the orders  $q^2 + 1$  and  $q^2 + 2$ . For each fixed  $q$ , the consecutive figures in each pair show consecutive alphabet sizes. Their near-real patterns suggest asking how the endpoint and local-interior classification change between these arithmetic families.

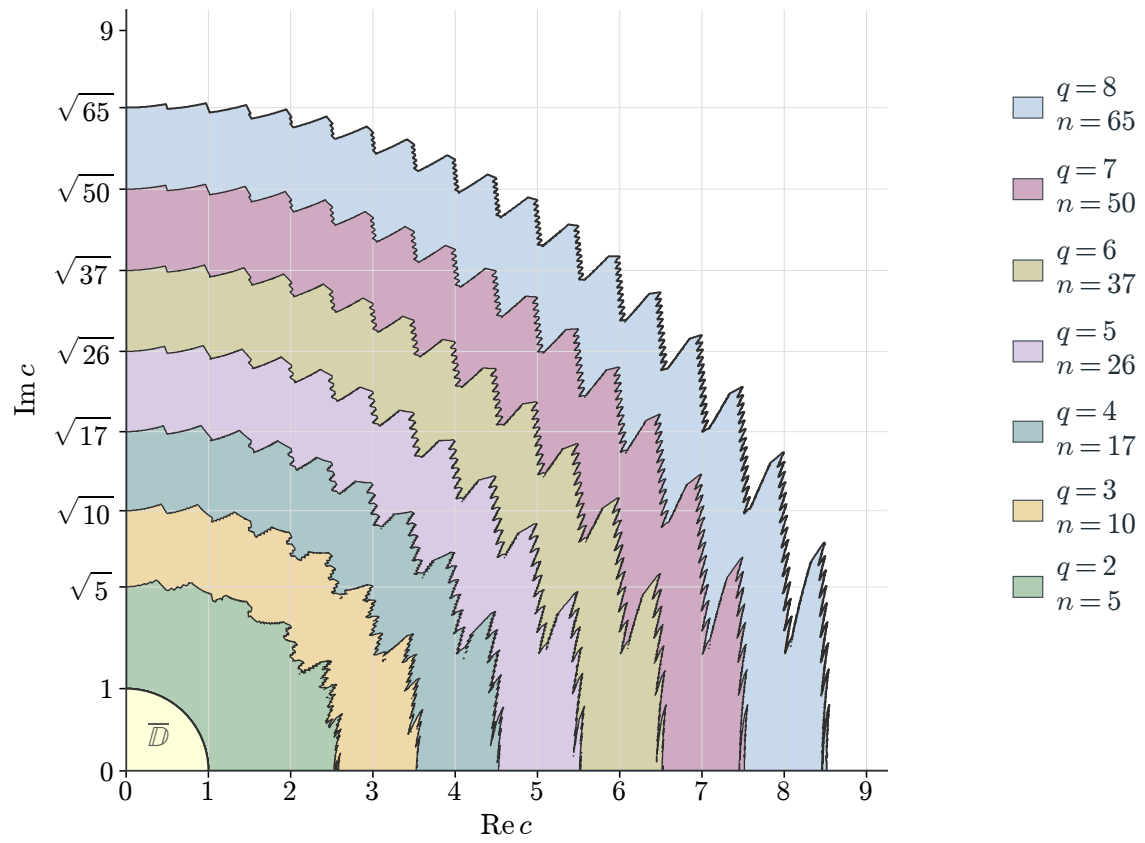
Each right-hand legend gives both  $q$  and the corresponding alphabet size  $n$ , with the same colour assigned to a fixed  $q$  throughout. Each order is sampled independently on a  $1600 \times 1600$  grid by depth-40 enclosure search with in-lens trap tests. The shaded regions and their outlines include parameters unresolved within the depth and queue limits; they do not certify boundaries, contacts, or counts of boundary features. All axes use the original parameter coordinates with equal horizontal and vertical units. The two figures on each page share aligned integer-abscissa guides. Vertical ticks are at  $y = \sqrt{n}$ , and the legend follows these levels from largest to smallest. The pale-yellow closed unit disk is excluded, and the real trace is omitted.



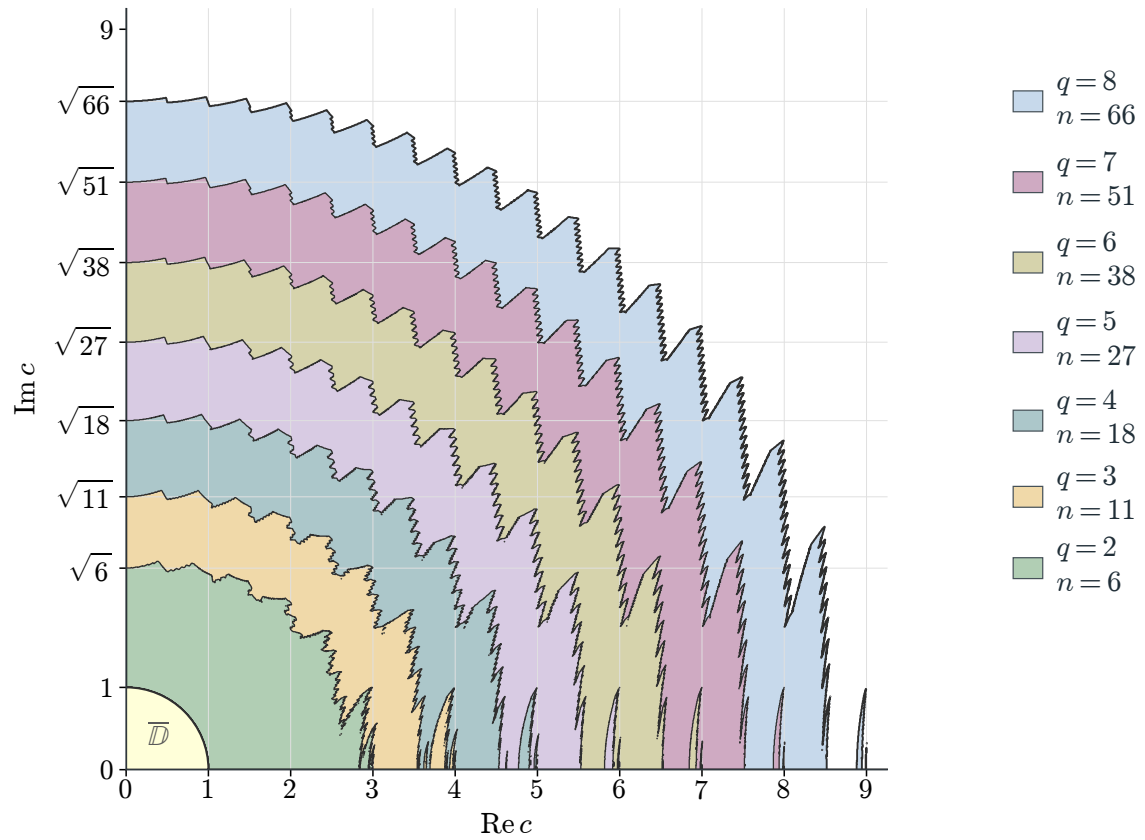
**Figure C.1.** Connectedness loci  $\mathcal{M}_{H_q}$  for  $H_q = q^2 - q + 1$ .



**Figure C.2.** Connectedness loci for the successor orders  $n = H_q + 1 = q^2 - q + 2$ .



**Figure C.3.** Connectedness loci for  $n = q^2 + 1$ .



**Figure C.4.** Connectedness loci for the successor orders  $n = q^2 + 2$ .

## C.5. Attractor topology and measures

The dendrites in Figure 3.1 exhaust the possible alphabet sizes, though not the possible parameters. The companion manuscript proves that  $E(c, n)$  is a dendrite for some non-real  $c$  if and only if  $2 \leq n \leq 6$ . Every connected  $E(c, n)$  with  $c \notin \mathbb{R}$  and  $n \geq 7$  contains a simple closed curve [Esp26c, Theorem 1.1].

The obstruction counts the first-level pieces of  $E(c, n)$  meeting  $2c + E(c, n)$ . Writing this contact count as  $k(c, n)$ , a dendrite requires  $k(c, n) = 1$ , whereas for every  $c \in \mathcal{M}_n \setminus \mathbb{R}$  and  $n \geq 7$ ,

$$k(c, n) \geq n - \left\lceil \sqrt{6(n-1)} \right\rceil + 1 \geq 2$$

[Esp26c, Theorem 1.2]. The remaining questions include classifying the dendrite parameters for  $2 \leq n \leq 6$  and determining the smallest possible contact counts at larger alphabet sizes.

For the natural measures, the same reciprocal series controls how closely distinct cylinder maps can approach one another. A separate manuscript [Esp26d] obtains an explicit radius excluding non-real double zeros, uniform sublevel estimates on compact subsets of the strict off-axis region beyond that radius, and eventual exponential separation outside a set of Hausdorff dimension zero. It then applies dimension theory to obtain the expected Hausdorff dimensions of the attractor and its equal-weight measure outside the exceptional set. The explicit radius and sublevel bounds give quantitative information within the transversality and entropy framework [PS00; Hoc14].

Absolute continuity already follows generically from Shmerkin and Solomyak's fixed-translation theorem: for non-real  $c$  with  $1 < |c| < \sqrt{n}$ , the equal-weight measure has a density in some  $L^q$ ,  $q > 1$ , outside a set of Hausdorff dimension zero [SS16, Theorem B]. Further work can therefore focus on specified parameters, an explicit description of the exceptional set, and effective bounds on the density and its regularity.





# Fractals col·lineals i conjunts de connexitat

*Topologia, captura finita i arrels de polinomis amb coeficients restringits*

*Com pot un contacte entre peces fractals revelar una identitat aritmètica? La tesi uneix aquestes dues perspectives i descobreix una estructura interior que es pot explorar capa a capa.*

Triem  $n$  posicions equiespaiades sobre una recta. Hi col·loquem còpies d'una figura, totes amb la mateixa reducció i el mateix gir, i repetim la regla indefinidament. Un nombre complex  $c$ , que podem representar com un punt del pla, controla aquests dos canvis. Si les peces veïnes s'intersequen, el fractal forma una sola peça: és connex; si totes són disjunctes, forma un conjunt de Cantor al pla complex. Els paràmetres que donen fractals connextos formen el conjunt de connexitat. En augmentar  $n$ , aquests conjunts formen una família nidificada que recorda el conjunt de Mandelbrot.

## 01 Del contacte a l'aritmètica

El cas binari, de dues peces, s'estudia des de fa dècades. La tesi desenvolupa sistemàticament la família per a qualsevol nombre de peces. Cada punt del fractal té almenys una adreça simbòlica infinita: una successió d'eleccions entre còpies a cada escala. Quan dues peces veïnes s'intersequen, un mateix punt té dues adreces diferents. Restant les expansions i dividint per l'espaiat, fixat en 2, s'obté una representació d'1 en base  $c$  amb dígitos signats. Si l'expansió és finita, eliminar els denominadors dona un polinomi amb coeficients enters que té  $c$  com a arrel. És mònic: el coeficient principal és 1 i els altres pertanyen a l'alfabet  $D_n = \{-(n-1), \dots, -1, 0, 1, \dots, n-1\}$ .

Si l'expansió no acaba,  $c$  és límit d'una successió d'arrels d'aquests polinomis. Fora del disc unitat tancat, el conjunt de connexitat coincideix amb la clausura d'aquest conjunt d'arrels: les arrels i tots els seus punts límit. És una igualtat exacta, no una semblança entre imatges. La tesi també estableix la connexitat i la connexitat local dels conjunts, en determina la intersecció amb l'eix real i n'obté cotes geomètriques.

## 02 L'interior, capa a capa

La captura finita treballa amb el conjunt de diferències entre punts del fractal. La connexitat equival al fet que el punt marcat  $2c$  pertanyi a aquest conjunt.

Fora de l'eix real, la cerca inversa descarta les branques que surten del paral·lelogram envolupant d'aquest conjunt. Dins d'una regió de paràmetres en forma de lent, l'entrada d'una branca en una trampa interior del conjunt de diferències certifica que la connexitat resisteix petites variacions de  $c$ . Si totes les branques s'esgoten, el fractal és desconnex; tot paràmetre no real exterior acaba sent exclòs. Una cerca aturada sense cap dels dos resultats no dona cap veredict.

La profunditat de captura revela una segona nidificació. Dins de la lent i fora de l'eix real, els paràmetres certificats en un màxim de  $k$  passos formen regions obertes que creixen amb  $k$ . A cada quadrant, cada nivell es descriu mitjançant un nombre finit de desigualtats polinòmiques: és semialgebraic. Els punts de la seva frontera dins d'aquest domini queden certificats al nivell  $k+2$ , com a màxim. Així, la frontera d'un nivell finit passa a formar part de l'interior certificat amb només dos passos addicionals.

Per a  $n \geq 20$ , la lent de dos discs conté tot el conjunt de connexitat fora de l'eix real. Cada paràmetre d'aquesta part és límit de paràmetres interiors amb certificats finits. Per a cada  $2 \leq n \leq 19$ , una arrel no real certificada pertany al conjunt però queda fora de la lent. Vint és el llindar exacte d'inclusió en aquesta lent, no un llindar general de regularitat.

## 03 Molts dígit, poques branques

El sistema de diferències té  $2n - 1$  dígitos possibles a cada pas: sense poda, una cerca de profunditat  $k$  tindria  $(2n - 1)^k$  candidats. La geometria en descarta gairebé tots. Per a  $n \geq 25$ , fora de l'eix real i amb  $|c| \geq \sqrt{n}$ , cada node de l'arbre admissible exacte té com a màxim tres fills. Aquesta regió conté tota la frontera no real. Un alfabet que creix amb  $n$  dona lloc, doncs, a una cerca com a màxim ternària.

El programari de codi obert i interactiu aprofita la poda geomètrica per explorar els conjunts de connexitat, els fractals, les capes de captura i els certificats. La computació orienta la descoberta; les conclusions es justifiquen amb demostracions i dades finites verificables independentment.

*El pont entre fractals i polinomis enters no només identifica un mateix paisatge: en revela l'estructura i proporciona certificats finits de connexitat i desconexió.*

# Fractales colineales y conjuntos de conexidad

*Topología, captura finita y raíces de polinomios con coeficientes restringidos*

*¿Cómo puede el contacto entre piezas fractales revelar una identidad aritmética? La tesis une estas dos perspectivas y descubre una estructura interior que se puede explorar capa a capa.*

Elegimos  $n$  posiciones equiespaciadas sobre una recta. Colocamos copias de una figura, todas con la misma reducción y el mismo giro, y repetimos la regla indefinidamente. Un número complejo  $c$ , representado por un punto del plano, controla ambos cambios. Si las piezas vecinas se intersecan, el fractal forma una sola pieza: es conexo; si todas son disjuntas, forma un conjunto de Cantor en el plano complejo. Los parámetros que dan fractales conexos forman el conjunto de conexidad. Al aumentar  $n$ , estos conjuntos forman una familia anidada que recuerda al conjunto de Mandelbrot.

## 01 Del contacto a la aritmética

El caso binario, de dos piezas, se estudia desde hace décadas. La tesis desarrolla sistemáticamente la familia para cualquier número de piezas. Cada punto del fractal tiene al menos una dirección simbólica infinita: una sucesión de elecciones entre copias a cada escala. Cuando dos piezas vecinas se intersecan, un mismo punto tiene dos direcciones distintas. Restando las expansiones y dividiendo por el espaciado, fijado en 2, se obtiene una representación de 1 en base  $c$  con dígitos con signo. Si la expansión es finita, eliminar los denominadores da un polinomio con coeficientes enteros que tiene a  $c$  como raíz. Es mónico: el coeficiente principal es 1 y los demás pertenecen al alfabeto  $D_n = \{-(n-1), \dots, -1, 0, 1, \dots, n-1\}$ .

Si la expansión no termina,  $c$  es límite de una sucesión de raíces de estos polinomios. Fuera del disco unidad cerrado, el conjunto de conexidad coincide con la clausura de este conjunto de raíces: las raíces y todos sus puntos límite. Es una igualdad exacta, no una semejanza entre imágenes. La tesis también establece la conexidad y la conexidad local de los conjuntos, determina su intersección con el eje real y obtiene cotas geométricas.

## 02 El interior, capa a capa

La captura finita trabaja con el conjunto de diferencias entre puntos del fractal. La conexidad equivale a que el punto marcado  $2c$  pertenezca a este conjunto.

Fuera del eje real, la búsqueda inversa descarta las ramas que salen del paralelogramo envolvente de este conjunto. Dentro de una región de parámetros con forma de lente, la entrada de una rama en una trampa interior del conjunto de diferencias certifica que la conexidad resiste pequeñas variaciones de  $c$ . Si todas las ramas se agotan, el fractal es desconexo; todo parámetro no real exterior acaba siendo excluido. Una búsqueda detenida sin ninguno de los dos resultados no da ningún veredicto.

La profundidad de captura revela una segunda anidación. Dentro de la lente y fuera del eje real, los parámetros certificados en un máximo de  $k$  pasos forman regiones abiertas que crecen con  $k$ . En cada cuadrante, cada nivel se describe mediante un número finito de desigualdades polinómicas: es semialgebraico. Los puntos de su frontera dentro de este dominio quedan certificados en el nivel  $k+2$ , como máximo. Así, la frontera de un nivel finito pasa a formar parte del interior certificado con solo dos pasos adicionales.

Para  $n \geq 20$ , la lente de dos discos contiene todo el conjunto de conexidad fuera del eje real. Cada parámetro de esta parte es límite de parámetros interiores con certificados finitos. Para cada  $2 \leq n \leq 19$ , una raíz no real certificada pertenece al conjunto pero queda fuera de la lente. Veinte es el umbral exacto de inclusión en esta lente, no un umbral general de regularidad.

## 03 Muchos dígitos, pocas ramas

El sistema de diferencias tiene  $2n-1$  dígitos posibles en cada paso: sin poda, una búsqueda de profundidad  $k$  tendría  $(2n-1)^k$  candidatos. La geometría descarta casi todos. Para  $n \geq 25$ , fuera del eje real y con  $|c| \geq \sqrt{n}$ , cada nodo del árbol admisible exacto tiene como máximo tres hijos. Esta región contiene toda la frontera no real. Un alfabeto que crece con  $n$  da lugar, pues, a una búsqueda a lo sumo ternaria.

El software interactivo de código abierto aprovecha la poda geométrica para explorar los conjuntos de conexidad, los fractales, las capas de captura y los certificados. La computación orienta el descubrimiento; las conclusiones se justifican con demostraciones y datos finitos verificables de forma independiente.

*El puente entre fractales y polinomios enteros no solo identifica un mismo paisaje: revela su estructura y proporciona certificados finitos de conexidad y desconexión.*

# Collinear Fractals and Connectedness Loci

*Topology, Finite Capture, and Restricted Polynomial Roots*

*How can contact between fractal pieces reveal an arithmetic identity? This thesis brings these two perspectives together and uncovers an interior structure that can be explored layer by layer.*

Choose  $n$  equally spaced positions on a line. Place copies of a figure there, all with the same contraction and rotation, and repeat the rule indefinitely. A complex number  $c$ , represented by a point in the plane, controls both transformations. If neighbouring pieces intersect, the fractal forms a single connected set; if all pieces are disjoint, it is a Cantor set in the complex plane. Parameters yielding connected fractals form the connectedness locus. As  $n$  increases, these loci form a nested family reminiscent of the Mandelbrot set.

## 01 From contact to arithmetic

The binary case, with two pieces, has been studied for decades. This thesis systematically develops the family for any number of pieces. Each point of the fractal has at least one infinite symbolic address: a sequence of choices between copies at each scale. When two neighbouring pieces intersect, the same point has two different addresses. Subtracting their expansions and dividing by the spacing, fixed at 2, gives an expansion of 1 in base  $c$  with signed digits. If the expansion is finite, clearing denominators gives an integer polynomial with  $c$  as a root. It is monic: the leading coefficient is 1, and the others belong to the alphabet  $D_n = \{-(n-1), \dots, -1, 0, 1, \dots, n-1\}$ .

If the expansion does not terminate,  $c$  is a limit of roots of these polynomials. Outside the closed unit disc, the connectedness locus equals the closure of this root set: the roots and all their limit points. This is an exact equality, not a resemblance between images. The thesis also establishes connectedness and local connectedness of the loci, determines their intersection with the real axis, and obtains geometric bounds.

## 02 The interior, layer by layer

Finite capture works with the difference set of the fractal. Connectedness is equivalent to the marked point  $2c$  belonging to this set.

Away from the real axis, backward search discards branches that leave an enclosing parallelogram of this set. Inside a lens-shaped parameter region, a branch entering an interior trap in the difference set certifies that connectedness persists under small changes in  $c$ . If all branches are exhausted, the fractal is disconnected; every non-real exterior parameter is eventually excluded. A search stopped without either outcome gives no verdict.

Capture depth reveals a second nesting. Inside the lens and away from the real axis, parameters certified in at most  $k$  steps form open regions that grow with  $k$ . In each quadrant, each level is described by finitely many polynomial inequalities: it is semialgebraic. Its boundary points within this domain are certified at level  $k+2$ , at the latest. Thus, the boundary of a finite level enters the certified interior with just two additional steps.

For  $n \geq 20$ , the two-disc lens contains the entire connectedness locus away from the real axis. Every parameter in this part is a limit of interior parameters with finite certificates. For each  $2 \leq n \leq 19$ , a certified non-real root belongs to the locus but lies outside the lens. Twenty is the exact threshold for inclusion in this lens, not a general threshold for regularity.

## 03 Many digits, few branches

The difference system has  $2n-1$  possible digits at each step: without pruning, a depth- $k$  search would have  $(2n-1)^k$  candidates. Geometry discards almost all of them. For  $n \geq 25$ , away from the real axis and with  $|c| \geq \sqrt{n}$ , every node of the exact admissible tree has at most three children. This region contains the entire non-real boundary. An alphabet that grows with  $n$  therefore leads to a search that is at most ternary.

Interactive, open-source software uses geometric pruning to explore the connectedness loci, fractals, capture layers, and certificates. Computation guides discovery; the conclusions are justified by proofs and independently verifiable finite data.

*The bridge between fractals and integer polynomials does more than identify a shared landscape: it reveals its structure and provides finite certificates of connectedness and disconnection.*