

A MÖBIUS DEFORMATION FROM THE SIERPIŃSKI GASKET TO THE EQUILATERAL APOLLONIAN GASKET

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ABSTRACT. We study an explicit one-parameter family of Möbius maps deforming the equilateral Sierpiński iterated function system into the parabolic generators of the equilateral Apollonian gasket. For every $t < 1$ we prove that the generators preserve the closed unit disk and satisfy an explicit Euclidean contraction bound, hence admit a unique Hutchinson attractor K_t . We also prove Hausdorff continuity of $t \mapsto K_t$ on each truncated interval $[0, 1 - \varepsilon]$. For the hyperbolic fibers we verify that the family is a conformal iterated function system and apply the projection-entropy formalism of Feng and Hu to obtain the exact variational formula $\dim_H K_t = \dim_B K_t = \sup_{m \in M_\sigma(\Sigma)} h_{\pi_t}(\sigma, m) / \chi_t(m)$ for $0 \leq t < 1$. The parabolic endpoint $t = 1$ is the equilateral Apollonian gasket, whose Hausdorff dimension was computed by Vytnova and Wormell.

1. INTRODUCTION

The Apollonian gasket is a classical conformal limit set and the Sierpiński gasket is the prototypical planar self-similar fractal. Although the two are often compared visually, their metric geometry is quite different. In particular,

$$\dim_H(S) = \frac{\log 3}{\log 2} = 1.58496\dots,$$

whereas the equilateral Apollonian gasket has Hausdorff dimension

$$\dim_H(A) = 1.3056867280498771846459862068510408911060\dots,$$

rigorously computed to 128 decimal places by Vytnova and Wormell [8]; see also [6, 7] for earlier dynamical and geometric viewpoints.

The purpose of this paper is to isolate the specific one-parameter iterated-function system implemented in the companion interactive demonstration [2]. The main point is that the deformation does *not* arise by post-composing a fixed coding map with a family of ambient homeomorphisms. Instead, the generators themselves vary with the parameter and deform the affine Sierpiński similarities into the parabolic Apollonian Möbius maps.

This conformal deformation should be compared with the affine-projective Rauzy–Sierpiński interpolation developed in the companion manuscript with Pierre Arnoux [3]. Its companion interactive demonstration is described in [1].

Theorem 1.1. *Let $v_1 = i$, $v_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$, and $v_3 = \frac{\sqrt{3}}{2} - \frac{i}{2}$. For $t \in [0, 1]$ define*

$$R_t = \frac{1-t}{2} + t(2\sqrt{3}-3), \quad \alpha_t = t \frac{1-\sqrt{3}}{2},$$

and for $i \in \{1, 2, 3\}$ define

$$(1) \quad \Phi_{i,t}(z) = R_t \frac{z + \alpha_t v_i}{1 + \alpha_t \overline{v_i} z} + (1 - R_t) v_i.$$

Then the following statements hold.

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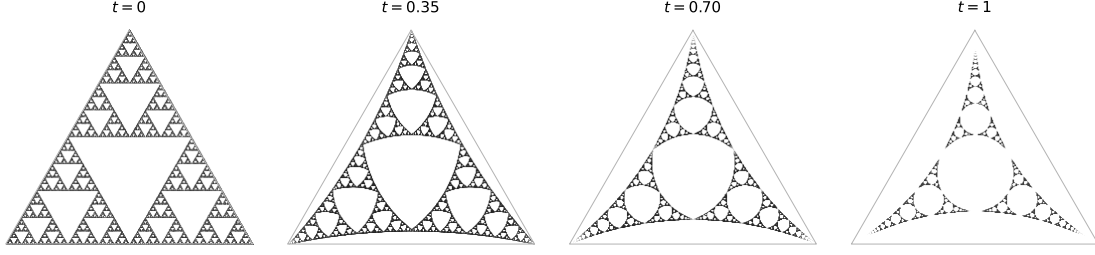


FIGURE 1. Representative fibers K_t at the four parameter values $t = 0, 0.35, 0.70$, and 1 . The left endpoint is the equilateral Sierpiński gasket; the right endpoint is the equilateral Apollonian gasket.

(i) At $t = 0$ one has

$$\Phi_{i,0}(z) = \frac{z + v_i}{2},$$

so the attractor is the standard equilateral Sierpiński gasket.

(ii) At $t = 1$ the maps $\Phi_{i,1}$ are the standard parabolic generators of the equilateral Apollonian gasket in the present normalization.

(iii) For every $t < 1$, each $\Phi_{i,t}$ maps the closed unit disk $\overline{\mathbb{D}}$ to itself and is a strict Euclidean contraction on $\overline{\mathbb{D}}$ with Lipschitz constant at most

$$L_t = R_t \frac{1 - \alpha_t}{1 + \alpha_t} < 1.$$

Hence there is a unique compact attractor $K_t \subset \overline{\mathbb{D}}$ satisfying

$$K_t = \Phi_{1,t}(K_t) \cup \Phi_{2,t}(K_t) \cup \Phi_{3,t}(K_t).$$

(iv) For every $\varepsilon \in (0, 1)$, the family $t \mapsto K_t$ is continuous in the Hausdorff metric on $[0, 1 - \varepsilon]$.

Representative fibers are shown in figure 1; the corresponding first-level carrier disks appear in figure 2. The continuity at the parabolic endpoint $t = 1$, suggested by the interactive visualization, is not proved here.

2. THE ONE-PARAMETER FAMILY

Let

$$\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

The vertices v_1, v_2, v_3 lie on the unit circle and form an equilateral triangle. The constants in (1) are chosen so that the affine endpoint and the parabolic endpoint are both recovered exactly, while the closed unit disk remains a natural carrier set throughout the deformation.

It is convenient to separate the two pieces of (1). Set

$$M_{i,t}(z) = \frac{z + \alpha_t v_i}{1 + \alpha_t \overline{v_i} z}, \quad A_{i,t}(w) = R_t w + (1 - R_t) v_i.$$

Then

$$\Phi_{i,t} = A_{i,t} \circ M_{i,t}.$$

For every $t \in [0, 1]$ one has $\alpha_t \in [-(\sqrt{3} - 1)/2, 0]$, hence $|\alpha_t| < 1$.

Proposition 2.1 (Endpoint formulas). *For all $i \in \{1, 2, 3\}$ and all $t \in [0, 1]$ one has $\Phi_{i,t}(v_i) = v_i$. Moreover:*

(a) at $t = 0$,

$$\Phi_{i,0}(z) = \frac{z + v_i}{2};$$

(b) at $t = 1$, each $\Phi_{i,1}$ is parabolic with neutral fixed point v_i .

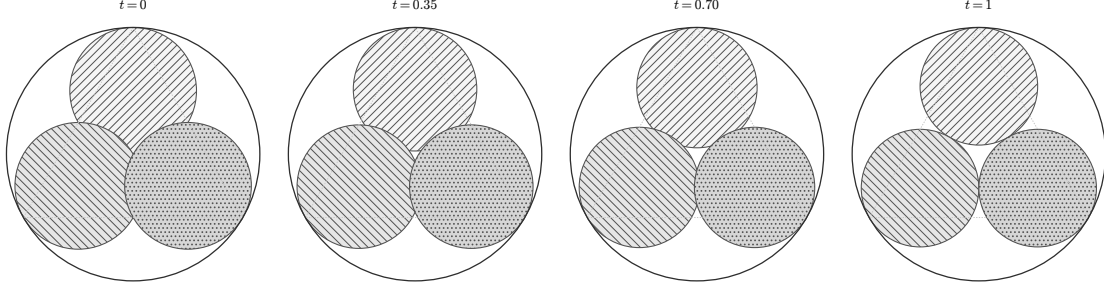


FIGURE 2. The three carrier disks $\Phi_{1,t}(\overline{\mathbb{D}})$, $\Phi_{2,t}(\overline{\mathbb{D}})$, and $\Phi_{3,t}(\overline{\mathbb{D}})$ inside the unit disk for the same four parameter values as in figure 1; distinct hatch patterns identify the three image disks uniformly across the four panels. This first-level picture records how the affine Sierpiński geometry bends toward the Apollonian disk configuration.

Proof. Since $|v_i| = 1$, one has

$$M_{i,t}(v_i) = \frac{v_i + \alpha_t v_i}{1 + \alpha_t \overline{v_i} v_i} = v_i,$$

and therefore $\Phi_{i,t}(v_i) = A_{i,t}(v_i) = v_i$.

At $t = 0$ one has $R_0 = 1/2$ and $\alpha_0 = 0$, so

$$\Phi_{i,0}(z) = \frac{1}{2}z + \frac{1}{2}v_i = \frac{z + v_i}{2}.$$

This is exactly the classical Sierpiński family.

A direct computation gives

$$(2) \quad \Phi'_{i,t}(z) = \frac{R_t(1 - \alpha_t^2)}{(1 + \alpha_t \overline{v_i} z)^2}.$$

Evaluating at the fixed point v_i yields

$$\Phi'_{i,t}(v_i) = R_t \frac{1 - \alpha_t}{1 + \alpha_t}.$$

At $t = 1$, the chosen constants satisfy

$$R_1 \frac{1 - \alpha_1}{1 + \alpha_1} = 1,$$

so $\Phi_{i,1}$ is parabolic at its fixed point. □

Remark 2.2. After the Euclidean rotation $z \mapsto -iz$, the endpoint $t = 1$ becomes the familiar Apollonian system generated by

$$z \mapsto \frac{(\sqrt{3} - 1)z + 1}{-z + \sqrt{3} + 1}$$

and its two $2\pi/3$ -rotates; see [6, 8]. Thus (1) is exactly a normalized Sierpiński-to-Apollonian deformation.

3. DISK INVARIANCE AND CONTRACTIVITY

The formula (1) is arranged so that the closed unit disk is invariant.

Proposition 3.1 (Disk invariance). *For every $t \in [0, 1]$ and every $i \in \{1, 2, 3\}$ one has*

$$\Phi_{i,t}(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}.$$

Proof. Because $|\alpha_t| < 1$, the map $M_{i,t}$ is a disk automorphism. Hence $M_{i,t}(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$. Now let $w \in \overline{\mathbb{D}}$. Since $|v_i| = 1$,

$$|A_{i,t}(w)| = |R_t w + (1 - R_t)v_i| \leq R_t|w| + (1 - R_t)|v_i| \leq R_t + (1 - R_t) = 1.$$

Therefore $A_{i,t}(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$, and the same holds for $\Phi_{i,t} = A_{i,t} \circ M_{i,t}$. $\square$

Proposition 3.2 (Explicit contraction bound). *For every $t \in [0, 1)$ and every $i \in \{1, 2, 3\}$, the map $\Phi_{i,t}$ is Lipschitz on $\overline{\mathbb{D}}$ with constant at most*

$$L_t = R_t \frac{1 - \alpha_t}{1 + \alpha_t}.$$

Moreover,

$$1 - L_t = \frac{(1 - t)(2 - (11\sqrt{3} - 19)t)}{2(2 - (\sqrt{3} - 1)t)} > 0,$$

so $L_t < 1$ for all $t < 1$ and $L_t \rightarrow 1$ as $t \rightarrow 1^-$.

Proof. By (2), for $z \in \overline{\mathbb{D}}$ we have

$$|\Phi'_{i,t}(z)| = R_t \frac{1 - \alpha_t^2}{|1 + \alpha_t \overline{v_i} z|^2}.$$

Since $\alpha_t \leq 0$ and $|z| \leq 1$, the reverse triangle inequality gives

$$|1 + \alpha_t \overline{v_i} z| \geq 1 + \alpha_t.$$

Therefore

$$|\Phi'_{i,t}(z)| \leq R_t \frac{1 - \alpha_t^2}{(1 + \alpha_t)^2} = R_t \frac{1 - \alpha_t}{1 + \alpha_t} = L_t.$$

The mean value theorem now yields the Lipschitz bound. The displayed identity for $1 - L_t$ is a direct algebraic simplification, and every factor in the numerator and denominator is positive for $0 \leq t < 1$. $\square$

Corollary 3.3 (Attractors for $t < 1$). *For every $t \in [0, 1)$ there is a unique nonempty compact set $K_t \subset \overline{\mathbb{D}}$ satisfying*

$$K_t = \Phi_{1,t}(K_t) \cup \Phi_{2,t}(K_t) \cup \Phi_{3,t}(K_t).$$

At $t = 0$, the set K_0 is the standard equilateral Sierpiński gasket.

Proof. This is the classical contraction mapping theorem for iterated function systems on the complete metric space of nonempty compact subsets of $\overline{\mathbb{D}}$ endowed with the Hausdorff metric [5]. The endpoint statement at $t = 0$ was already proved in theorem 2.1. $\square$

4. CONTINUITY AWAY FROM THE PARABOLIC ENDPOINT

The loss of uniform contraction at $t = 1$ is precisely what makes the Apollonian endpoint subtle. Nevertheless the family is well behaved on each truncated interval.

Proposition 4.1. *Fix $\varepsilon \in (0, 1)$. Then there exists $0 < c_\varepsilon < 1$ such that every generator $\Phi_{i,t}$ with $t \in [0, 1 - \varepsilon]$ is c_ε -Lipschitz on $\overline{\mathbb{D}}$. Consequently the attractor map*

$$[0, 1 - \varepsilon] \ni t \mapsto K_t \in \mathcal{K}(\overline{\mathbb{D}})$$

into the compact subsets of $\overline{\mathbb{D}}$ endowed with the Hausdorff metric is continuous.

Proof. By theorem 3.2, the function $t \mapsto L_t$ is continuous and strictly less than 1 on the compact interval $[0, 1 - \varepsilon]$, so it attains a maximum $c_\varepsilon < 1$ there. Thus the Hutchinson operators

$$\mathcal{H}_t(K) = \Phi_{1,t}(K) \cup \Phi_{2,t}(K) \cup \Phi_{3,t}(K)$$

are uniformly contracting on $\mathcal{K}(\overline{\mathbb{D}})$ for $t \in [0, 1 - \varepsilon]$. Since the maps $\Phi_{i,t}$ depend continuously on (z, t) , the operators $\mathcal{H}_t$ depend continuously on t in the Hausdorff metric. Standard

fixed-point stability for contractions therefore implies continuity of the unique fixed point K_t . $\square$

Remark 4.2. The interactive three-dimensional visualization strongly suggests that $K_t \rightarrow K_1$ in the Hausdorff metric as $t \rightarrow 1^-$, where K_1 is the classical equilateral Apollonian gasket generated by the parabolic endpoint maps $\Phi_{i,1}$. A proof of this endpoint continuity is left open here.

5. DIMENSION THEORY OF THE HYPERBOLIC FIBERS

For $t \in [0, 1)$ let

$$\Sigma = \{1, 2, 3\}^{\mathbb{N}}, \quad \sigma((x_n)_{n \geq 1}) = (x_{n+1})_{n \geq 1},$$

and let $\pi_t : \Sigma \rightarrow K_t$ denote the coding map

$$\pi_t(x) = \lim_{n \rightarrow \infty} \Phi_{x_1, t} \circ \cdots \circ \Phi_{x_n, t}(0).$$

Let $\mathcal{P} = \{[1], [2], [3]\}$ be the one-step cylinder partition of Σ , and let $\gamma_{\mathbb{C}}$ denote the Borel σ -algebra of $\mathbb{C}$. For $m \in M_{\sigma}(\Sigma)$ define the projection entropy

$$h_{\pi_t}(\sigma, m) := H_m(\mathcal{P} \mid \sigma^{-1} \pi_t^{-1} \gamma_{\mathbb{C}}) - H_m(\mathcal{P} \mid \pi_t^{-1} \gamma_{\mathbb{C}}),$$

and the Lyapunov exponent

$$\chi_t(m) := - \int_{\Sigma} \log |\Phi'_{x_1, t}(\pi_t(\sigma x))| dm(x).$$

Theorem 5.1 (Exact Hausdorff dimension of every hyperbolic fiber). *For every $t \in [0, 1)$ the family $\{\Phi_{1,t}, \Phi_{2,t}, \Phi_{3,t}\}$ is a conformal IFS on the closed unit disk. Consequently:*

- (i) *for every ergodic $m \in M_{\sigma}(\Sigma)$, the projected measure $\mu_{t,m} := m \circ \pi_t^{-1}$ is exact-dimensional and*

$$\dim_{\text{H}} \mu_{t,m} = \frac{h_{\pi_t}(\sigma, m)}{\chi_t(m)};$$

- (ii) *the attractor itself satisfies the exact variational formula*

$$\dim_{\text{H}} K_t = \dim_B K_t = \sup_{m \in M_{\sigma}(\Sigma)} \frac{h_{\pi_t}(\sigma, m)}{\chi_t(m)} = \sup_{\substack{m \in M_{\sigma}(\Sigma) \\ m \text{ ergodic}}} \dim_{\text{H}} \mu_{t,m}.$$

Proof. Fix $t < 1$. By theorem 3.2, each $\Phi_{i,t}$ is a strict contraction of $\overline{\mathbb{D}}$. Since $\Phi_{i,t}$ is a Möbius map with pole at $-1/(\alpha_t \bar{v}_i)$ and $|\alpha_t| < 1$, this pole lies strictly outside $\overline{\mathbb{D}}$. Hence each branch extends holomorphically to an open neighborhood of $\overline{\mathbb{D}}$. Its derivative is given by (2), so it never vanishes there. Therefore the family is a conformal IFS in the sense of Feng and Hu [4, Definition 2.9]. Part (i) is exactly [4, Theorem 2.8], and part (ii) is [4, Theorem 2.13]. $\square$

Corollary 5.2 (The Sierpiński endpoint). *At $t = 0$ one has*

$$\dim_{\text{H}}(K_0) = \frac{\log 3}{\log 2}.$$

Proof. At $t = 0$ the branches are the three similarities $z \mapsto (z + v_i)/2$ with strong separation on the open triangle interior. Hence the classical similarity-dimension formula applies. $\square$

Corollary 5.3 (The Apollonian endpoint). *At $t = 1$ one has*

$$\dim_{\text{H}}(K_1) = 1.3056867280498771846459862068510408911060 \dots \pm 10^{-129}.$$

Proof. At $t = 1$ the family is the classical equilateral Apollonian parabolic system, so this is exactly the theorem of Vytnova and Wormell [8]. Earlier transfer-operator approximations go back to McMullen [6]. $\square$

Remark 5.4. Theorem 5.1 gives an exact formula for every intermediate fiber, but it is variational rather than closed-form. What remains outside the scope of this paper is a more explicit evaluation of the supremum as a function of t , analogous to the much subtler parabolic computation at $t = 1$.

COMPANION INTERACTIVE DEMONSTRATION

A companion interactive demonstration implementing the exact family (1), together with a three-dimensional visualization of the deformation, is available at <https://complextrees.com/sierpinski-apollonian-stack>; see [2]. The optional bend control in the application is a display-only visualization mode and is not part of the mathematical statements of the paper. For comparison with the affine-projective Rauzy–Sierpiński deformation, see [3, 1].

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