

AN EXPLICIT AFFINE-PROJECTIVE INTERPOLATION BETWEEN THE SIERPIŃSKI GASKET AND THE RAUZY GASKET

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ABSTRACT. We study an explicit one-parameter family of projective maps on the standard simplex that interpolates between the affine iterated function system of the Sierpiński gasket at $t = 0$ and the Rauzy system at $t = 1$. We prove that the first-level decomposition is independent of the parameter and that, for every $t < 1$, the system is uniformly contracting in an explicit affine chart. For $t < 1$ the symbolic address relation is the classical Sierpiński relation, while the Rauzy endpoint fits continuously into the same quotient; consequently the fibers K_t vary continuously in the Hausdorff metric on $[0, 1]$ and form a three-dimensional stack homeomorphic to $S \times [0, 1]$. Along constant symbolic rays we obtain closed formulas that distinguish exponential collapse for $t < 1$ from polynomial $O(n^{-1})$ collapse at $t = 1$. Finally, for every $0 < t < 1$ we prove that $\dim_H K_t = s_A(\Gamma_t)$, where $s_A(\Gamma_t)$ is the projective affinity dimension of the normalized semigroup; moreover $s_A(\Gamma_t) < 2$ throughout the interpolation.

1. INTRODUCTION

The Rauzy gasket is a classical self-projective fractal arising in multidimensional continued fractions, interval exchange transformations, and Arnoux–Rauzy symbolic dynamics. Arnoux and Starosta proved that it is homeomorphic to the usual Sierpiński gasket via a higher-dimensional analogue of Minkowski’s question-mark function [1]. On the metric side, Avila, Hubert, and Skripchenko proved that its Hausdorff dimension is strictly smaller than 2 [2]; Gutiérrez-Romo and Matheus obtained a nontrivial lower bound [6]; Pollicott and Sewell sharpened the upper bound [9]; and Jurga proved that the Hausdorff dimension equals the projective affinity dimension [8].

The present paper studies an explicit one-parameter family of projective maps that connects the affine Sierpiński system at $t = 0$ to the projective Rauzy system at $t = 1$. A companion interactive web application is included in [3]. For an analogous Möbius interpolation from the Sierpiński gasket to the equilateral Apollonian gasket, see the companion manuscript [5] and its supplementary interactive application [4]. The endpoint homeomorphism is therefore not the main novelty here; rather, the point is that the interpolation itself can be written in elementary closed form and analyzed directly.

Our main contributions are the following.

- (i) The three first-level cells are independent of t : the central hole is always the same open medial triangle.
- (ii) For every $t < 1$, the maps are strict contractions in an explicit affine chart, so the attractor K_t exists by Hutchinson’s theorem [7].
- (iii) For $t < 1$, the address equivalence relation is the standard Sierpiński relation; moreover the Rauzy endpoint fits continuously into the same symbolic quotient.
- (iv) The resulting three-dimensional stack is homeomorphic to $S \times [0, 1]$.

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- (v) Constant-address cells admit closed formulas. Their size decays exponentially for $t < 1$ but only like $1/n$ at $t = 1$, which gives an exact signature of the parabolic endpoint.
- (vi) For every $0 < t < 1$, the normalized semigroup is projectively conjugate to a positive SL_3 -system, is strongly irreducible and Zariski dense, and therefore satisfies

$$\dim_H K_t = s_A(\Gamma_t).$$

In particular, the ambient cut-off at dimension 2 never occurs along the interpolation.

The geometric viewpoint behind the interpolation was communicated by Pierre Arnoux to the first author at ICERM in 2019.

2. THE INTERPOLATING FAMILY

2.1. Simplex coordinates. Let

$$\Delta := \{(x_1, x_2, x_3) \in \mathbb{R}_{\geq 0}^3 : x_1 + x_2 + x_3 = 1\}$$

be the standard 2-simplex, with vertices

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

We use the affine chart

$$\iota : T \rightarrow \Delta, \quad \iota(u, v) = (1 - u - v, u, v),$$

where

$$T := \{(u, v) \in \mathbb{R}^2 : u \geq 0, v \geq 0, u + v \leq 1\}.$$

In this chart the vertices are

$$e_1 = (0, 0), \quad e_2 = (1, 0), \quad e_3 = (0, 1).$$

For display purposes we also use the affine equilateral-triangle embedding

$$\mathcal{E}(u, v) := \left(u + \frac{v}{2} - \frac{1}{2}, \frac{\sqrt{3}}{2}v\right) \in \mathbb{R}^2,$$

which sends e_1, e_2, e_3 to the vertices of an equilateral triangle. This changes only the presentation of the figures: all formulas and proofs continue to use the chart T . For $i \neq j$ let

$$m_{ij} := \frac{e_i + e_j}{2}.$$

We also write

$$T_1 := \{u + v \leq 1/2\}, \quad T_2 := \{u \geq 1/2\}, \quad T_3 := \{v \geq 1/2\},$$

so that $T = T_1 \cup T_2 \cup T_3 \cup H$, where the open central hole is

$$H := \{u < 1/2, v < 1/2, u + v > 1/2\}.$$

See figure 1.

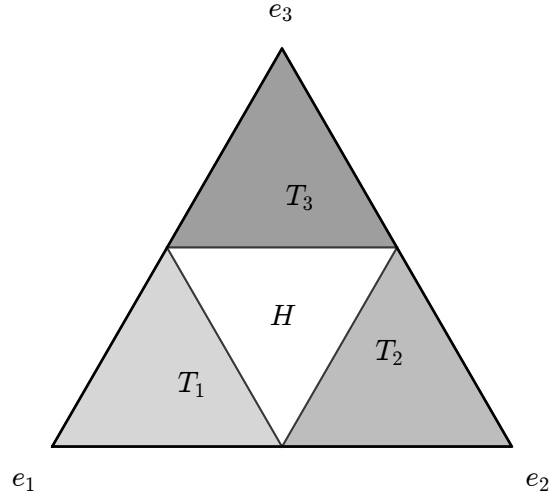


FIGURE 1. The parameter-independent first-level decomposition $T = T_1 \cup T_2 \cup T_3 \cup H$, displayed through the equilateral embedding $\mathcal{E}$. The central hole H is the same for every $t \in [0, 1]$.

2.2. Definition of the maps. For $t \in [0, 1]$ define matrices

$$M_1(t) = \begin{pmatrix} 2-t & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2-t & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2-t \end{pmatrix}.$$

Since $\det M_i(t) = 2-t > 0$, each $M_i(t)$ is invertible. Let $\mathbf{1} = (1, 1, 1)^\top$ and set

$$f_{i,t}(x) := \frac{M_i(t)x}{\mathbf{1}^\top M_i(t)x}, \quad x \in \Delta.$$

In the chart T these maps become

$$\begin{aligned} (1) \quad f_{1,t}(u, v) &= \left(\frac{u}{2-t+t(u+v)}, \frac{v}{2-t+t(u+v)} \right), \\ (2) \quad f_{2,t}(u, v) &= \left(\frac{1+(1-t)u}{2-tu}, \frac{v}{2-tu} \right), \\ (3) \quad f_{3,t}(u, v) &= \left(\frac{u}{2-tv}, \frac{1+(1-t)v}{2-tv} \right). \end{aligned}$$

At $t = 0$ one recovers the affine Sierpiński maps

$$g_1(u, v) = \left(\frac{u}{2}, \frac{v}{2} \right), \quad g_2(u, v) = \left(\frac{1+u}{2}, \frac{v}{2} \right), \quad g_3(u, v) = \left(\frac{u}{2}, \frac{1+v}{2} \right),$$

while at $t = 1$ one obtains the classical Rauzy inverse branches from [1, §3.1].

Proposition 2.1 (Parameter-independent first-level geometry). *For every $t \in [0, 1]$ one has*

$$f_{1,t}(T) = T_1, \quad f_{2,t}(T) = T_2, \quad f_{3,t}(T) = T_3.$$

Equivalently, in barycentric coordinates,

$$f_{i,t}(\Delta) = \Delta_i := \{x \in \Delta : x_i \geq 1/2\}.$$

Moreover,

$$T_i \cap T_j = \{m_{ij}\} \quad (i \neq j).$$

Proof. For $f_{1,t}$, let $s = u + v$. Then (1) gives

$$f_{1,t}(u, v) = \frac{1}{2 - t + ts}(u, v), \quad s' := u' + v' = \frac{s}{2 - t + ts}.$$

As s varies in $[0, 1]$, the function $s \mapsto s/(2 - t + ts)$ is increasing and maps $[0, 1]$ onto $[0, 1/2]$. The ratio v/u is preserved, so every point of T_1 is reached along the unique ray from the origin through that point. Hence $f_{1,t}(T) = T_1$.

The proofs for $f_{2,t}$ and $f_{3,t}$ are identical after permuting coordinates. The intersection statement is then immediate from the explicit descriptions of T_1, T_2, T_3 . $\square$

The piecewise inverse map F_t on $T_1 \cup T_2 \cup T_3$ is therefore well defined by

$$F_t|_{T_i} = f_{i,t}^{-1}.$$

Explicitly,

$$F_t(u, v) = \begin{cases} \left(\frac{(2-t)u}{1-t(u+v)}, \frac{(2-t)v}{1-t(u+v)} \right), & (u, v) \in T_1, \\ \left(\frac{2u-1}{1-t+tu}, \frac{(2-t)v}{1-t+tu} \right), & (u, v) \in T_2, \\ \left(\frac{(2-t)u}{1-t+tv}, \frac{2v-1}{1-t+tv} \right), & (u, v) \in T_3. \end{cases}$$

At $t = 0$ this is the usual expanding map for the Sierpiński gasket; at $t = 1$ it is the Rauzy map.

3. ATTRACTORS IN THE CONTRACTING REGIME

Proposition 3.1 (Uniform contraction for $t < 1$). *Fix $t \in [0, 1)$ and set*

$$q_t := \frac{1}{2-t} < 1.$$

Then each map $f_{i,t} : T \rightarrow T$ is q_t -Lipschitz for the norm

$$\|(u, v)\|_\infty := \max\{|u|, |v|\}.$$

Consequently, the Hutchinson operator

$$\mathcal{H}_t(X) := f_{1,t}(X) \cup f_{2,t}(X) \cup f_{3,t}(X)$$

is a strict contraction on the space of nonempty compact subsets of T endowed with the Hausdorff metric induced by $\|\cdot\|_\infty$.

Proof. For $f_{1,t}$ let $D = 2 - t + t(u + v)$. Differentiating (1) yields

$$Df_{1,t}(u, v) = \frac{1}{D^2} \begin{pmatrix} 2 - t + tv & -tu \\ -tv & 2 - t + tu \end{pmatrix}.$$

The absolute row sums are

$$\frac{2 - t + t(u + v)}{D^2} = \frac{1}{D} \leq \frac{1}{2-t} = q_t.$$

Thus $\|Df_{1,t}(u, v)\|_{\infty \rightarrow \infty} \leq q_t$.

For $f_{2,t}$ let $E = 2 - tu$. Differentiating (2) gives

$$Df_{2,t}(u, v) = \frac{1}{E^2} \begin{pmatrix} 2 - t & 0 \\ tv & E \end{pmatrix}.$$

The first absolute row sum is $(2 - t)/E^2 \leq 1/(2 - t)$. For the second row,

$$\frac{tv + E}{E^2} \leq \frac{E + (E - (2 - t))}{E^2} = \frac{2E - (2 - t)}{E^2} \leq \frac{1}{2 - t},$$

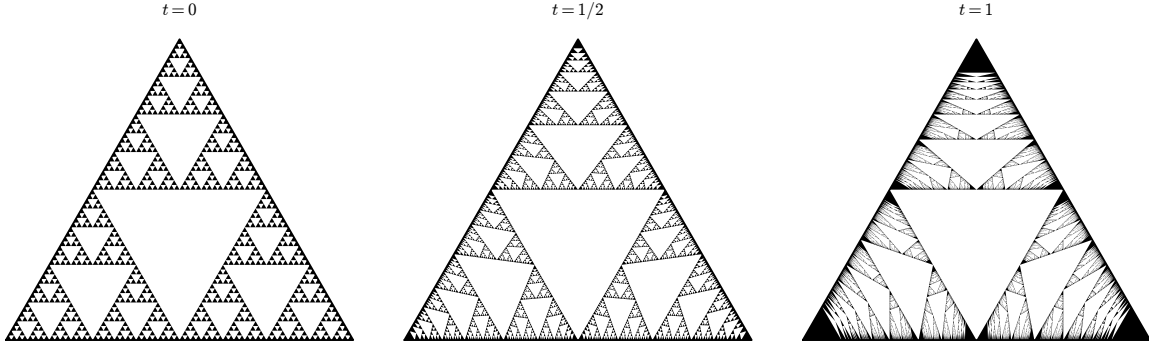


FIGURE 2. Finite-level approximants for $t = 0$, $t = 1/2$, and $t = 1$, displayed in the same equilateral embedding $\mathcal{E}$. The first-level hole is unchanged, but the geometry of deeper cells drifts from affine to projective. The endpoint $t = 1$ is the Rauzy fiber.

because $tv \leq t(1 - u) = E - (2 - t)$ and the function $x \mapsto (2x - (2 - t))/x^2$ is decreasing on $[2 - t, 2]$. The proof for $f_{3,t}$ is symmetric.

The claim about $\mathcal{H}_t$ follows from the usual Hausdorff-metric estimate for finite unions of Lipschitz maps. $\square$

Definition 3.2. For $t \in [0, 1)$, let $K_t \subset T$ denote the unique nonempty compact set satisfying

$$K_t = f_{1,t}(K_t) \cup f_{2,t}(K_t) \cup f_{3,t}(K_t).$$

We also set $K_0 =: S$, the Sierpiński gasket.

4. CODING AND TOPOLOGICAL CONSTANCY BELOW THE RAUZY ENDPOINT

Let $\Sigma := \{1, 2, 3\}^{\mathbb{N}}$ be the full one-sided shift. For a finite word

$$w = i_1 \cdots i_n \in \{1, 2, 3\}^{<\infty}$$

we write

$$f_{w,t} := f_{i_1,t} \circ \cdots \circ f_{i_n,t}.$$

Proposition 4.1 (Coding maps). *For every $t \in [0, 1)$ and every $\omega = (\omega_n)_{n \geq 1} \in \Sigma$, the limit*

$$\pi_t(\omega) := \lim_{n \rightarrow \infty} f_{\omega_1,t} \circ \cdots \circ f_{\omega_n,t}(x_*)$$

exists, is independent of the base point $x_ \in T$, and belongs to K_t . The map $\pi_t : \Sigma \rightarrow K_t$ is continuous and onto.*

Proof. Fix $x_* \in T$. By Proposition 3.1, for every $n, p \geq 0$,

$$\|f_{\omega_1,t} \circ \cdots \circ f_{\omega_{n+p},t}(x_*) - f_{\omega_1,t} \circ \cdots \circ f_{\omega_n,t}(x_*)\|_{\infty} \leq q_t^n \text{diam}(T),$$

so the sequence is Cauchy and the limit exists. The same estimate shows independence of x_* and continuity of π_t . Surjectivity is standard for attractors of contractive IFSs. $\square$

Definition 4.2. Define an equivalence relation $\sim$ on Σ by declaring that

$$wij^{\infty} \sim wji^{\infty} \quad (i \neq j, w \in \{1, 2, 3\}^{<\infty})$$

and taking the smallest equivalence relation generated by these pairs.

Theorem 4.3 (Parameter-independent address relation). *For every $t \in [0, 1]$ and every $\omega, \eta \in \Sigma$ one has*

$$\pi_t(\omega) = \pi_t(\eta) \iff \omega \sim \eta.$$

In particular, the quotient $Q := \Sigma / \sim$ is canonically homeomorphic to K_t for every $t < 1$.

Proof. The implication $\omega \sim \eta \Rightarrow \pi_t(\omega) = \pi_t(\eta)$ is immediate from

$$f_{i,t}(e_j) = m_{ij} = f_{j,t}(e_i), \quad f_{i,t}(e_i) = e_i.$$

Indeed,

$$\pi_t(wij^\infty) = f_{w,t}(f_{i,t}(e_j)) = f_{w,t}(m_{ij}) = f_{w,t}(f_{j,t}(e_i)) = \pi_t(wji^\infty).$$

Conversely, assume $\pi_t(\omega) = \pi_t(\eta)$ and $\omega \neq \eta$. Let w be their maximal common prefix, and write

$$\omega = wi\alpha, \quad \eta = wj\beta,$$

with $i \neq j$. Then

$$\pi_t(\omega) \in f_{w,t}(T_i) \cap f_{w,t}(T_j).$$

By Proposition 2.1, $T_i \cap T_j = \{m_{ij}\}$, and since $f_{w,t}$ is injective we obtain

$$\pi_t(\omega) = \pi_t(\eta) = f_{w,t}(m_{ij}).$$

Applying $f_{w,t}^{-1}$ and then $f_{i,t}^{-1}$ yields

$$\pi_t(\alpha) = e_j.$$

But $e_j \in T_j$ and $e_j \notin T_k$ for $k \neq j$, so every coding of e_j must begin with j . Iterating this argument shows that the only coding of e_j is j^∞ . Hence $\alpha = j^\infty$ and similarly $\beta = i^\infty$. Therefore $\omega \sim \eta$. $\square$

Corollary 4.4 (Canonical homeomorphisms with the Sierpiński gasket). *For each $t \in [0, 1]$ there exists a canonical homeomorphism*

$$H_t : S \rightarrow K_t$$

obtained by matching symbolic addresses. Equivalently,

$$H_t(\pi_0(\omega)) = \pi_t(\omega) \quad (\omega \in \Sigma).$$

Proof. By Theorem 4.3, both π_0 and π_t factor through the same quotient $Q = \Sigma / \sim$. The induced maps $\bar{\pi}_0 : Q \rightarrow S$ and $\bar{\pi}_t : Q \rightarrow K_t$ are continuous bijections from a compact space to Hausdorff spaces, hence homeomorphisms. Set $H_t = \bar{\pi}_t \circ \bar{\pi}_0^{-1}$. $\square$

Remark 4.5 (The Rauzy endpoint). At $t = 1$ the system ceases to be uniformly contractive, but the endpoint fiber is classical: by [1, Theorem 2 and §4], the unique invariant compact set is the Rauzy gasket R , and the corresponding coding relation is again the standard Sierpiński one. We write

$$K_1 := R,$$

and denote by $\pi_1 : \Sigma \rightarrow R$ the Arnoux–Starosta coding map, which factors through the same quotient $Q = \Sigma / \sim$.

5. GLOBAL CONTINUITY UP TO THE RAUZY ENDPOINT

For a finite word $w \in \{1, 2, 3\}^{<\infty}$ and $t \in [0, 1]$, let

$$C_w(t) := f_{w,t}(T).$$

For an infinite sequence $\omega \in \Sigma$ we write $\omega|_n$ for its prefix of length n .

Lemma 5.1 (Finite cylinders vary continuously). *For every finite word w , the map $t \mapsto C_w(t)$ is continuous on $[0, 1]$ for the Hausdorff metric.*

Proof. The map $(x, t) \mapsto f_{w,t}(x)$ is continuous on the compact set $T \times [0, 1]$. Hence it is uniformly continuous. Standard compactness then implies that the images $f_{w,t}(T)$ vary continuously in the Hausdorff metric. $\square$

Theorem 5.2 (Continuity of the coding map on the full parameter interval). *Define*

$$\Pi : \Sigma \times [0, 1] \rightarrow T, \quad \Pi(\omega, t) := \pi_t(\omega),$$

with π_t as in Proposition 4.1 for $t < 1$ and π_1 as in Remark 4.5. Then Π is continuous on the whole compact space $\Sigma \times [0, 1]$.

Proof. On $\Sigma \times [0, 1 - \varepsilon]$, continuity follows from the uniform contraction argument used in Proposition 4.1: the finite approximants

$$\Pi_n(\omega, t) := f_{\omega_1,t} \circ \cdots \circ f_{\omega_n,t}(x_*)$$

converge uniformly to Π on every set of the form $\Sigma \times [0, 1 - \varepsilon]$.

It remains to prove continuity at points $(\omega, 1)$. Fix such a point and let $\varepsilon > 0$. Since the Rauzy coding identifies the same quotient $Q = \Sigma / \sim$ by Remark 4.5, the nested compact cylinders

$$C_{\omega|_1}(1) \supset C_{\omega|_2}(1) \supset \cdots$$

have singleton intersection $\{\pi_1(\omega)\}$. Therefore

$$\text{diam } C_{\omega|_n}(1) \longrightarrow 0.$$

Choose n so large that

$$\text{diam } C_{\omega|_n}(1) < \varepsilon/3.$$

By Lemma 5.1, there exists $\delta > 0$ such that

$$d_H(C_{\omega|_n}(t), C_{\omega|_n}(1)) < \varepsilon/3$$

whenever $|t - 1| < \delta$.

Now let $\eta \in \Sigma$ have the same prefix of length n as ω , and let $|t - 1| < \delta$. Then

$$\pi_t(\eta) \in C_{\eta|_n}(t) = C_{\omega|_n}(t).$$

Hence there exists $y \in C_{\omega|_n}(1)$ such that

$$\|\pi_t(\eta) - y\|_\infty < \varepsilon/3.$$

Since both y and $\pi_1(\omega)$ lie in $C_{\omega|_n}(1)$, we also have

$$\|y - \pi_1(\omega)\|_\infty \leq \text{diam } C_{\omega|_n}(1) < \varepsilon/3.$$

Therefore

$$\|\pi_t(\eta) - \pi_1(\omega)\|_\infty < 2\varepsilon/3.$$

This proves continuity at $(\omega, 1)$. $\square$

Corollary 5.3 (Hausdorff continuity of the fibers on the full interval). *The map*

$$[0, 1] \rightarrow \mathcal{K}(T), \quad t \mapsto K_t,$$

is continuous for the Hausdorff metric, where $\mathcal{K}(T)$ denotes the space of nonempty compact subsets of T .

Proof. By Theorem 4.3, Remark 4.5, and Theorem 5.2, the map $\bar{\Pi}$ is continuous and constant on the equivalence classes of $\sim$ for every $t \in [0, 1]$. It therefore descends to a continuous map

$$\bar{\Pi} : Q \times [0, 1] \rightarrow T, \quad \bar{\Pi}(q, t) = \bar{\pi}_t(q),$$

whose image at time t is precisely K_t . Since $Q \times [0, 1]$ is compact, $\bar{\Pi}$ is uniformly continuous, so

$$d_H(K_s, K_t) \leq \sup_{q \in Q} \|\bar{\Pi}(q, s) - \bar{\Pi}(q, t)\|_\infty \xrightarrow{s \rightarrow t} 0.$$

□

6. CONSTANT-ADDRESS CELLS AND THE PARABOLIC RAUZY LIMIT

The interpolation becomes especially transparent along constant symbolic rays.

Proposition 6.1 (Closed formula for constant-address iterates). *Fix $t \in [0, 1]$ and let $a := 2 - t$. For every $n \geq 1$ and every $(u, v) \in T$,*

$$f_{1,t}^n(u, v) = \lambda_{n,t}(u + v)(u, v),$$

where, for $s \in [0, 1]$,

$$\lambda_{n,t}(s) = \begin{cases} \frac{1}{a^n + t \frac{a^n - 1}{1-t} s}, & 0 \leq t < 1, \\ \frac{1}{1 + ns}, & t = 1. \end{cases}$$

In particular,

$$f_{1,t}^n(T) = \{u + v \leq \sigma_n(t)\},$$

with

$$\sigma_n(t) = \begin{cases} \frac{1-t}{(2-t)^n - t}, & 0 \leq t < 1, \\ \frac{1}{n+1}, & t = 1. \end{cases}$$

The analogous formulas hold for $f_{2,t}^n$ and $f_{3,t}^n$ by symmetry.

Proof. Let $s = u + v$. Since (1) preserves every ray from the origin,

$$f_{1,t}(u, v) = \frac{1}{a + ts}(u, v).$$

Hence $f_{1,t}^n(u, v) = \lambda_n(s)(u, v)$ for some scalar factor $\lambda_n(s)$. Writing $\mu_n(s) := \lambda_n(s)^{-1}$, the recursion becomes

$$\mu_{n+1}(s) = a \mu_n(s) + ts, \quad \mu_1(s) = a + ts.$$

Solving this affine recurrence yields

$$\mu_n(s) = a^n + t(1 + a + \cdots + a^{n-1})s = a^n + t \frac{a^n - 1}{a - 1} s,$$

which is the stated formula for $t < 1$. The case $t = 1$ is the limit $a \rightarrow 1$ and gives $\mu_n(s) = 1 + ns$.

Finally, $f_{1,t}^n(T)$ consists exactly of those points on rays from the origin whose radial parameter s is at most the image of $s = 1$, namely $\sigma_n(t)$. □

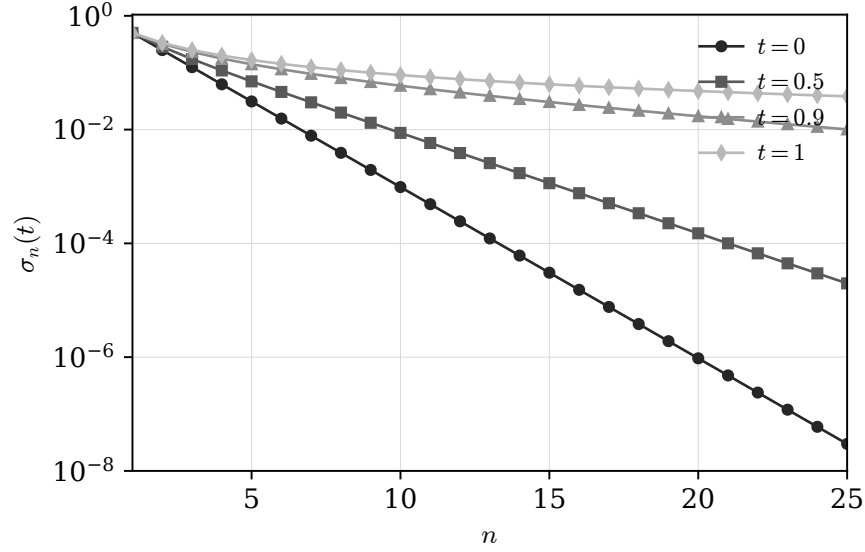


FIGURE 3. The explicit formula for $\sigma_n(t)$ shows exponential decay for every $t < 1$ and polynomial decay at $t = 1$.

Corollary 6.2 (Exponential versus polynomial collapse). *For each fixed $t < 1$ one has $\sigma_n(t) \asymp (2-t)^{-n}$ as $n \rightarrow \infty$, whereas*

$$\sigma_n(1) = \frac{1}{n+1}.$$

Thus the repeated-cell collapse is exponential on every contracting fiber and polynomial at the Rauzy endpoint.

This dichotomy is one of the simplest exact signatures of the passage from the affine Sierpiński end to the parabolic Rauzy end; see figure 3.

7. THE THREE-DIMENSIONAL STACK

To assemble the family into a single closed subset of $\mathbb{R}^3$, we place the fiber K_t in the horizontal plane of height

$$Y = 2(1-t),$$

using the three-dimensional equilateral-triangle embedding

$$E_t(x_1, x_2, x_3) = \left(\frac{\sqrt{3}}{2}(x_3 - x_2), 2(1-t), \frac{3}{2}x_1 - \frac{1}{2} \right) \in \mathbb{R}^3.$$

The bottom face $Y = 0$ is the Rauzy fiber R , while the top face $Y = 2$ is the Sierpiński gasket.

Theorem 7.1 (The stack is a product). *The set*

$$\mathfrak{S} := \{E_t(x) : x \in K_t, t \in [0, 1]\} \subset \mathbb{R}^3$$

is homeomorphic to $S \times [0, 1]$.

Proof. Let $Q = \Sigma/\sim$, which is homeomorphic to S by Corollary 4.4. By Theorem 5.2, the coding map descends to a continuous map

$$\bar{\Pi} : Q \times [0, 1] \rightarrow T, \quad \bar{\Pi}(q, t) = \bar{\pi}_t(q).$$

Composing with E_t gives

$$\Psi : Q \times [0, 1] \rightarrow \mathfrak{S}, \quad \Psi(q, t) = E_t(\bar{\pi}_t(q)).$$

This map is continuous. It is injective because the second coordinate of $E_t(x)$ is exactly $2(1-t)$, so different parameters t land in different horizontal planes; within a fixed plane, injectivity follows from the fact that the coding relation is the same quotient Q for every $t \in [0, 1]$. It is surjective by construction. Since $Q \times [0, 1]$ is compact and $\mathfrak{S} \subset \mathbb{R}^3$ is Hausdorff, Ψ is a homeomorphism. Finally $Q \cong S$. $\square$

The theorem records the topological content of the three-dimensional stack itself. For a dynamic visualization of the embedding E_t and of the resulting closed stack $\mathfrak{S}$, see the companion interactive demonstration [3].

8. HAUSDORFF DIMENSION

This section identifies the exact Hausdorff dimension of every intermediate fiber. The endpoint case $t = 1$ is covered by Jurga's theorem for the Rauzy gasket [8]; here we show that every $0 < t < 1$ falls into the same positive-semigroup framework, and that the ambient truncation at dimension 2 never occurs.

For $t \in (0, 1)$ write

$$a := 2 - t \in (1, 2)$$

and define

$$A_i(t) := a^{-1/3} M_i(t) \in SL_3(\mathbb{R}), \quad \Gamma_t := \langle A_1(t), A_2(t), A_3(t) \rangle.$$

The projective action of $A_i(t)$ on Δ is exactly $f_{i,t}$, since multiplying by a positive scalar does not change projective coordinates. We write $s_A(\Gamma_t)$ for the projective affinity dimension of Γ_t in the sense of Jurga [8]. Thus, if $\alpha_1(g) \geq \alpha_2(g) \geq \alpha_3(g)$ denote the singular values of $g \in SL_3(\mathbb{R})$, then for $0 \leq s \leq 2$

$$\varphi^s(g) := \begin{cases} (\alpha_2(g)/\alpha_1(g))^s, & 0 \leq s \leq 1, \\ (\alpha_2(g)/\alpha_1(g))(\alpha_3(g)/\alpha_1(g))^{s-1}, & 1 \leq s \leq 2. \end{cases}$$

For a word $w = i_1 \cdots i_n$ we write $A_w(t) := A_{i_1}(t) \cdots A_{i_n}(t)$, and we may characterize $s_A(\Gamma_t)$ as the critical exponent of the word series

$$\sum_{n \geq 1} \sum_{|w|=n} \varphi^s(A_w(t)).$$

This is the formulation used below to verify directly that the ambient cut-off at 2 never occurs.

Proposition 8.1 (Explicit positive conjugation). *Fix $t \in (0, 1)$ and write $a = 2 - t$. For every $c \in (1/a, 1)$ sufficiently close to $1/a$, the matrix*

$$P_{a,c} := \begin{pmatrix} 1 & -c & -c \\ -c & 1 & -c \\ -c & -c & 1 \end{pmatrix}$$

has the property that each conjugate

$$B_i(t, c) := P_{a,c}^{-1} A_i(t) P_{a,c}$$

has strictly positive entries.

Proof. Since $A_i(t) = a^{-1/3}M_i(t)$, it is enough to prove positivity for $P_{a,c}^{-1}M_i(t)P_{a,c}$. A direct calculation gives

$$P_{a,c}^{-1}M_1(t)P_{a,c} = \frac{1}{2c^2 + c - 1} \begin{pmatrix} c(a+2) - a & (1-c)(ac-1) & (1-c)(ac-1) \\ c(2c+1-a) & c^2(a+2) - 1 & c(ac-1) \\ c(2c+1-a) & c(ac-1) & c^2(a+2) - 1 \end{pmatrix},$$

and the formulas for $M_2(t)$ and $M_3(t)$ are obtained by cyclic permutation of the coordinates. At the boundary value $c = 1/a$ this becomes

$$\begin{pmatrix} a & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

while the other two conjugates become its cyclic permutations. Since

$$2(1/a)^2 + 1/a - 1 = \frac{a+2-a^2}{a^2} > 0$$

for $1 < a < 2$, the denominator stays positive near $1/a$. The entries that are already positive at $c = 1/a$ remain positive by continuity, and the zero entries are exactly the ones carrying the factor $ac - 1$, so they become positive as soon as $c > 1/a$. Hence every conjugate is strictly positive for $c > 1/a$ close enough to $1/a$. The same then holds for $B_i(t, c) = a^{-1/3}P_{a,c}^{-1}M_i(t)P_{a,c}$. $\square$

Lemma 8.2 (Strong irreducibility). *For every $t \in (0, 1)$, the semigroup Γ_t is strongly irreducible.*

Proof. Scalar normalization does not change invariant subspaces, so it is enough to work with

$$S_t := \langle M_1(t), M_2(t), M_3(t) \rangle.$$

Set $a = 2 - t \in (1, 2)$ and $b = (a - 1)^{-1}$. Then

$$\begin{aligned} a^{-n}M_1(t)^n &\longrightarrow L_1 = \begin{pmatrix} 1 & b & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ a^{-n}M_2(t)^n &\longrightarrow L_2 = \begin{pmatrix} 0 & 0 & 0 \\ b & 1 & b \\ 0 & 0 & 0 \end{pmatrix}, \\ a^{-n}M_3(t)^n &\longrightarrow L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b & 1 \end{pmatrix}. \end{aligned}$$

The row vectors $(1, b, b)$, $(b, 1, b)$, and $(b, b, 1)$ are linearly independent because their determinant is $(1-b)^2(1+2b) \neq 0$. Hence for every nonzero $x \in \mathbb{R}^3$, at least one of L_1x, L_2x, L_3x is nonzero.

Assume that F is a finite S_t -invariant set of projective lines. Choose $\ell = [x] \in F$ with $x \neq 0$. For some i , one has $L_ix \neq 0$. Then $M_i(t)^n\ell \in F$ for all n , and

$$[M_i(t)^n x] \longrightarrow [L_ix] = [e_i].$$

Because F is finite, some infinite subsequence $M_i(t)^{n_k}\ell$ is constant; its limit therefore belongs to F . Thus one coordinate line lies in F . Applying suitable powers of the other generators shows that all three coordinate lines belong to F ; for instance,

$$M_2(t)^n e_1 = e_1 + \frac{a^n - 1}{a - 1} e_2 \longrightarrow [e_2], \quad M_3(t)^n e_1 = e_1 + \frac{a^n - 1}{a - 1} e_3 \longrightarrow [e_3].$$

Now $[e_1 + e_2] = M_2(t)[e_1] \in F$, but

$$M_3(t)^n(e_1 + e_2) = e_1 + e_2 + 2\frac{a^n - 1}{a - 1}e_3$$

yields infinitely many distinct projective lines, contradicting finiteness of F . So no finite invariant set of lines exists.

The same argument applied to the transpose semigroup $S_t^\top$ excludes finite invariant unions of planes, since such unions correspond by orthogonal complement to finite invariant unions of lines for $S_t^\top$. Therefore S_t , and hence Γ_t , is strongly irreducible. $\square$

Lemma 8.3 (Absence of invariant quadratic forms). *For $t \in (0, 1)$ there is no nonzero symmetric matrix Q such that*

$$A_i(t)^\top Q A_i(t) = Q \quad (i = 1, 2, 3).$$

Proof. Write $a = 2 - t \in (1, 2)$. Since $A_i(t) = a^{-1/3}M_i(t)$, the displayed equation is equivalent to

$$M_i(t)^\top Q M_i(t) = a^{2/3}Q \quad (i = 1, 2, 3).$$

Let $Q = (q_{jk})_{1 \leq j, k \leq 3}$. Comparing the $(1, 1)$ -entry in the equation for $i = 1$, the $(2, 2)$ -entry in the equation for $i = 2$, and the $(3, 3)$ -entry in the equation for $i = 3$, one obtains

$$(a^2 - a^{2/3})q_{11} = 0, \quad (a^2 - a^{2/3})q_{22} = 0, \quad (a^2 - a^{2/3})q_{33} = 0.$$

Because $1 < a < 2$, we have $a^2 \neq a^{2/3}$, so $q_{11} = q_{22} = q_{33} = 0$. The $(2, 2)$ -entry in the equation for $i = 1$ then gives $2q_{12} = 0$, the $(3, 3)$ -entry in the equation for $i = 2$ gives $2q_{23} = 0$, and the $(1, 1)$ -entry in the equation for $i = 3$ gives $2q_{13} = 0$. Hence $Q = 0$. $\square$

Proposition 8.4 (Zariski density). *For every $t \in (0, 1)$, the semigroup Γ_t is Zariski dense in $SL_3(\mathbb{R})$.*

Proof. Let G be the Zariski closure of Γ_t . We first claim that the identity component G° acts irreducibly on $\mathbb{R}^3$. Indeed, if W were a nonzero proper G° -invariant subspace, then the finite union

$$\bigcup_{g \in G/G^\circ} gW$$

would be Γ_t -invariant, contradicting Lemma 8.2.

The unipotent radical of G° is therefore trivial: its fixed-point space is nonzero by Lie–Kolchin, and because the radical is normal that fixed-point space is G° -invariant, hence all of $\mathbb{R}^3$; faithfulness then forces the radical to be trivial. So G° is reductive. By Schur’s lemma, the connected center of G° acts by scalars, and since $G^\circ \subset SL_3(\mathbb{R})$, that connected center is trivial. Thus G° is semisimple.

Complexifying the Lie algebra, one obtains a semisimple Lie subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ with a faithful irreducible 3-dimensional representation. The only possibilities are $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$. If the first occurs, then $G^\circ = SL_3(\mathbb{R})$ and we are done. In the second case the representation is the symmetric-square representation of SL_2 , which preserves a nondegenerate quadratic form. That is impossible by Lemma 8.3. Therefore $G^\circ = SL_3(\mathbb{R})$, and hence $G = SL_3(\mathbb{R})$. $\square$

Proposition 8.5 (The affinity dimension is strictly smaller than 2). *For every $t \in (0, 1)$ one has*

$$s_A(\Gamma_t) < 2.$$

Proof. Let $a = 2 - t \in (1, 2)$, so $A_i(t) = a^{-1/3}M_i(t) \in SL_3(\mathbb{R})$. For $s = 2$ one has

$$\varphi^2(A) = \frac{\alpha_2(A)\alpha_3(A)}{\alpha_1(A)^2} = \frac{1}{\alpha_1(A)^3}$$

for every $A \in SL_3(\mathbb{R})$, because $\alpha_1(A)\alpha_2(A)\alpha_3(A) = 1$.

Each matrix $M_i(t)$ is permutation-conjugate to $M_1(t)$, so it is enough to estimate one of them. Since

$$M_1(t)^\top \mathbf{1} = (a, 2, 2)^\top,$$

we obtain

$$\alpha_1(M_i(t)) \geq \frac{\|M_i(t)^\top \mathbf{1}\|_2}{\|\mathbf{1}\|_2} = \sqrt{\frac{a^2 + 8}{3}} \quad (i = 1, 2, 3).$$

Therefore

$$\varphi^2(A_i(t)) = \frac{1}{\alpha_1(A_i(t))^3} = \frac{a}{\alpha_1(M_i(t))^3} \leq \frac{3^{3/2}a}{(a^2 + 8)^{3/2}}.$$

Summing over the three generators gives

$$\sum_{i=1}^3 \varphi^2(A_i(t)) \leq \frac{9\sqrt{3}a}{(a^2 + 8)^{3/2}}.$$

The right-hand side is increasing on $a \in [1, 2]$, because its derivative has the sign of $4 - a^2$. Hence

$$\sum_{i=1}^3 \varphi^2(A_i(t)) \leq \frac{9\sqrt{3} \cdot 2}{(2^2 + 8)^{3/2}} = \frac{3}{4} < 1.$$

By submultiplicativity of φ^2 ,

$$\sum_{|w|=n} \varphi^2(A_w(t)) \leq \left(\sum_{i=1}^3 \varphi^2(A_i(t)) \right)^n \leq \left(\frac{3}{4} \right)^n.$$

Hence the defining series for $s_A(\Gamma_t)$ converges at $s = 2$, so $s_A(\Gamma_t) < 2$. □

Theorem 8.6 (Exact dimension of the intermediate fibers). *For every $t \in (0, 1)$,*

$$\dim_H K_t = s_A(\Gamma_t).$$

Proof. Write $a = 2 - t$ and choose c as in Proposition 8.1. Set

$$\tilde{\Gamma}_t := P_{a,c}^{-1} \Gamma_t P_{a,c}.$$

By Proposition 8.1, the generators of $\tilde{\Gamma}_t$ lie in $SL_3(\mathbb{R})_{>0}$. By Proposition 2.1, the original system satisfies the strong open set condition with open set $\text{int}(T)$: the sets $f_{i,t}(\text{int } T)$ are pairwise disjoint subsets of $\text{int } T$, and $f_{123,t}(T) \subset \text{int } T$ because the product matrix $M_1(t)M_2(t)M_3(t)$ has strictly positive entries. Projective conjugacy transfers this property to $\tilde{\Gamma}_t$. By Proposition 8.4, the semigroup $\tilde{\Gamma}_t$ is Zariski dense because linear conjugacy changes the Zariski closure only by conjugation.

Jurga's dimension theorem for positive SL_3 -semigroups satisfying the strong open set condition [8, Theorem 1.3] therefore gives

$$\dim_H \tilde{K}_t = \min\{2, s_A(\tilde{\Gamma}_t)\},$$

where $\tilde{K}_t$ denotes the projective attractor of $\tilde{\Gamma}_t$. Linear conjugacy changes singular values only by multiplicative constants depending on $P_{a,c}$, so $s_A(\tilde{\Gamma}_t) = s_A(\Gamma_t)$. The projective automorphism

of $\mathbb{RP}^2$ induced by $P_{a,c}$ is a smooth diffeomorphism, hence bi-Lipschitz on compact subsets in affine charts, and it carries $\widetilde{K}_t$ onto K_t . Consequently,

$$\dim_H K_t = \min\{2, s_A(\Gamma_t)\}.$$

The ambient truncation is absent by Proposition 8.5, so in fact $\dim_H K_t = s_A(\Gamma_t)$. $\square$

Corollary 8.7 (Dimensions on the full parameter interval). *Let $\Gamma_1 := \langle M_1(1), M_2(1), M_3(1) \rangle$ be the Rauzy semigroup. Then*

$$\dim_H K_t = \begin{cases} \frac{\log 3}{\log 2}, & t = 0, \\ s_A(\Gamma_t), & 0 < t < 1, \\ s_A(\Gamma_1), & t = 1. \end{cases}$$

Proof. The case $t = 0$ is the classical Hausdorff-dimension computation for the Sierpiński gasket [7]. The range $0 < t < 1$ is exactly Theorem 8.6. The identity at $t = 1$ is Jurga’s theorem for the Rauzy gasket [8]. $\square$

Remark 8.8. The formula for $\dim_H K_t$ is exact, but it remains implicit through the projective affinity dimension $s_A(\Gamma_t)$. Natural next questions are the regularity, monotonicity, and effective computation of the map $t \mapsto s_A(\Gamma_t)$.

9. CONCLUDING REMARKS

The family considered here packages the affine Sierpiński system, the projective Rauzy system, and an explicit interpolation between them into a single elementary formula. Within that family the first-level hole is fixed, the symbolic quotient is stable all the way to the Rauzy endpoint, the three-dimensional stack is a product $S \times [0, 1]$, constant-address cells display a sharp exponential-to-parabolic transition, and every intermediate fiber has exact Hausdorff dimension equal to the projective affinity dimension of its normalized semigroup.

Several natural questions remain. The most immediate is to understand the function $t \mapsto s_A(\Gamma_t)$ itself, including its regularity, possible monotonicity, and direct dynamical meaning. The analogous Möbius deformation between the Sierpiński and equilateral Apollonian gaskets developed in [5], together with its supplementary interactive companion [4], suggests a broader program: it would be interesting to construct similarly explicit interpolations for other projective fractals arising from multidimensional continued fraction algorithms.

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