A Family of Fern-like Ternary Complex Trees

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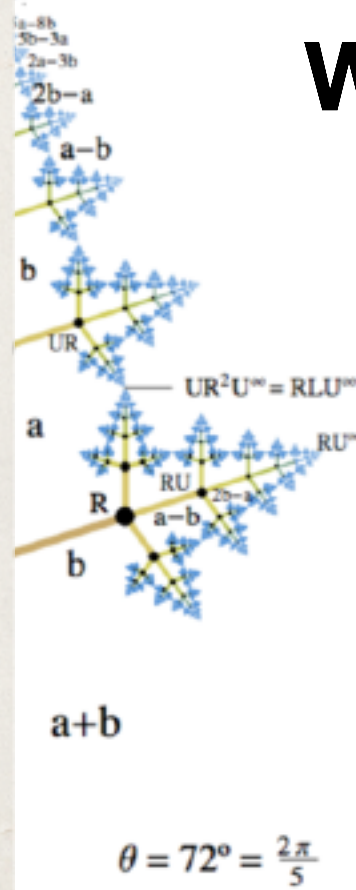
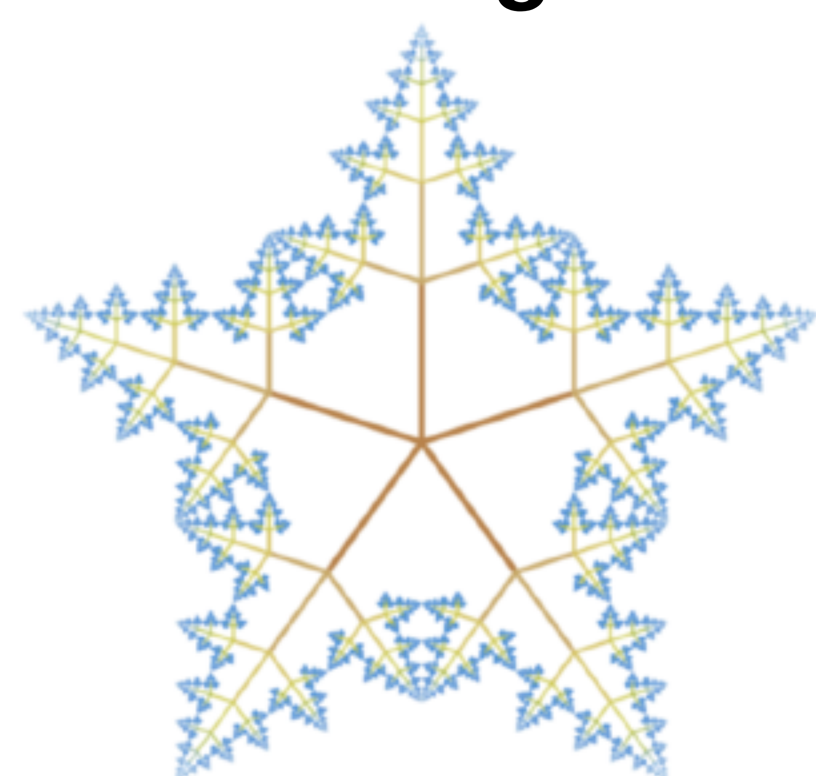
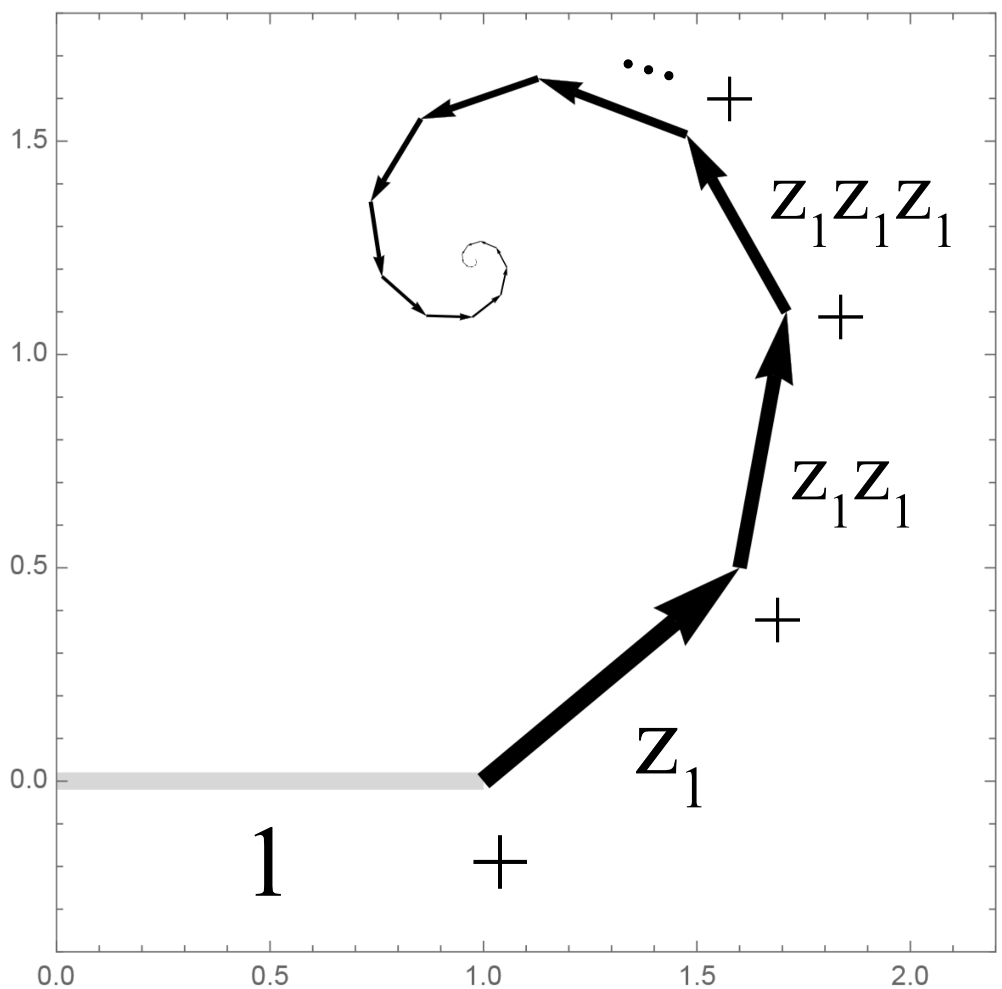
(Left to right: Antonio Garijo, Núria Fagella, Dan P., Anna M. Benini, Jordi Canela, Xavier Jarque, Bernat Espigulé and Robert Florido). Course 2017-18

Wolfram Blog 2014





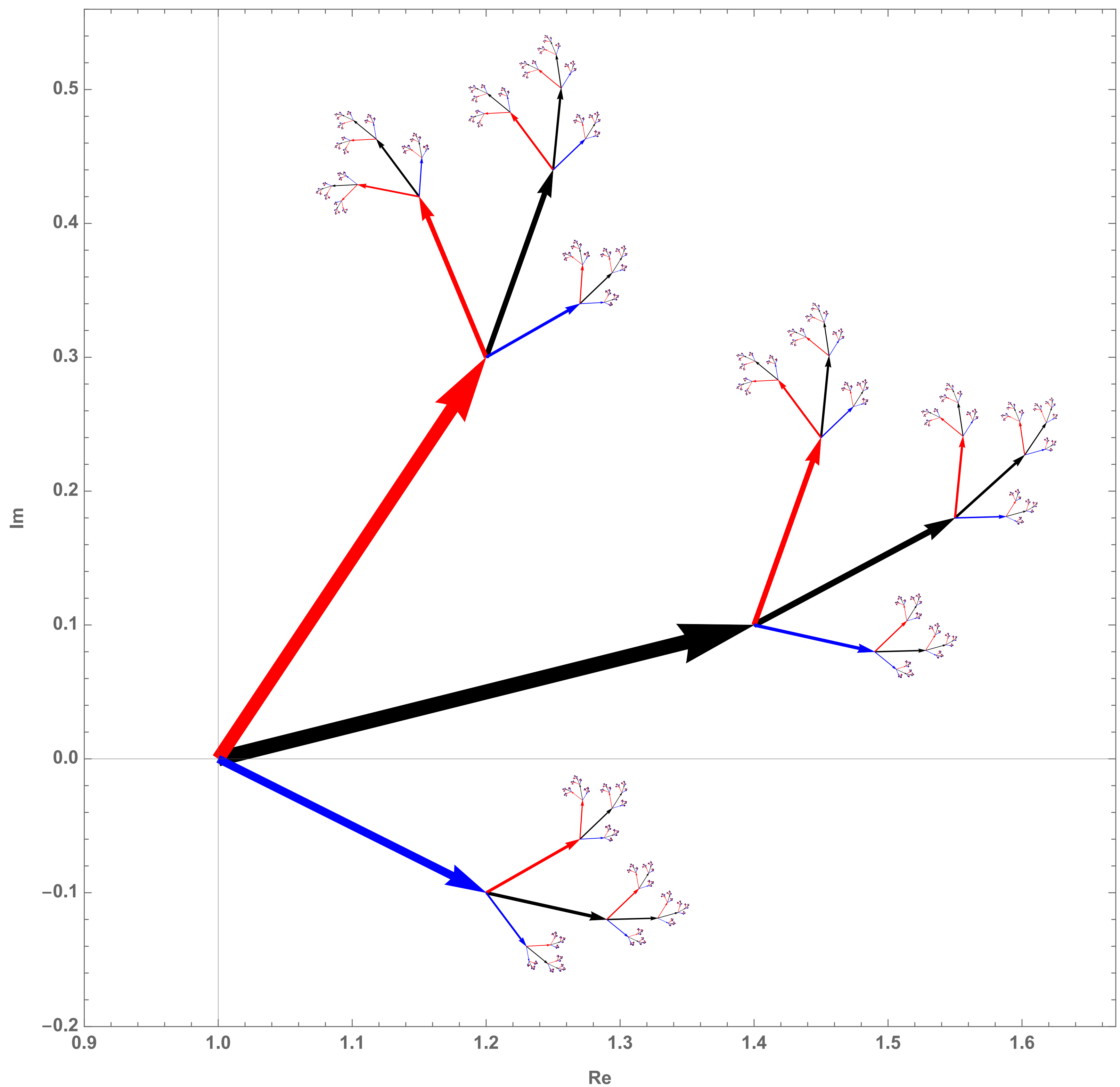
Geometric series ⟶ Complex tree



$$1 + z_1 + z_1^2 + z_1^3 + \ldots = \frac{1}{1 - z_1}$$

$$T_A = T\{z_1\}$$

$$T_A = T\{c_1, {\color{red}c_2}, {\color{blue}c_3}\}$$



complex-valued alphabet $A = \{c_1, c_2\}$

letters $w_1, w_2, \ldots, w_m \in A = \{1 \to c_1, 2 \to c_2\}$

word $w = w_1 w_2 \ldots w_m$

$$\phi(w) = 1 + w_1 + w_1 w_2 + \ldots + w_1 w_2 \ldots w_m$$

$$\phi(1) = 1 + c_1$$

$$\phi(11) = 1 + c_1 + c_1^2$$

$$\phi(112) = 1 + c_1 + c_1^2 + c_1^2 c_2$$

nodes

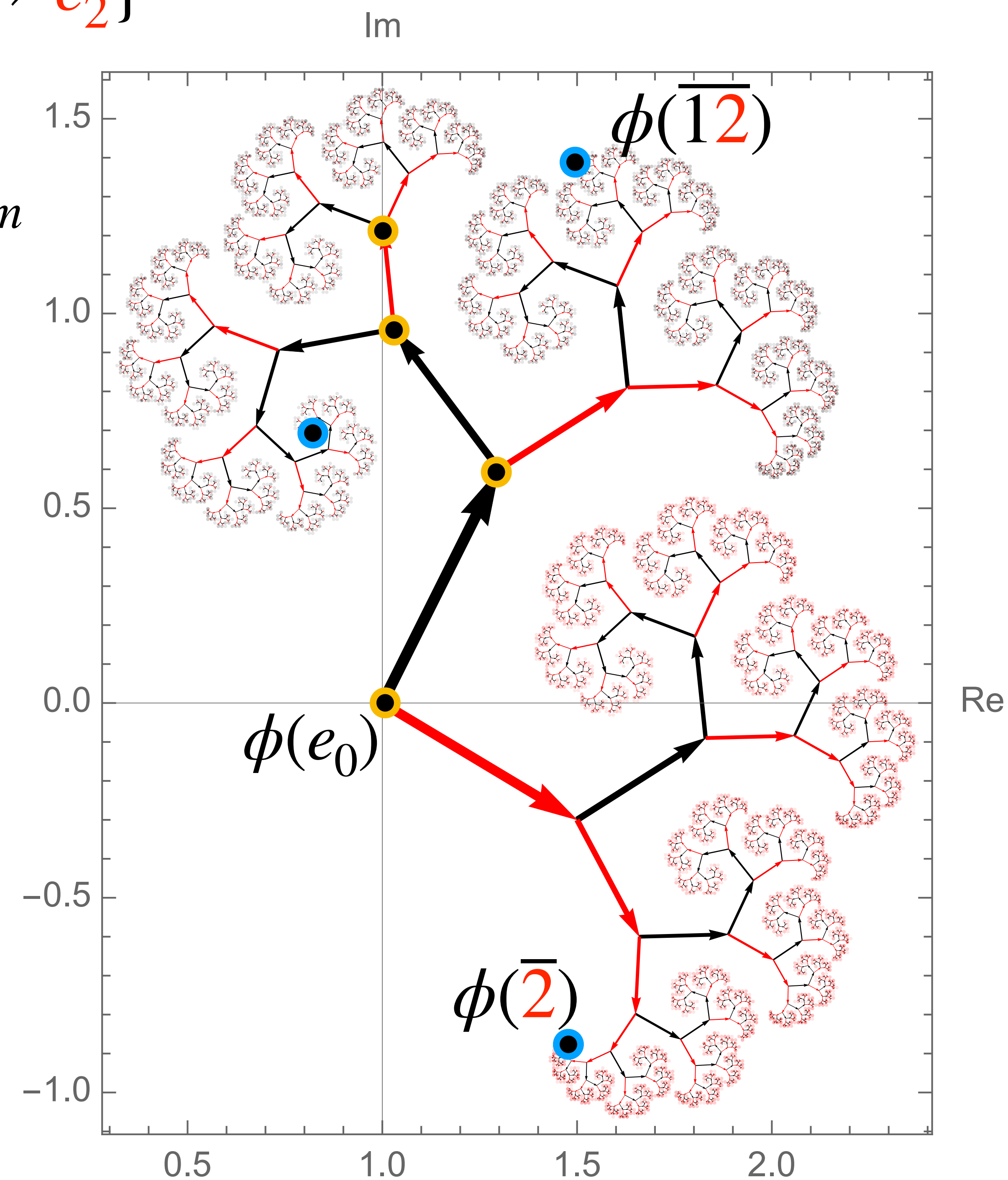
empty string e_0 **root** $\phi(e_0) = 1$

$$w = w_1 w_2 w_3 \ldots w_k \ldots \in \{c_1, c_2\}^\infty = A^\infty$$

tip points $\phi(w)$

$$\phi(111\ldots) = \phi(\overline{1}) = \sum_{k=0}^{\infty} c_1^k = \frac{1}{1 - c_1}$$

$$\phi(121212\ldots) = \phi(\overline{12}) = \frac{1 + c_1}{1 - c_1 c_2}$$

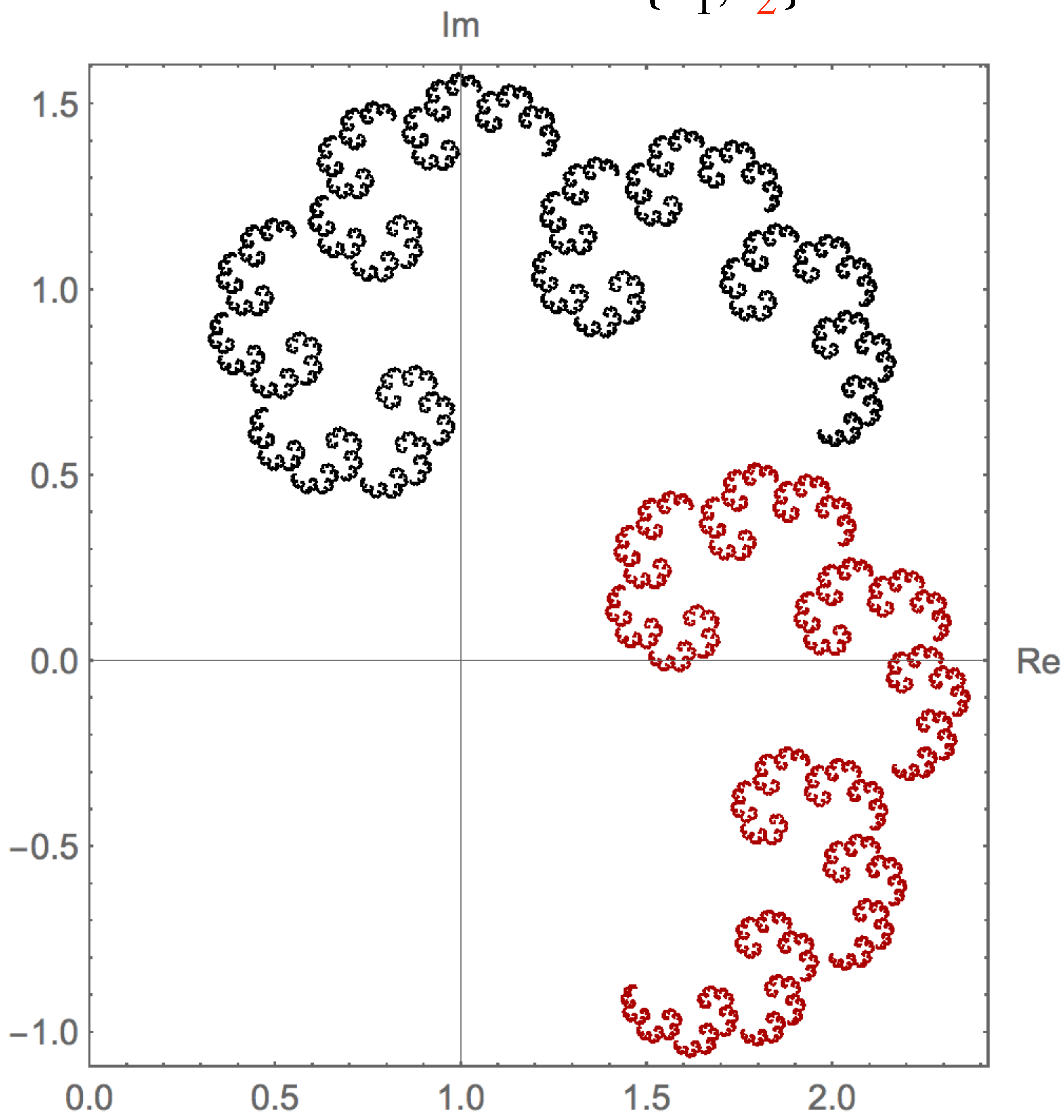


complex tree

$$T_A = T\{c_1, c_2\} = T\{.3 + i.6,\ .5 - i.3\}$$

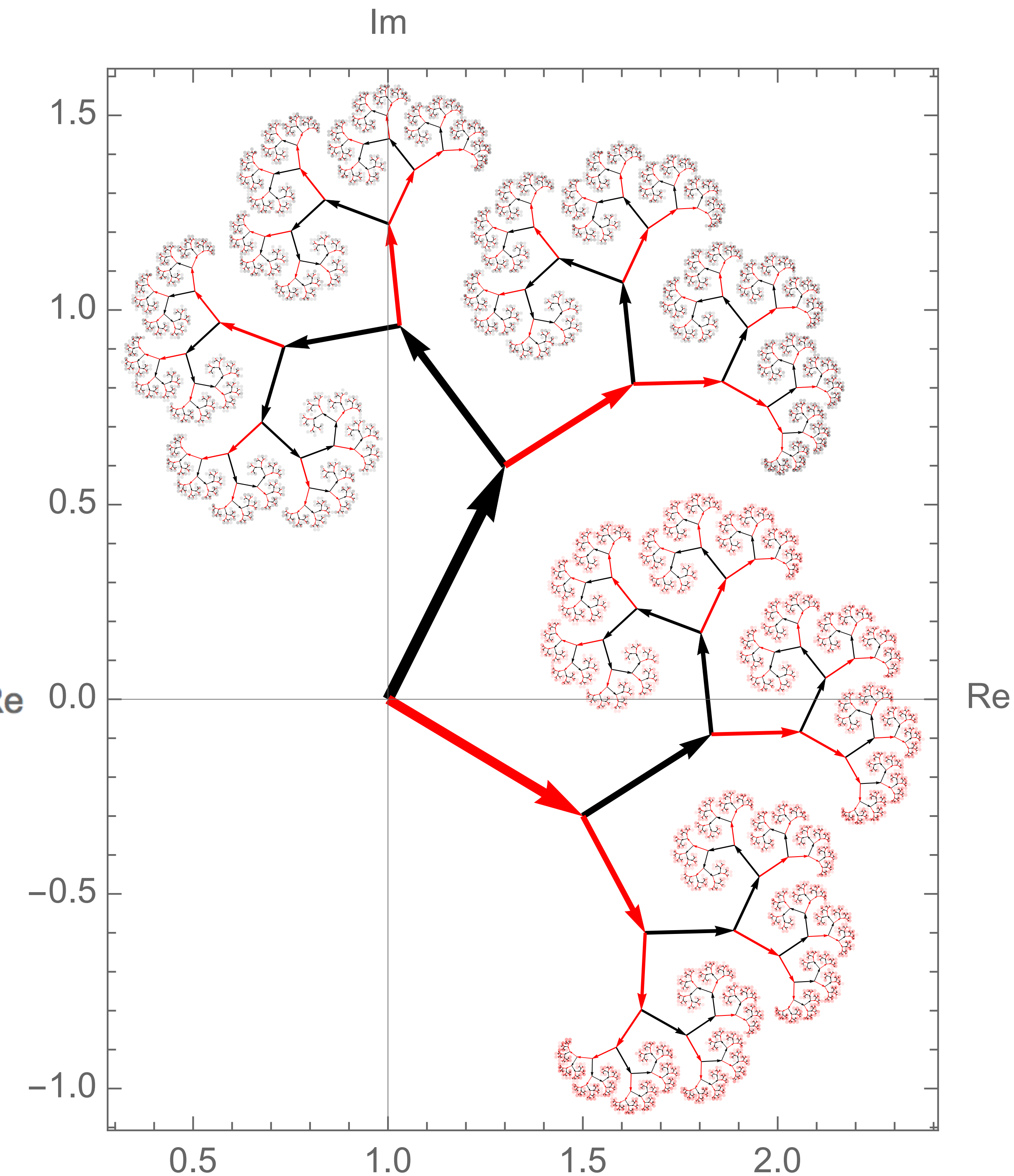
$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \phi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$



tipset

$F\{.3 + i.6,\ .5 - i.3\}$

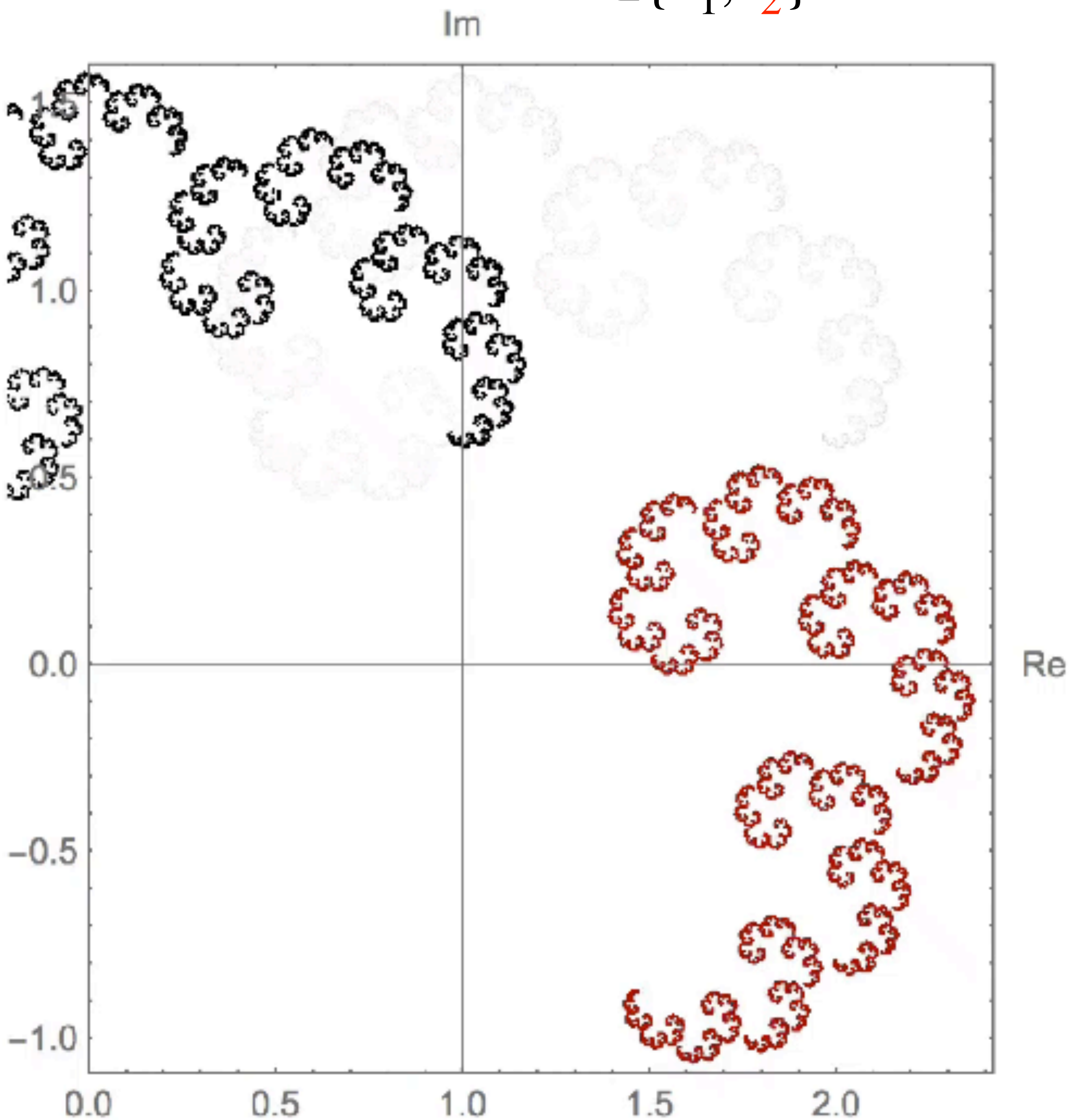


complex tree

$T\{.3 + i.6,\ .5 - i.3\}$

$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \phi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$



$$F_A = f_1(F_A) \cup f_2(F_A)$$

First-level pieces

$$F_{1A} := f_1(F_A) = 1 + c_1 \cdot F_A$$

$$F_{2A} := f_2(F_A) = 1 + c_2 \cdot F_A$$

$$r_2 = |c_2| \approx 0.58$$

$$\theta_2 = \arg(c_2) \approx -31^\circ$$

$$r_1 = |c_1| \approx 0.67$$

$$\theta_1 = \arg(c_1) \approx 63^\circ$$

The tipset is a self-similar set

$$F_{1A} \cap F_{2A} = \varnothing$$

Key idea: “tip-to-tip gluing”

$$\phi(23\bar{1}) = 1+c_2+\frac{c_2c_3}{1-c_1}$$

$$\phi(122\bar{1}) = 1+c_1+c_1c_2+\frac{c_1{c_2}^2}{1-c_1}$$

$$\phi(133\bar{1}) = 1+c_1+c_1c_3+\frac{c_1{c_3}^2}{1-c_1}$$

$$\phi(32\bar{1}) = 1+c_3+\frac{c_3c_2}{1-c_1}$$

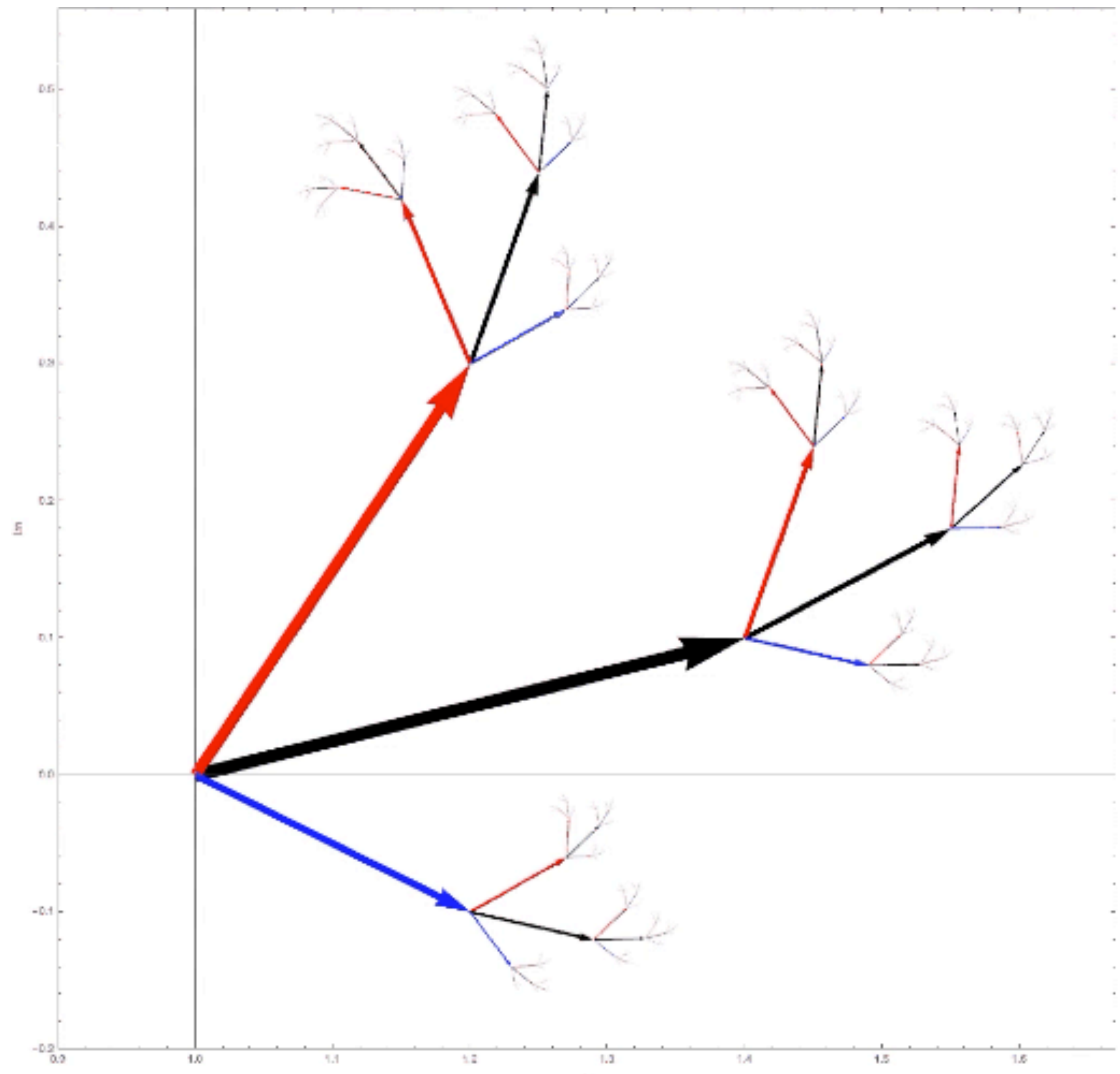
$$\begin{cases} \phi(23\bar{1}) = \phi(122\bar{1}) \\ \phi(32\bar{1}) = \phi(133\bar{1}) \end{cases} = \begin{cases} c_2+\frac{c_2c_3}{1-c_1} = c_1+c_1c_2+\frac{c_1{c_2}^2}{1-c_1} \\ c_3+\frac{c_3c_2}{1-c_1} = c_1+c_1c_3+\frac{c_1{c_3}^2}{1-c_1} \end{cases}$$

$$c_1(z) := z$$

$$c_2(z) := \frac{1-z-z^2+z^3+\sqrt{1-2z-z^2-z^4+2z^5+z^6}}{2(z+z^2)}$$

$$c_3(z) := 2+\frac{1}{z}+z-c_2(z)$$

$$T_A(z) := T\{c_1(z), c_2(z), c_3(z)\}$$



A Family of Fern-like Ternary Complex Trees

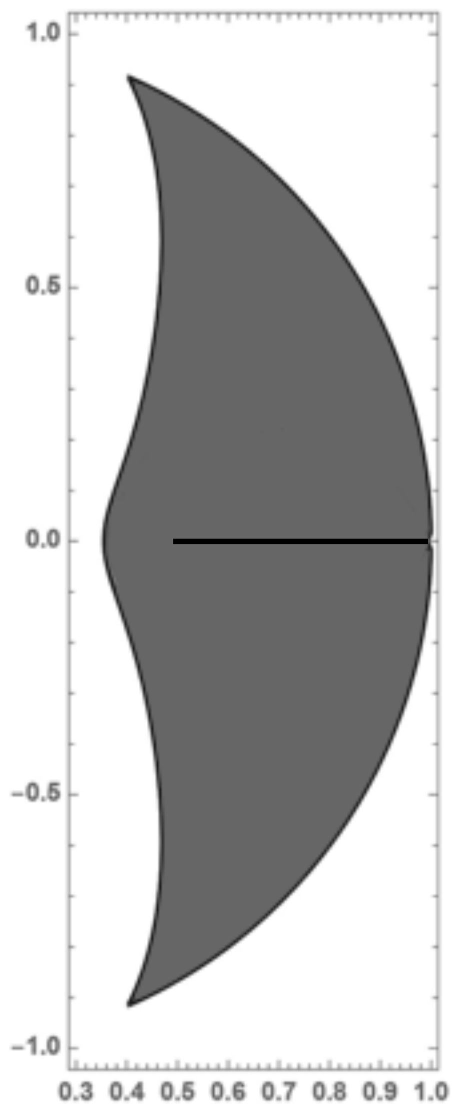
$z \in \mathbb{R}$

$T_A(z) := T\{c_1(z), {\color{red}c_2}(z), {\color{blue}c_3}(z)\}$

$0 < |z| < 1$

$0 < |{\color{red}c_2}(z)| < 1$

$0 < |{\color{blue}c_3}(z)| < 1$



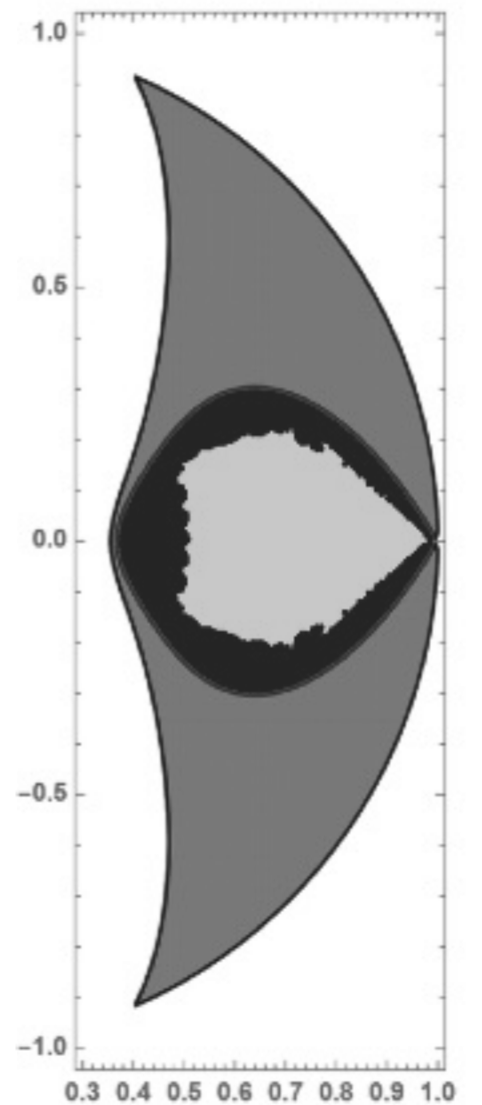
$$\begin{cases} \phi({\color{red}2}{\color{blue}3}\bar{1}) = \phi(1{\color{red}22}\bar{1}) \\ \phi({\color{blue}3}{\color{red}2}\bar{1}) = \phi(1{\color{blue}33}\bar{1}) \end{cases}$$

$c_1(z) := z$

$${\color{red}c_2}(z) := \frac{1 - z - z^2 + z^3 + \sqrt{1 - 2z - z^2 - z^4 + 2z^5 + z^6}}{2(z + z^2)}$$

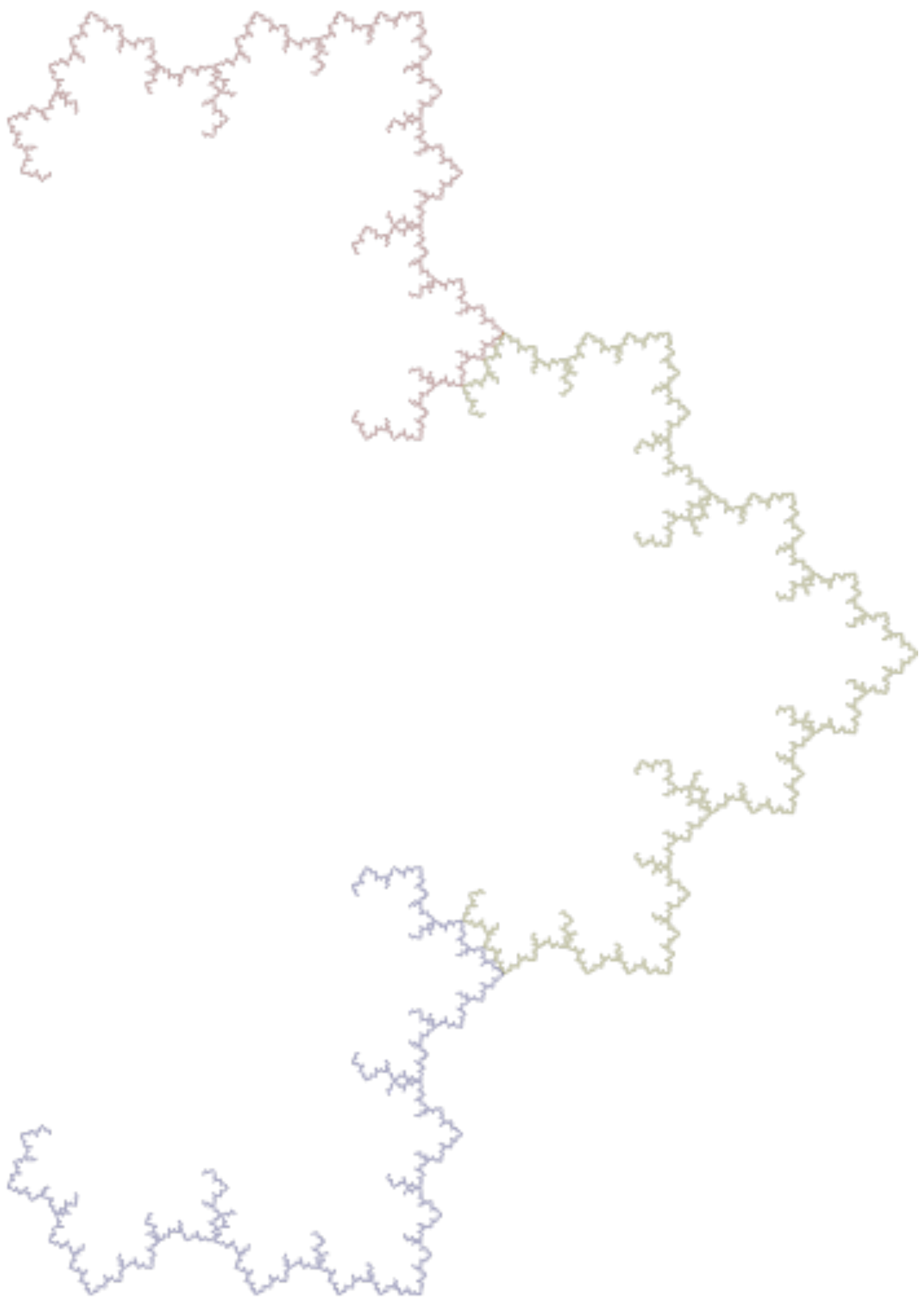
$${\color{blue}c_3}(z) := 2 + \frac{1}{z} + z - {\color{red}c_2}(z)$$

A Family of Fern-like Ternary Complex Trees



$\{23\overline{1} \sim 122\overline{1}, 32\overline{1} \sim 133\overline{1}\}$

$T_A(z) := T\{c_1(z), c_2(z), c_3(z)\}$



$c_1(z) := z$

$$c_2(z) := \frac{1 - z - z^2 + z^3 + \sqrt{1 - 2z - z^2 - z^4 + 2z^5 + z^6}}{2(z + z^2)}$$

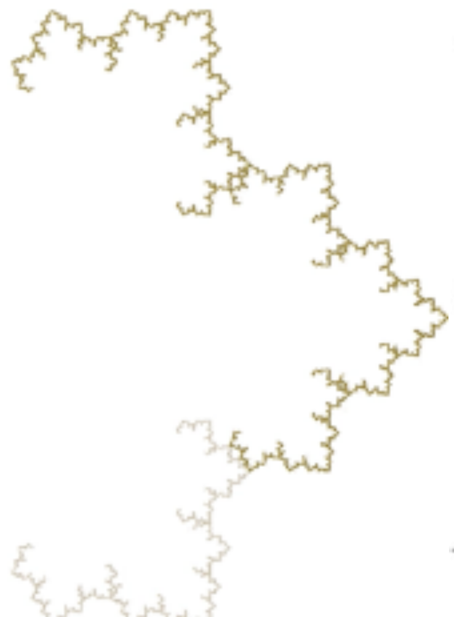
$$c_3(z) := 2 + \frac{1}{z} + z - c_2(z)$$

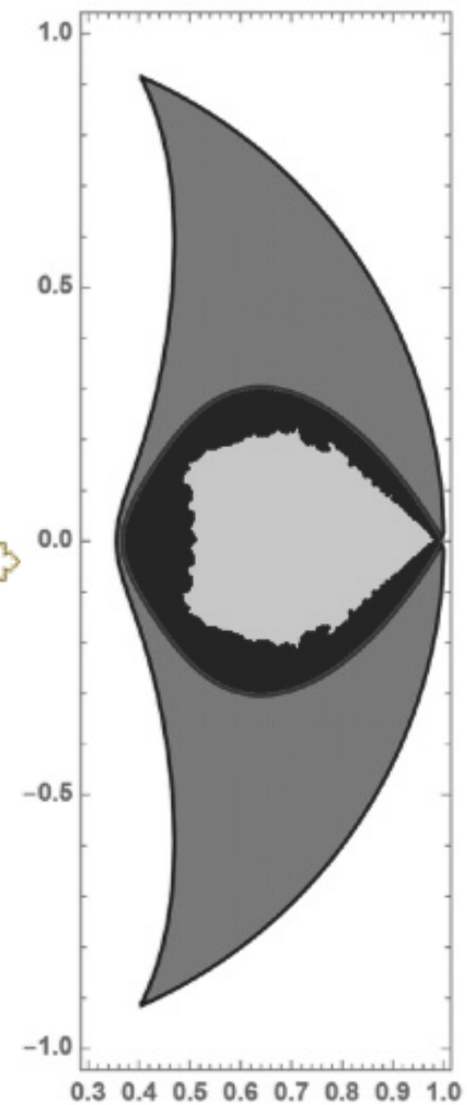
A Family of Fern-like Ternary Complex Trees

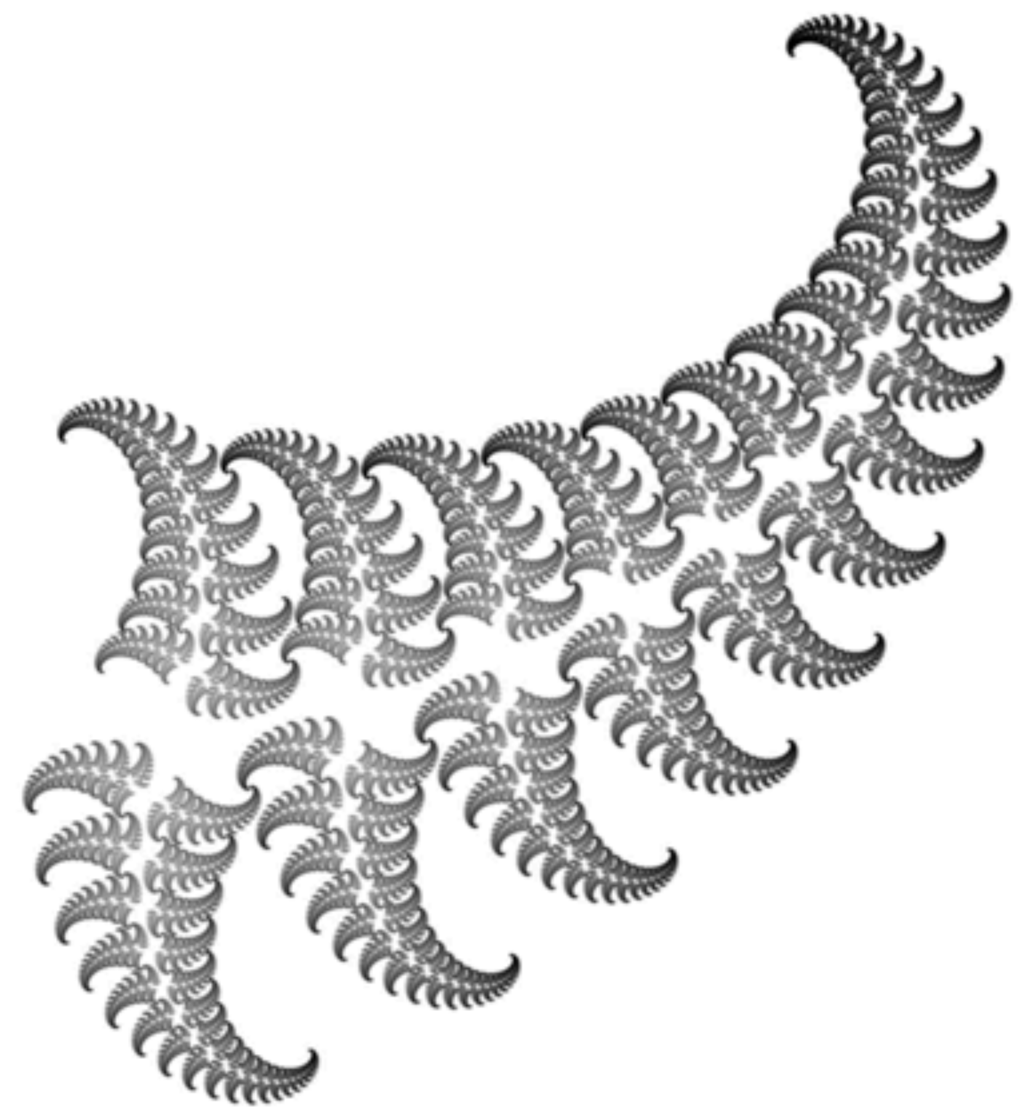
$$1 - z^2 - z^3 - z^4 + z^5 + z^6 + z^7$$

$$1 - z^2 - z^4 - z^5 + z^6 + z^7 + z^9$$

$$T_A(z) := T\{c_1(z), c_2(z), c_3(z)\}$$







$$c_1(z) := z$$

$$c_2(z) := \frac{1 - z - z^2 + z^3 + \sqrt{1 - 2z - z^2 - z^4 + 2z^5 + z^6}}{2(z + z^2)}$$

$$c_3(z) := 2 + \frac{1}{z} + z - c_2(z)$$