

complex-valued alphabet $A = \{c_1, c_2\}$

letters $w_1, w_2, \dots, w_m \in A = \{1 \rightarrow c_1, 2 \rightarrow c_2\}$

word $w = w_1 w_2 \dots w_m$

$$\phi(w) = 1 + w_1 + w_1 w_2 + \dots + w_1 w_2 \dots w_m$$

$$\phi(1) = 1 + c_1$$

$$\phi(11) = 1 + c_1 + c_1^2$$

$$\phi(112) = 1 + c_1 + c_1^2 + c_1^2 c_2$$

nodes

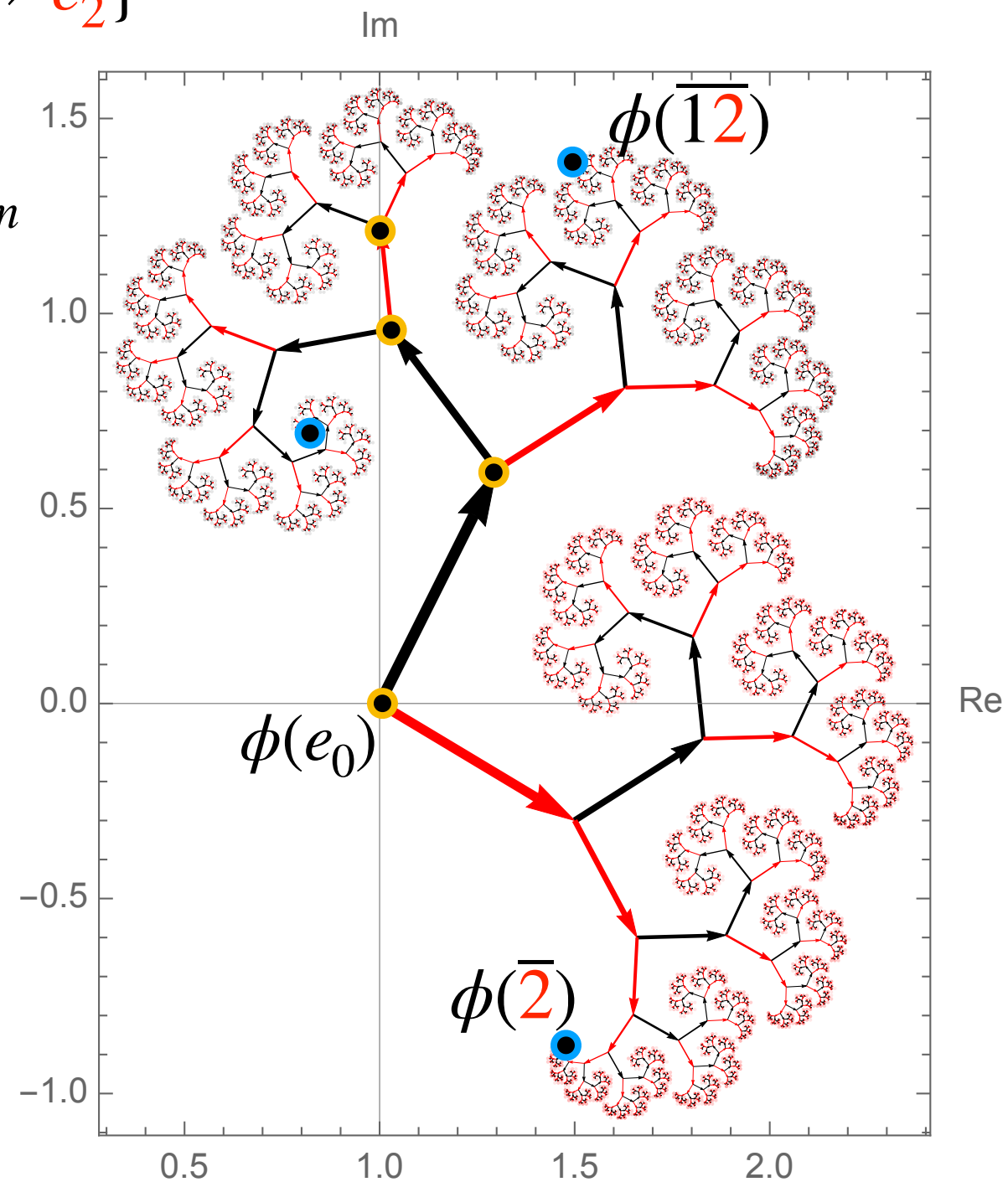
empty string e_0 **root** $\phi(e_0) = 1$

$$w = w_1 w_2 w_3 \dots w_k \dots \in \{c_1, c_2\}^\infty = A^\infty$$

tip points $\phi(w)$

$$\phi(111\dots) = \phi(\bar{1}) = \sum_{k=0}^{\infty} c_1^k = \frac{1}{1 - c_1}$$

$$\phi(1212\dots) = \phi(\overline{12}) = \frac{1 + c_1}{1 - c_1 c_2}$$

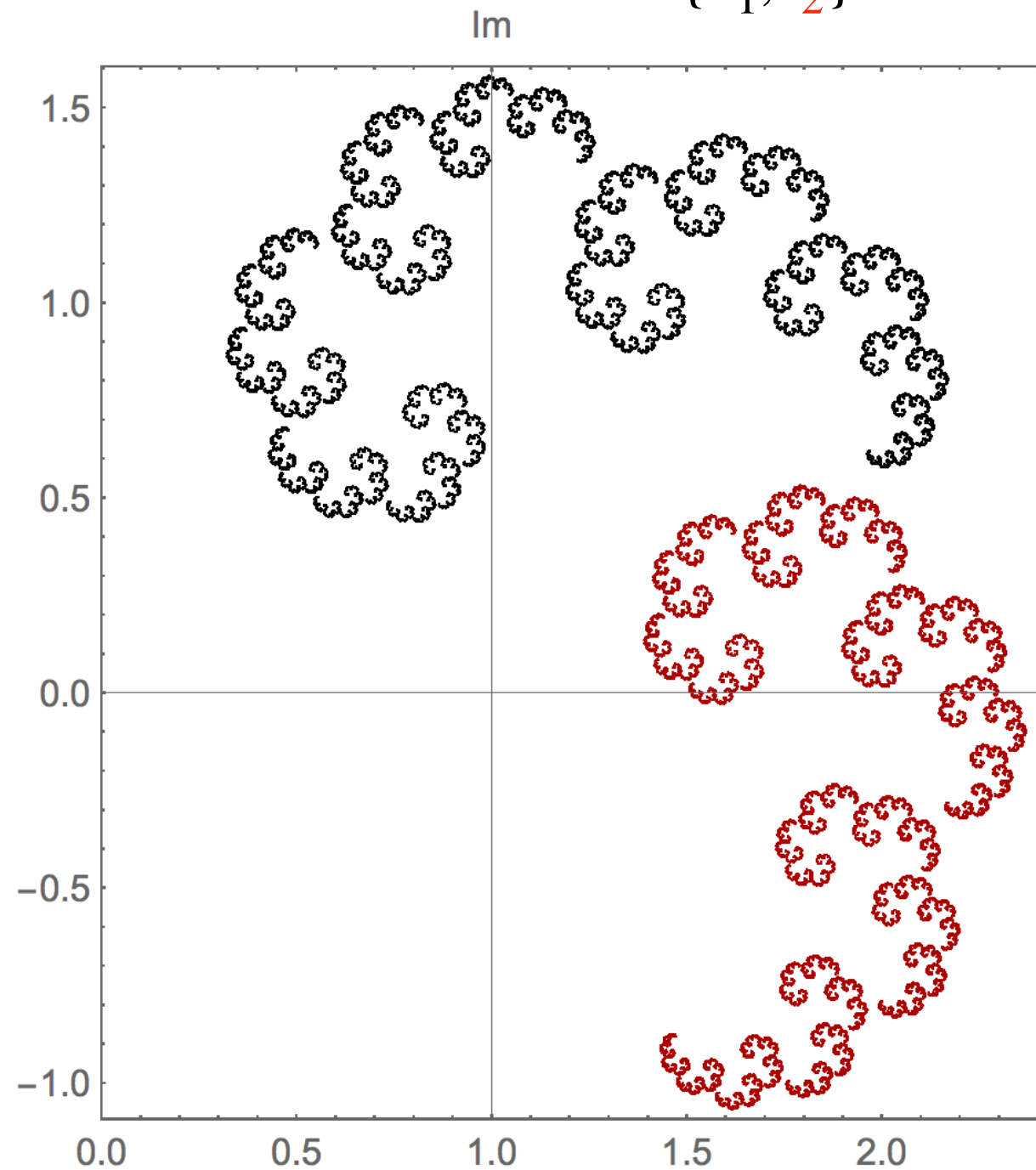


complex tree

$$T_A = T\{c_1, c_2\} = T\{.3 + i.6, .5 - i.3\}$$

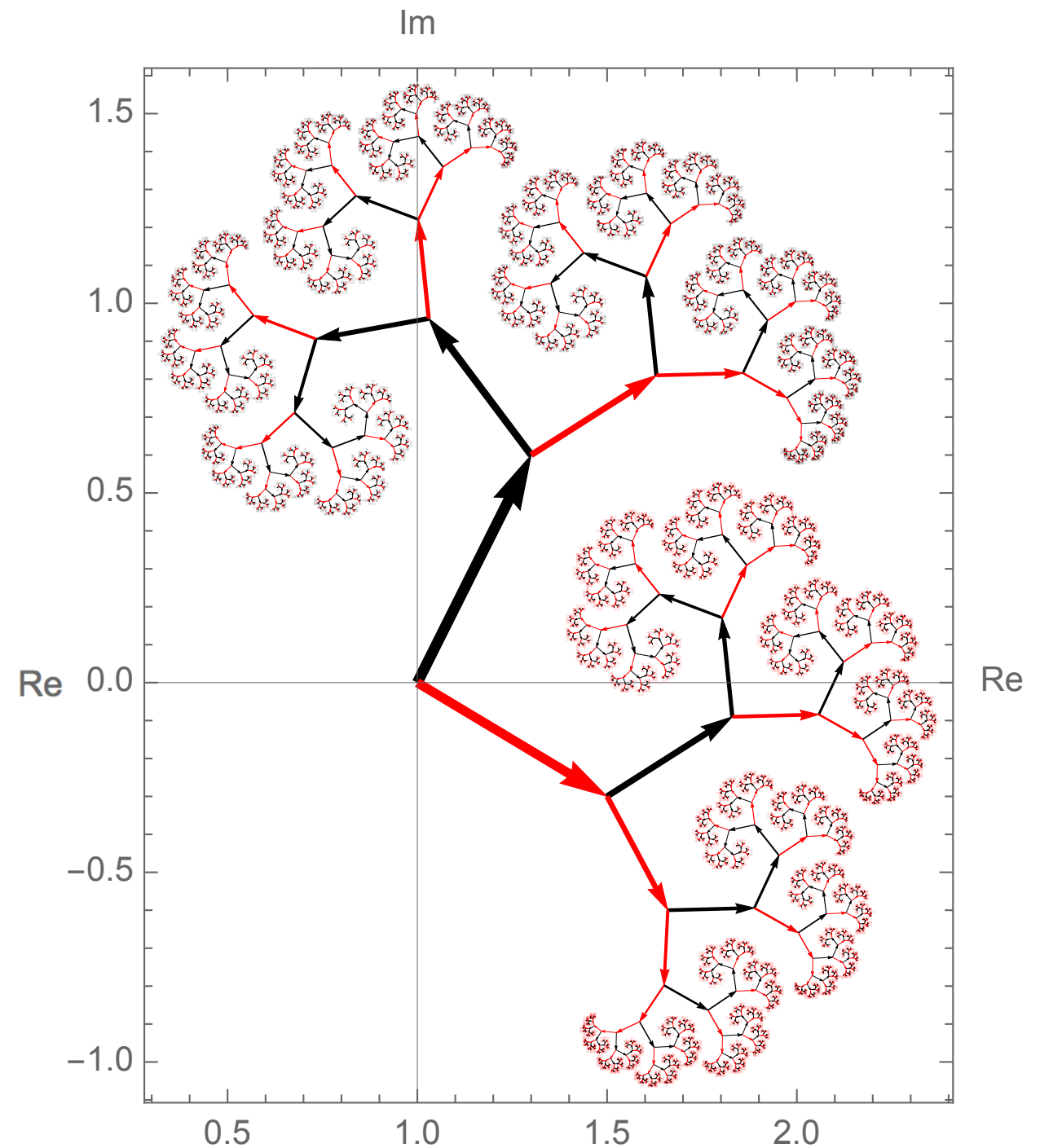
$$F_A = F\{c_1, \textcolor{red}{c}_2\} := \bigcup_{w \in \{c_1, \textcolor{red}{c}_2\}^\infty} \phi(w)$$

$$\{c_1, \textcolor{red}{c}_2\} = \{.3 + i.6, \textcolor{red}{.5} - i.3\}$$



tipset

$$F\{.3 + i.6, \textcolor{red}{.5} - i.3\}$$



complex tree

$$T\{.3 + i.6, \textcolor{red}{.5} - i.3\}$$

$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \phi(w)$$

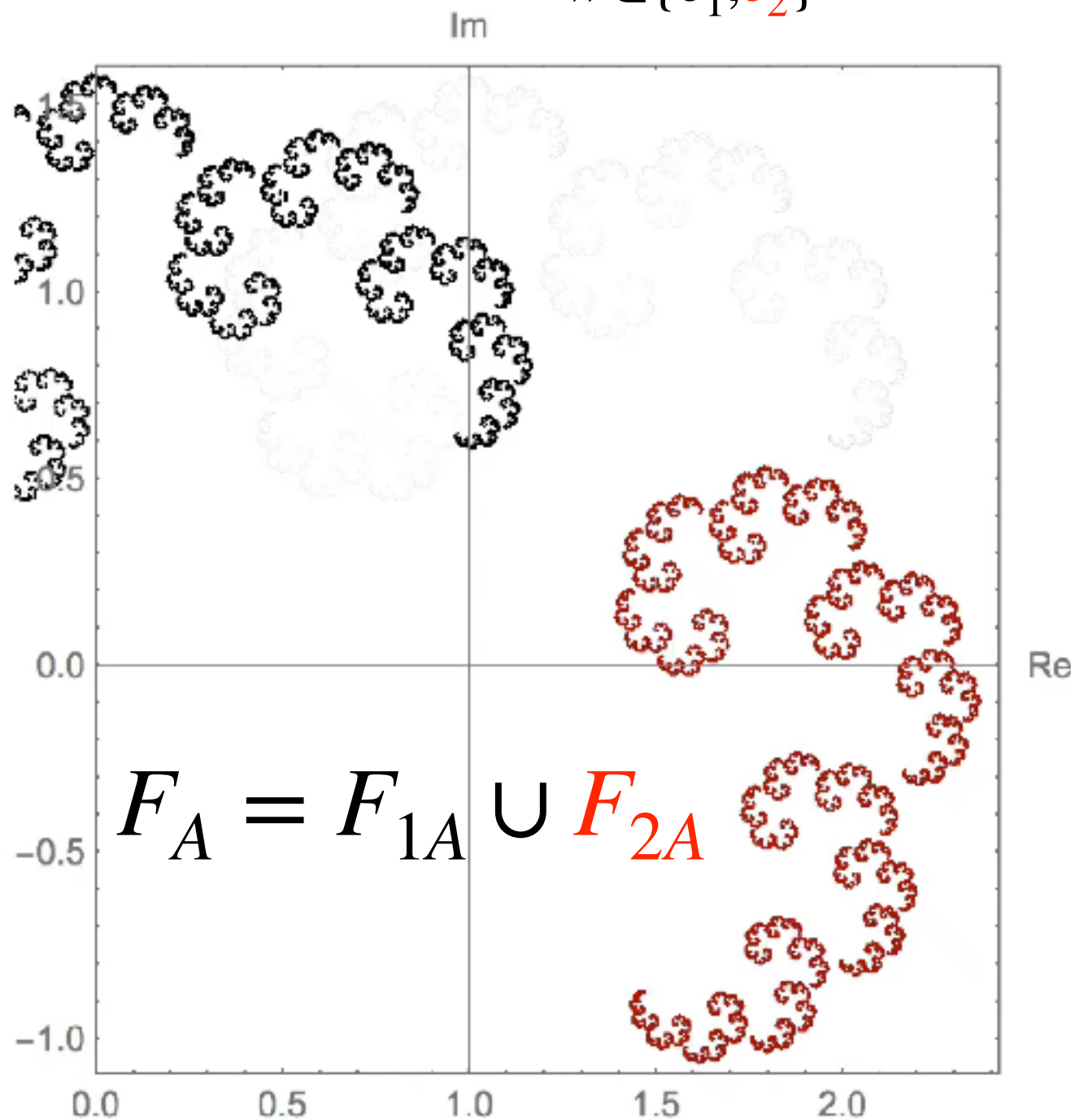
$$\{c_1, c_2\} = \{.3 + i.6, .5 - i.3\}$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$

First-level pieces

$$F_{1A} := f_1(F_A) = 1 + c_1 \cdot F_A$$

$$F_{2A} := f_2(F_A) = 1 + c_2 \cdot F_A$$



$$r_2 = |c_2| \approx 0.58$$

$$\theta_2 = \arg(c_2) \approx -31^\circ$$

$$r_1 = |c_1| \approx 0.67$$

$$\theta_1 = \arg(c_1) \approx 63^\circ$$

The tipset is a self-similar set

$$F_{1A} \cap F_{2A} = \emptyset$$

Key idea: “tip-to-tip gluing”

$$\phi(13\bar{2}) = 1 + c_1 + c_1 c_3 \sum_{k=0}^{\infty} c_2^k = 1 + c_1 + \frac{c_1 c_3}{1 - c_2}$$

$$\phi(21\bar{2}) = 1 + c_2 + \frac{c_2 c_1}{1 - c_2}$$

$$\phi(23\bar{2}) = 1 + c_2 + \frac{c_2 c_3}{1 - c_2}$$

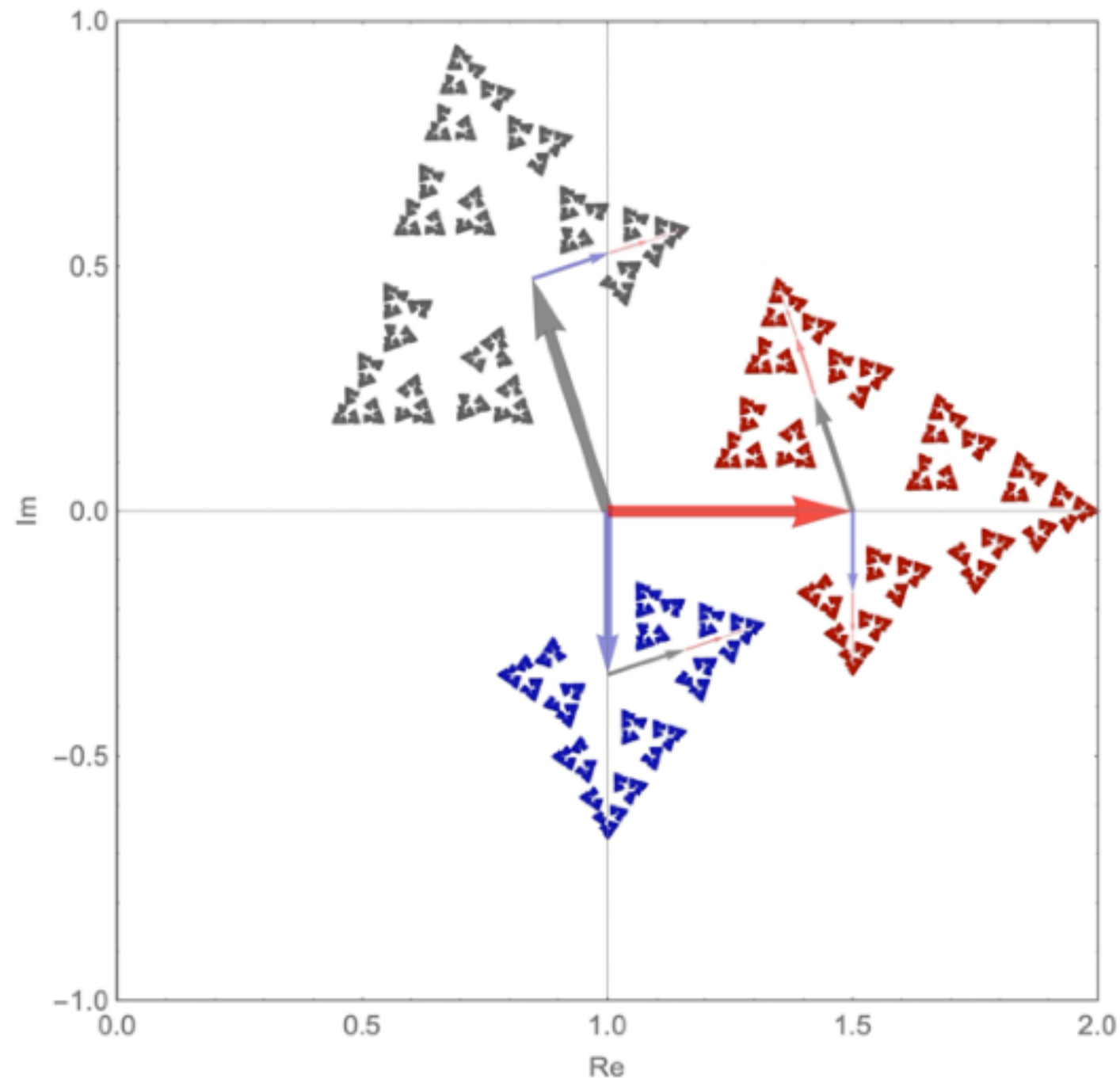
$$\phi(31\bar{2}) = 1 + c_3 + \frac{c_3 c_1}{1 - c_2}$$

$$\begin{cases} \phi(13\bar{2}) = \phi(21\bar{2}) \\ \phi(23\bar{2}) = \phi(31\bar{2}) \end{cases} \Rightarrow \begin{cases} c_2 = 1/2 \\ c_3 = 1/4 c_1 \end{cases}$$

$$c_1(z) := z, \quad c_2(z) := 1/2, \quad c_3(z) := 1/4 z$$

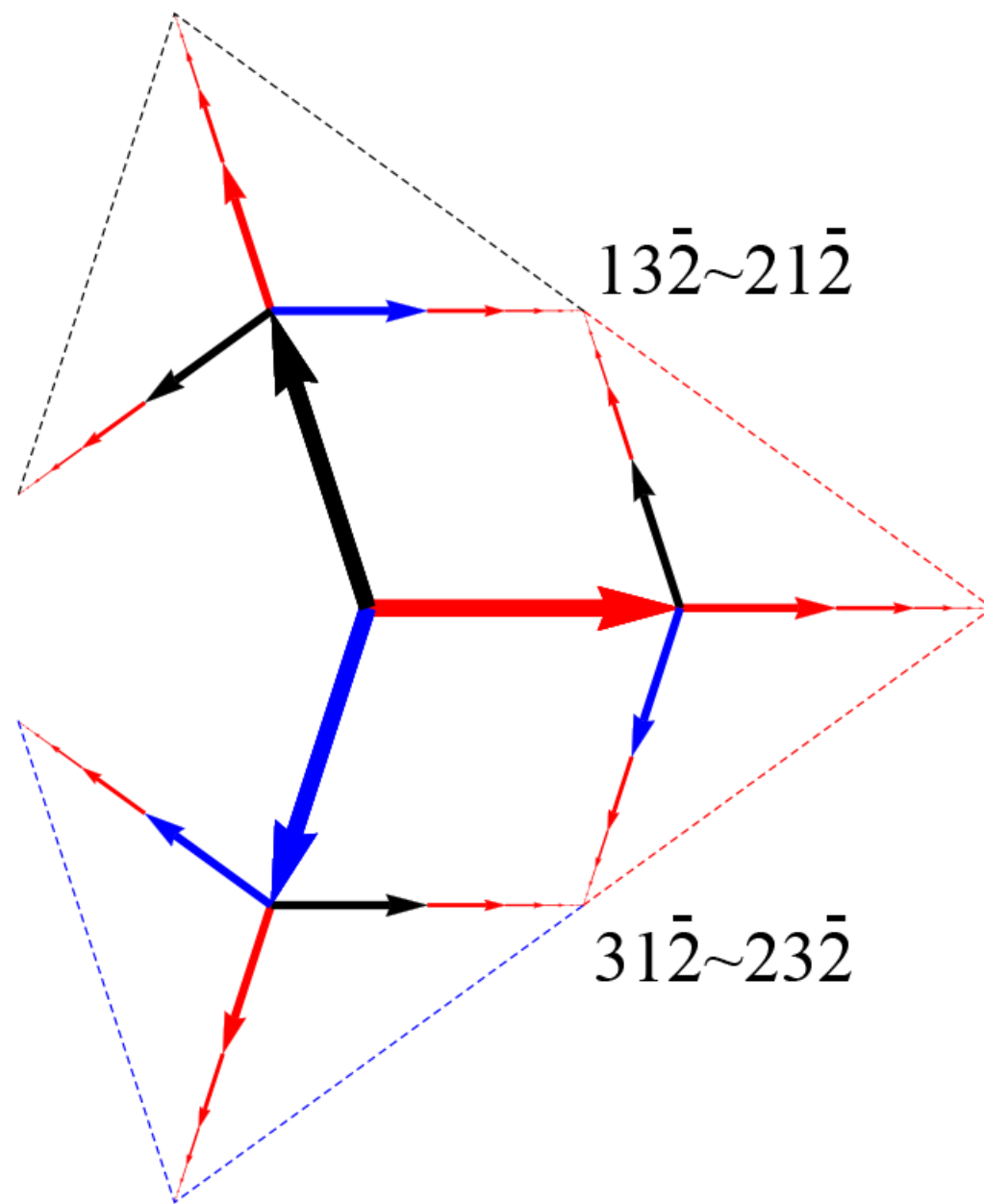
$$T_A(z) = T\{c_1(z), c_2(z), c_3(z)\} = T\{z, 1/2, 1/4 z\}$$

$$T_A(i^{6/5}/2) = T\left\{\frac{i^{6/5}}{2}, \frac{1}{2}, -\frac{i^{4/5}}{2}\right\}$$



Topological set

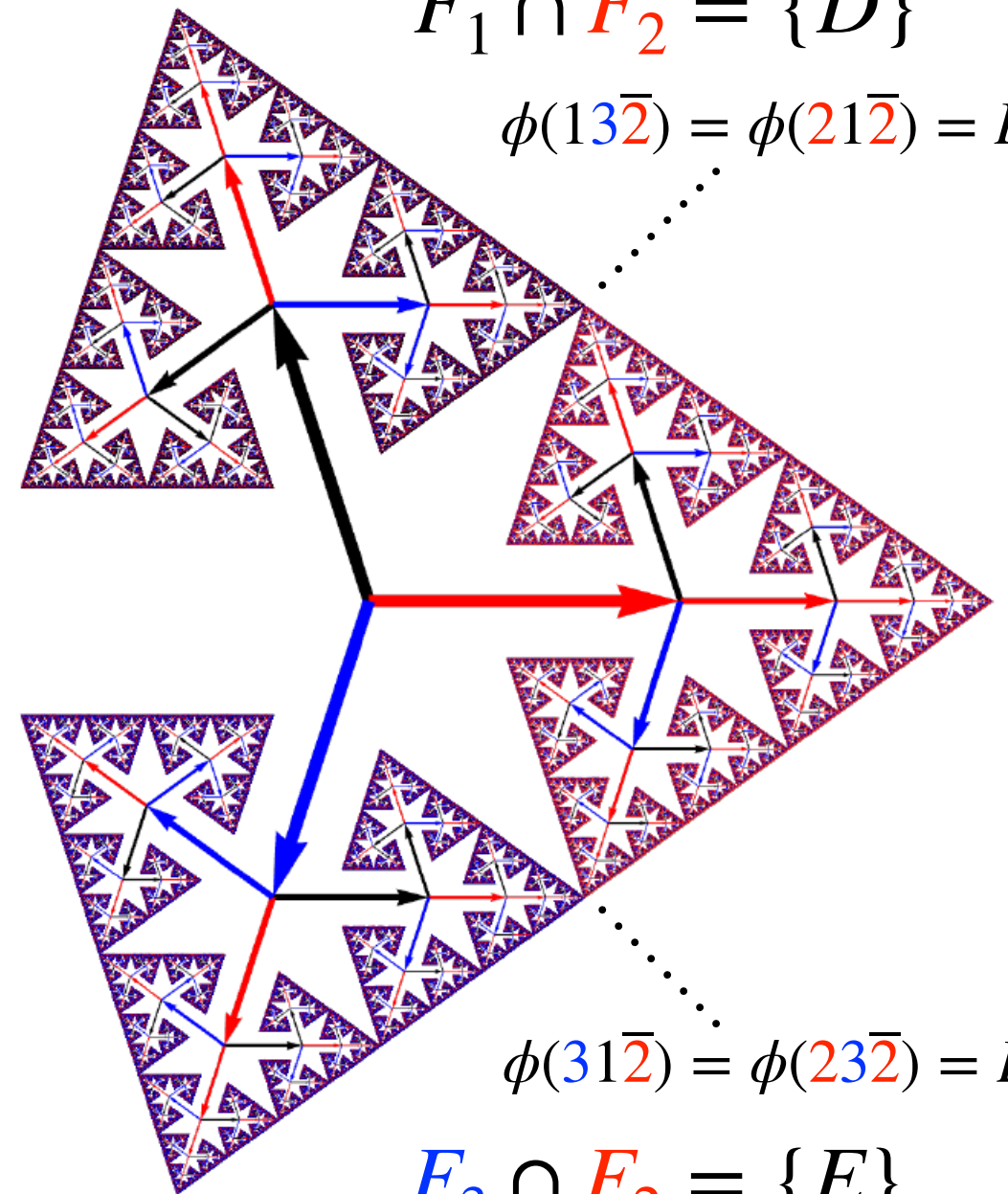
$$Q_A(i^{6/5}/2) = \{13\bar{2} \sim 21\bar{2}, 31\bar{2} \sim 23\bar{2}\}$$



$$T_A(i^{6/5}/2) = T\left\{\frac{i^{6/5}}{2}, \frac{1}{2}, -\frac{i^{4/5}}{2}\right\}$$

$$F_1 \cap F_2 = \{D\}$$

$$\phi(1\bar{3}\bar{2}) = \phi(2\bar{1}\bar{2}) = D$$

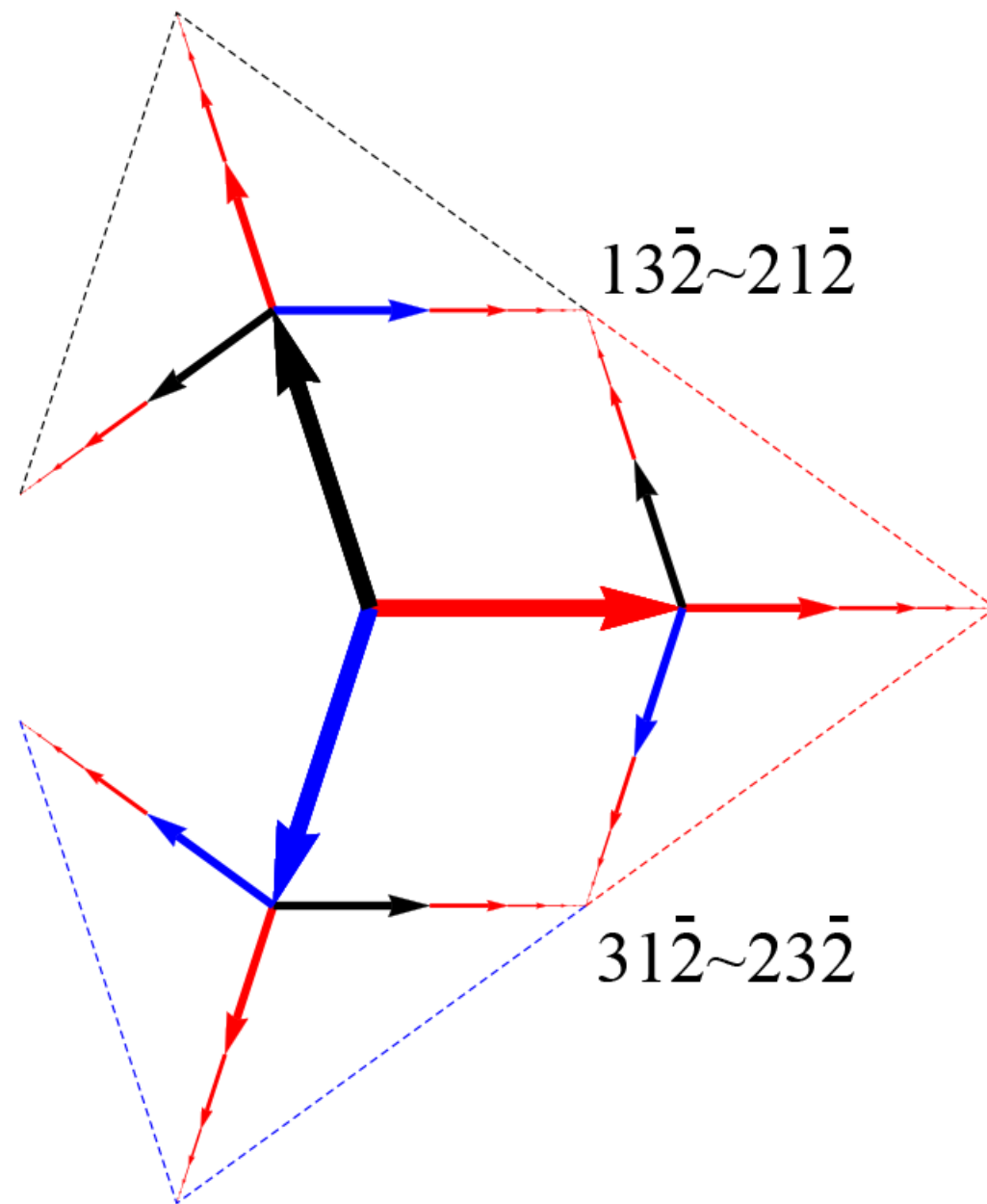


$$\phi(3\bar{1}\bar{2}) = \phi(2\bar{3}\bar{2}) = E$$

$$F_3 \cap F_2 = \{E\}$$

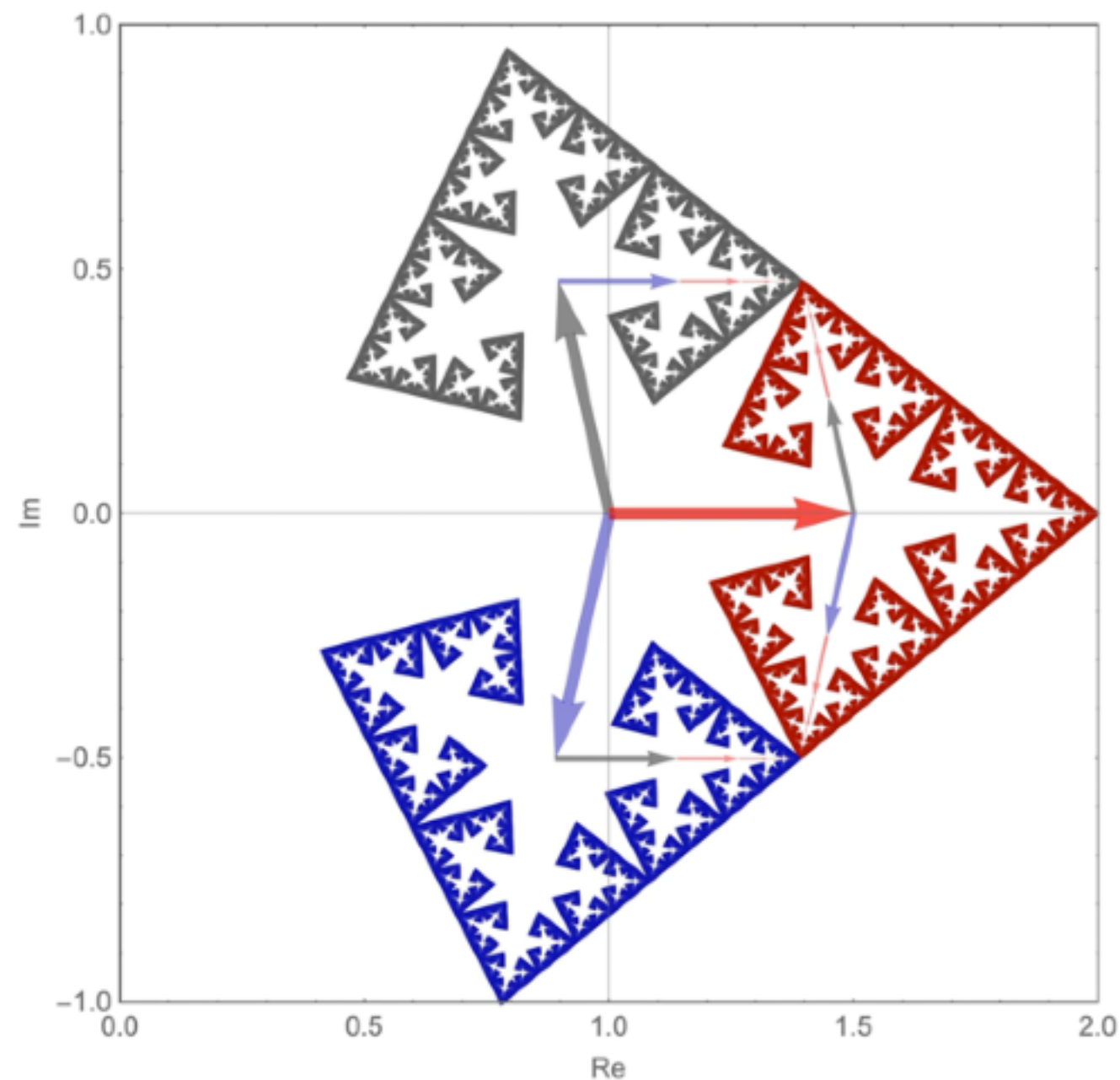
Topological set

$$Q_A(i^{6/5}/2 + \epsilon) = \{13\bar{2} \sim 21\bar{2}, 31\bar{2} \sim 23\bar{2}\}$$



Structurally stable tree

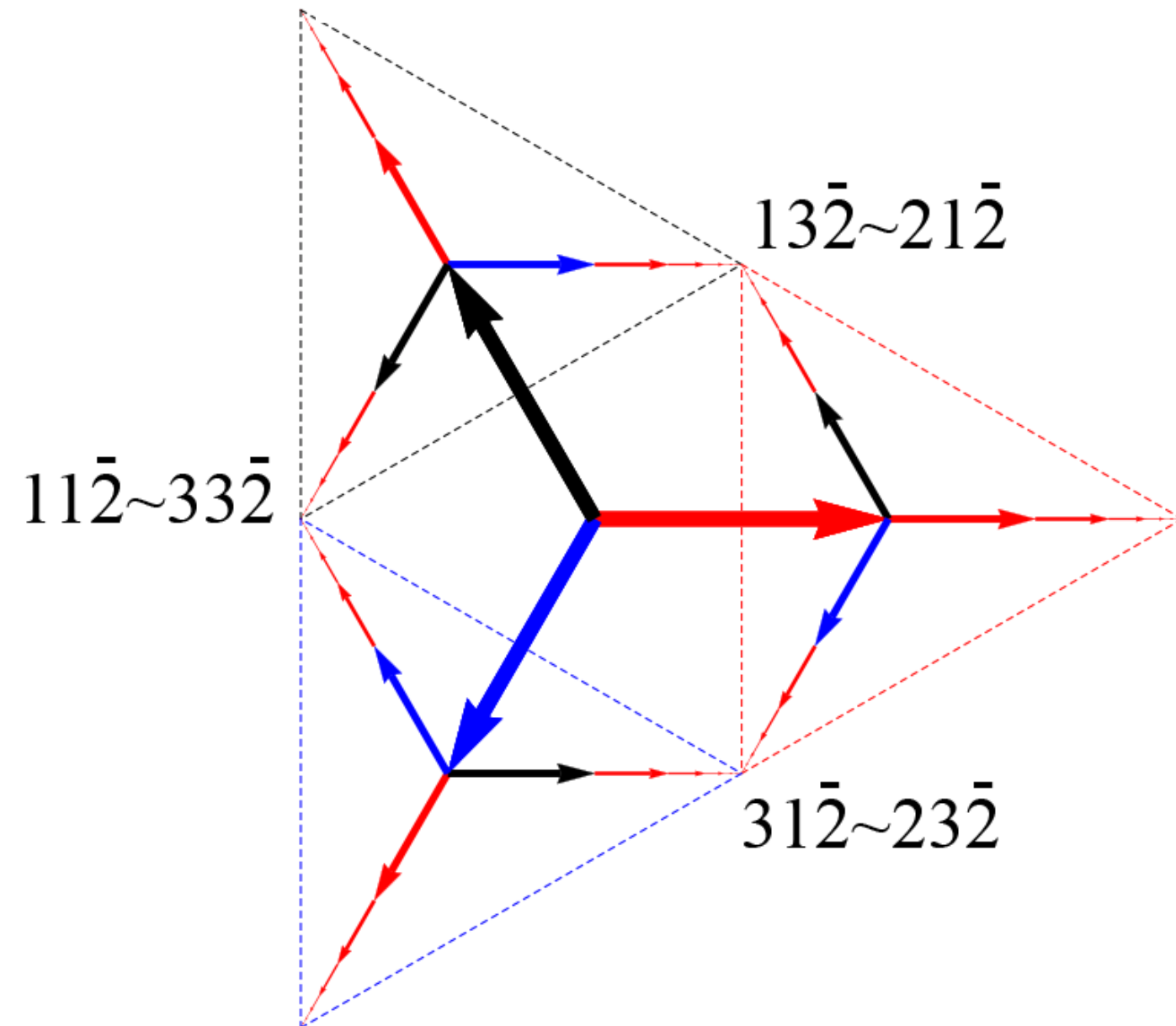
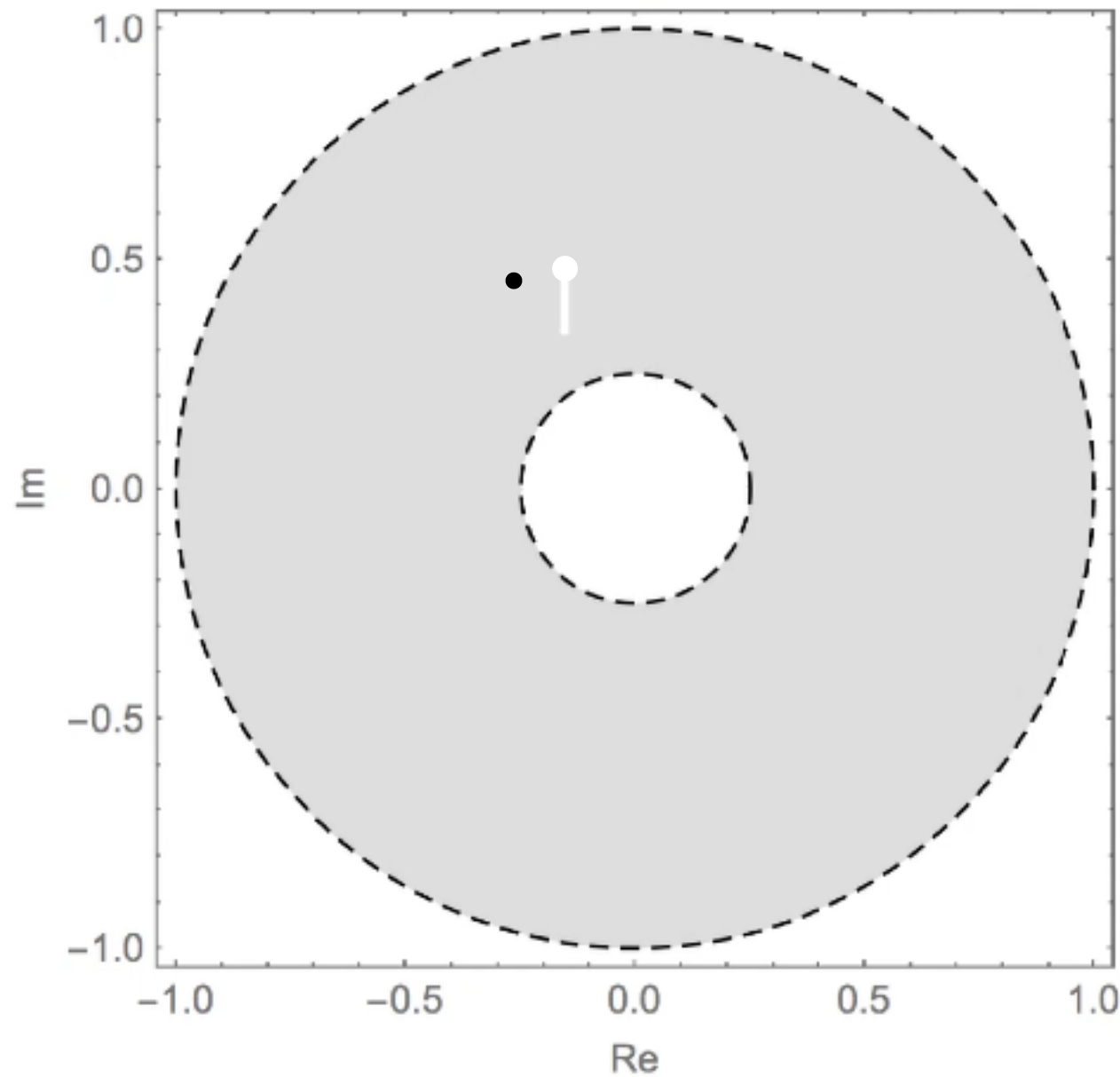
$$T_A(i^{6/5}/2 + \epsilon) = T\{i^{6/5}/2 + \epsilon, \frac{1}{2}, c_3(i^{6/5}/2 + \epsilon)\}$$



A family of tipset connected trees $T_A(z) := T\{z, \frac{1}{2}, \frac{1}{4z}\}$

$$Q_A(-1/4 + i\sqrt{3}/4) = \{13\bar{2} \sim 21\bar{2}, 31\bar{2} \sim 23\bar{2}, \underline{11\bar{2} \sim 33\bar{2}}\}$$

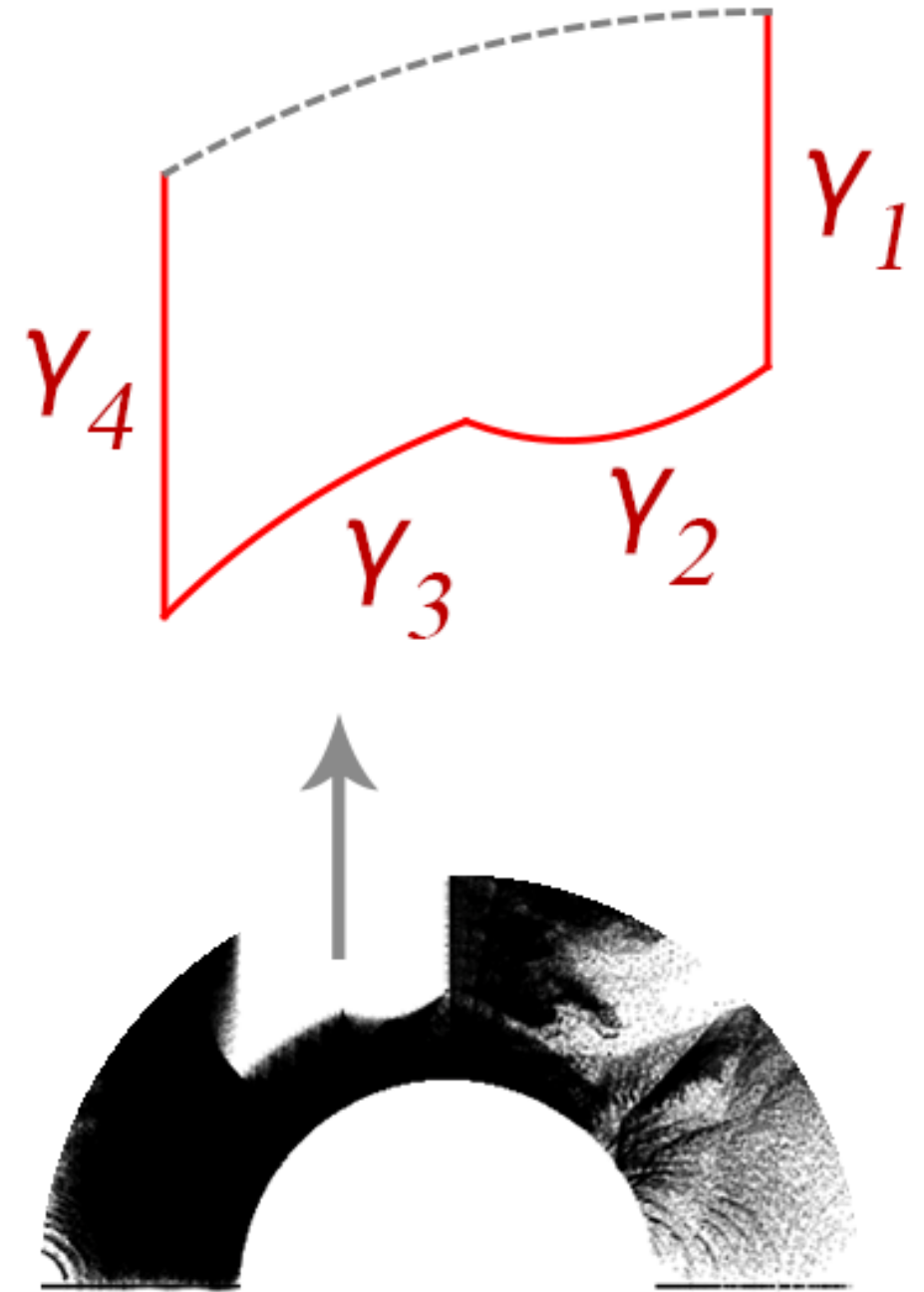
$$z = -0.16 + 0.34i$$



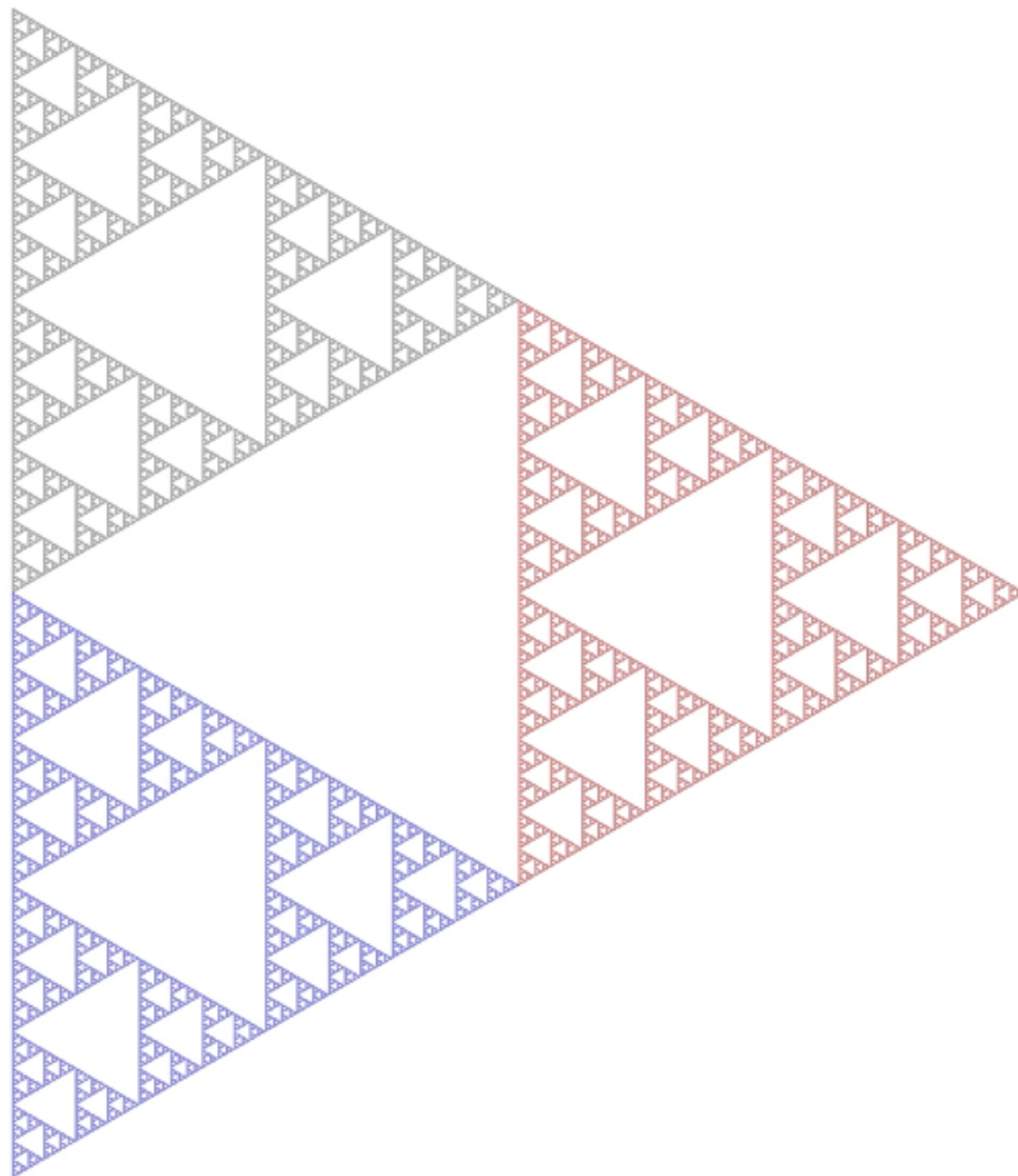
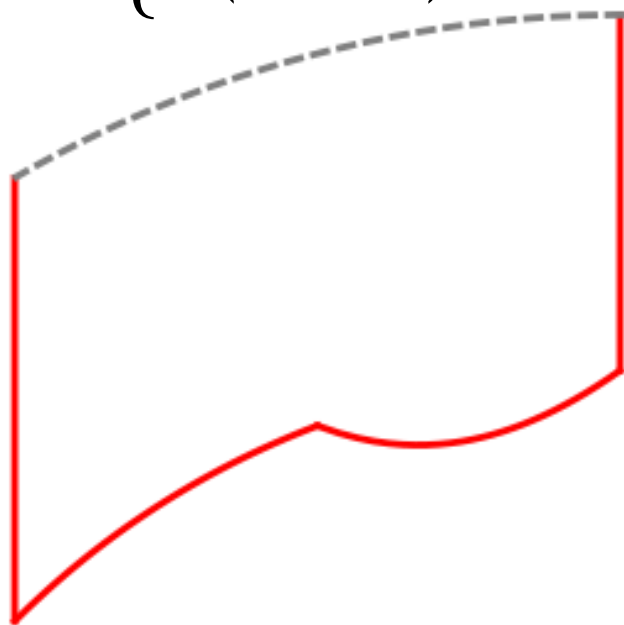
$$\mathcal{R} := \{z \in \mathbb{C} : 1/4 < |z| < 1\}$$

$$K \cup K^* \subseteq \{z \in \mathcal{R} : Q_A(z) = \{13\bar{2} \sim 21\bar{2}, 31\bar{2} \sim 23\bar{2}\}\} \quad T_A(z) := T\left\{z, \frac{1}{2}, \frac{1}{4z}\right\}$$

$$\gamma(t) := \begin{cases} -it - \frac{1}{4} & -\frac{\sqrt{3}}{4} \leq t \leq -\frac{1}{4} \\ t + \frac{i\sqrt{1-8t^2}}{2\sqrt{2}} & -\frac{1}{4} < t \leq -\frac{1}{8} \\ t + \frac{i\sqrt{24t^2+4t+1}}{2\sqrt{2}} & -\frac{1}{8} < t < 0 \\ i\left(t + \frac{1}{2\sqrt{2}}\right) & 0 \leq t \leq \frac{1}{4}(2-\sqrt{2}) \end{cases}$$



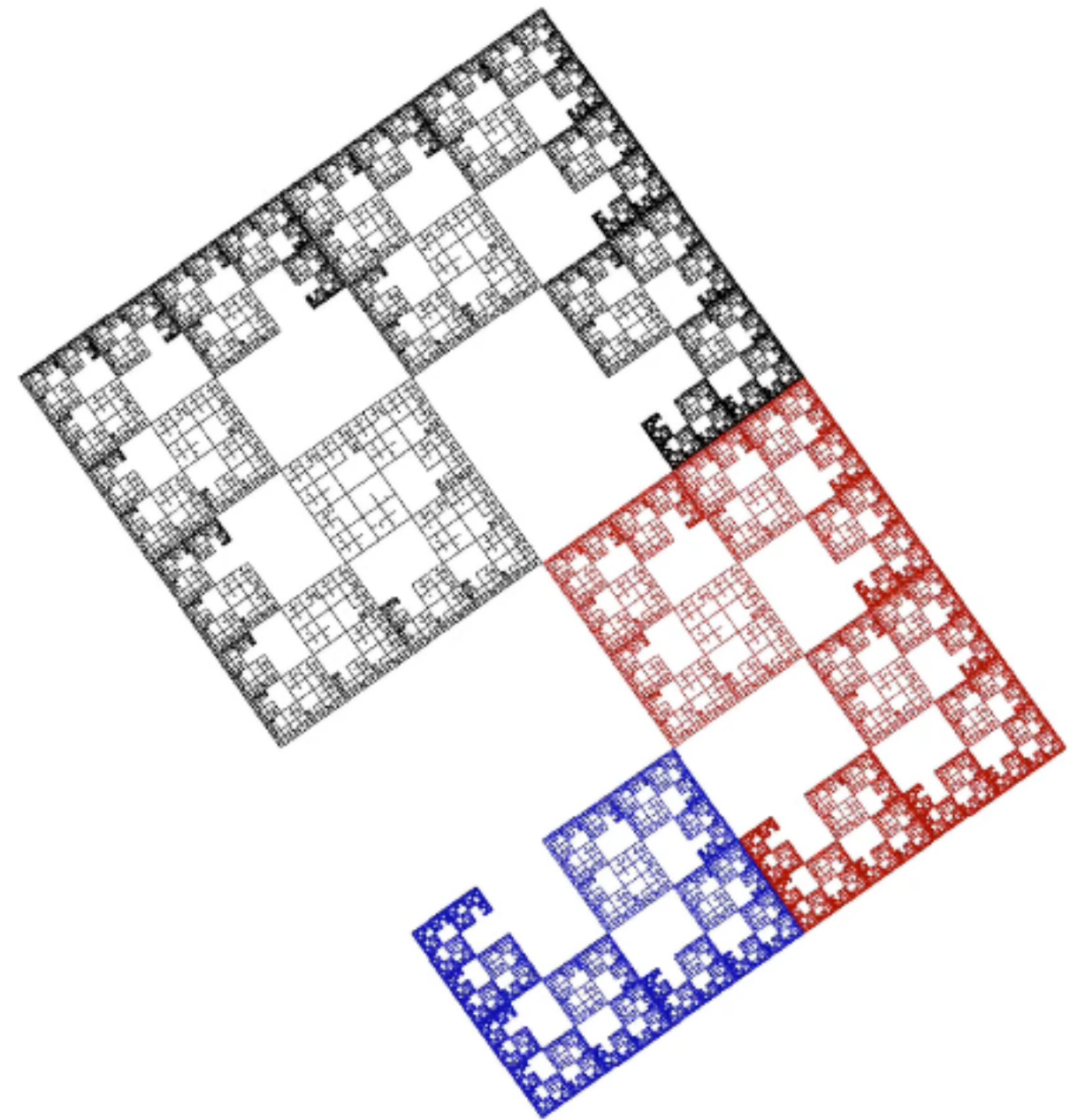
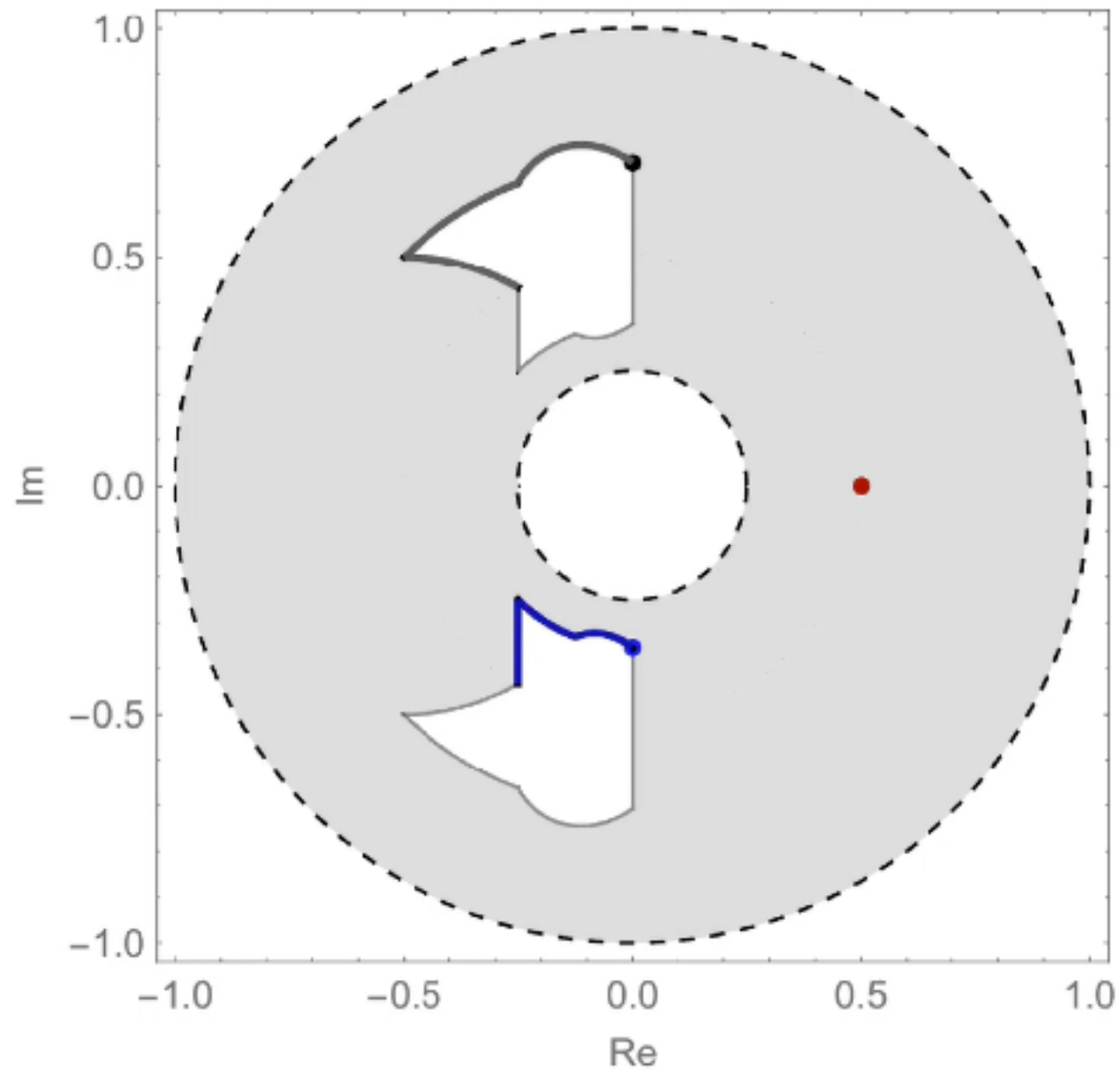
$$\gamma(t) := \begin{cases} -it - \frac{1}{4} & -\frac{\sqrt{3}}{4} \leq t \leq -\frac{1}{4} \\ t + \frac{i\sqrt{1-8t^2}}{2\sqrt{2}} & -\frac{1}{4} < t \leq -\frac{1}{8} \\ t + \frac{i\sqrt{24t^2+4t+1}}{2\sqrt{2}} & -\frac{1}{8} < t < 0 \\ i\left(t + \frac{1}{2\sqrt{2}}\right) & 0 \leq t \leq \frac{1}{4}(2-\sqrt{2}) \end{cases}$$



A family of tipset connected trees $T_A(z) := T\{z, \frac{1}{2}, \frac{1}{4z}\}$

$$Q_A(z) \supset \{13\bar{2} \sim 21\bar{2}, 31\bar{2} \sim 23\bar{2}\}$$

$$z = 0.0000 + 0.7065i$$



$$\mathcal{R} := \{z \in \mathbb{C} : 1/4 < |z| < 1\}$$

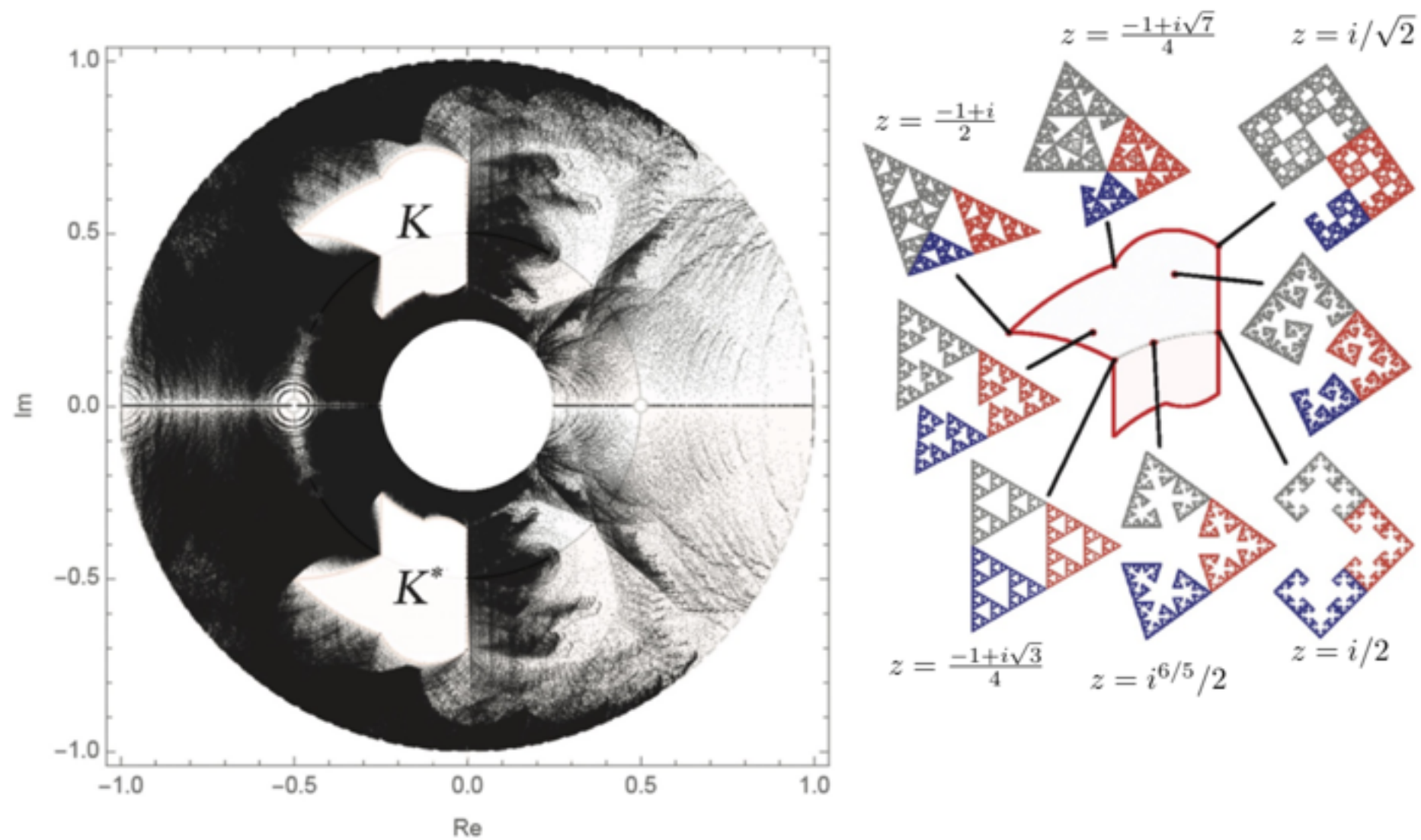
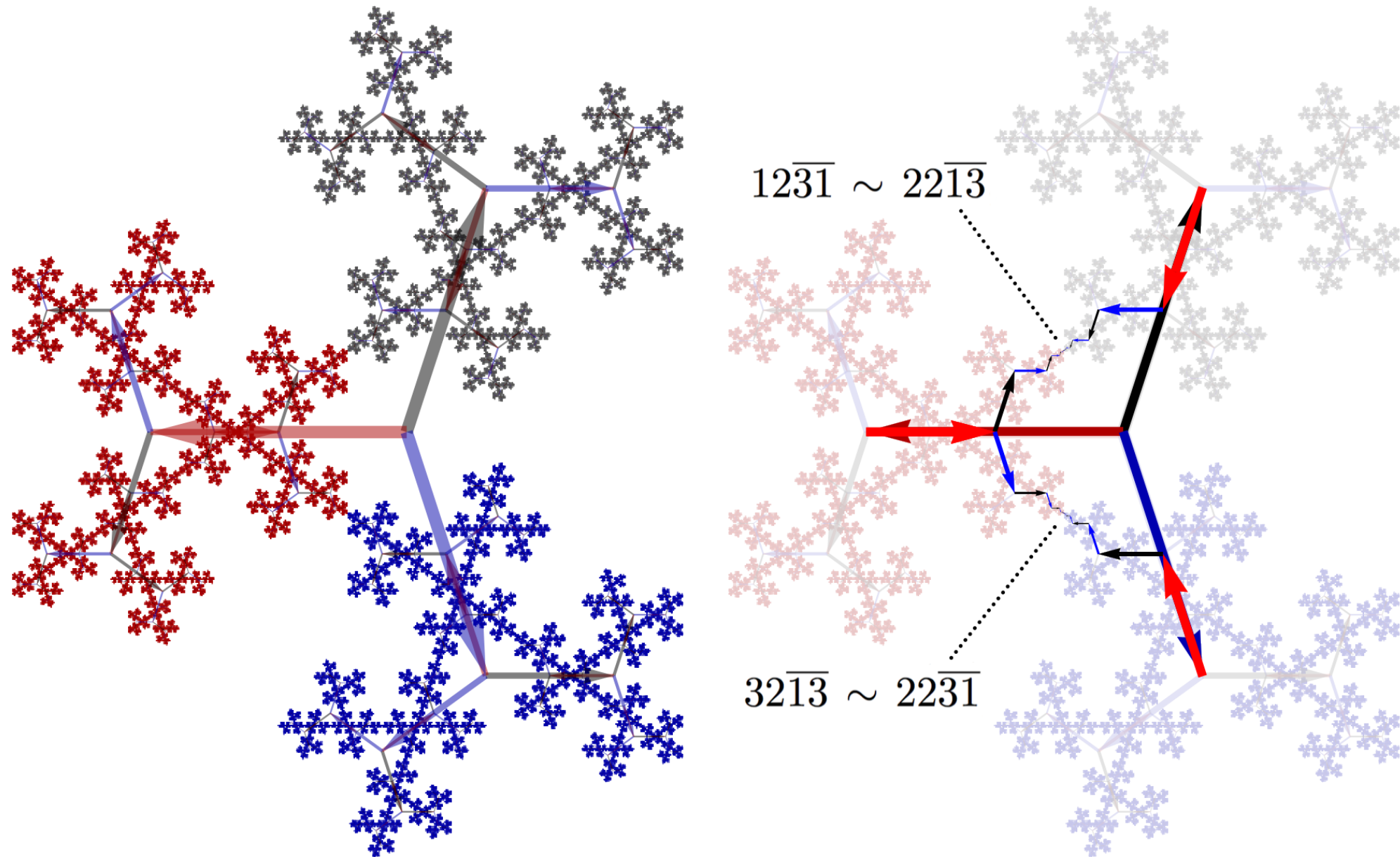


Figure 9: The unstable set $\mathcal{M}$ for $T\{z, 1/2, 1/4z\}$ with a pair of open regions K and K^* contained in the stable set $\mathcal{K}$. On the right, examples of tipsets $F_{\mathcal{A}}(z)$, five of which have their parameters z taken from the boundary of $\mathcal{M}$.

A family of tipset connected trees $T_A(z) := T\{z, -\frac{1}{2}, \frac{1}{4z}\}$



$$\begin{cases} \phi(12\overline{31}) = \phi(22\overline{13}) \\ \phi(32\overline{13}) = \phi(22\overline{31}) \end{cases} = \begin{cases} c_1 + c_1 c_2 \left(\frac{1+c_3}{1-c_1 c_3} \right) = c_2 + c_2^2 \left(\frac{1+c_1}{1-c_3 c_1} \right) \\ c_3 + c_3 c_2 \left(\frac{1+c_1}{1-c_1 c_3} \right) = c_2 + c_2^2 \left(\frac{1+c_3}{1-c_3 c_1} \right) \end{cases} \Rightarrow \begin{cases} c_2 = -1/2 \\ c_3 = 1/4 c_1 \end{cases}$$

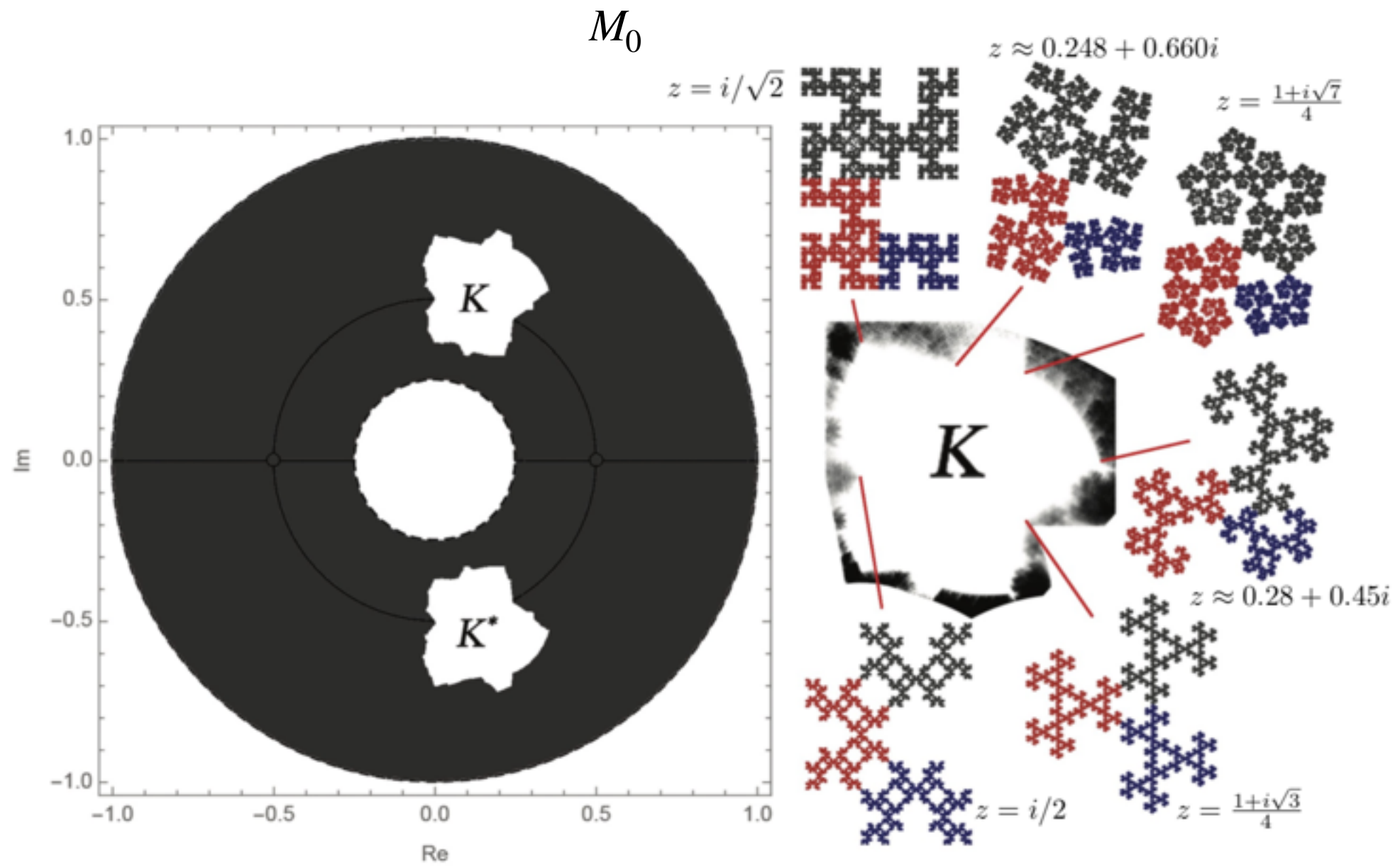


Figure 10: The unstable set $\mathcal{M}$ for $T\{z, -1/2, 1/4z\}$ with six examples of tipsets $F_{\mathcal{A}}(z)$ found in $\partial\mathcal{M}$.