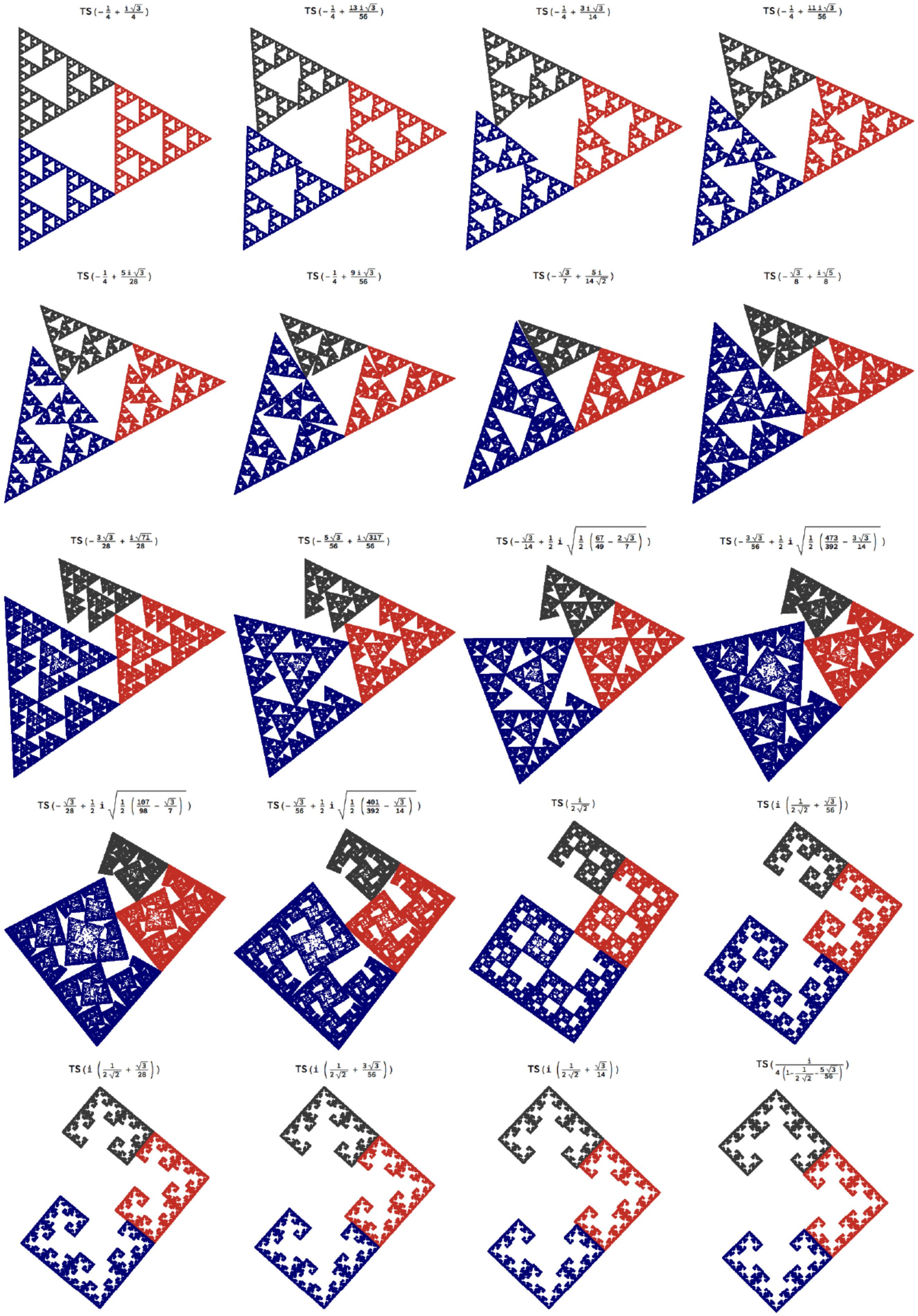
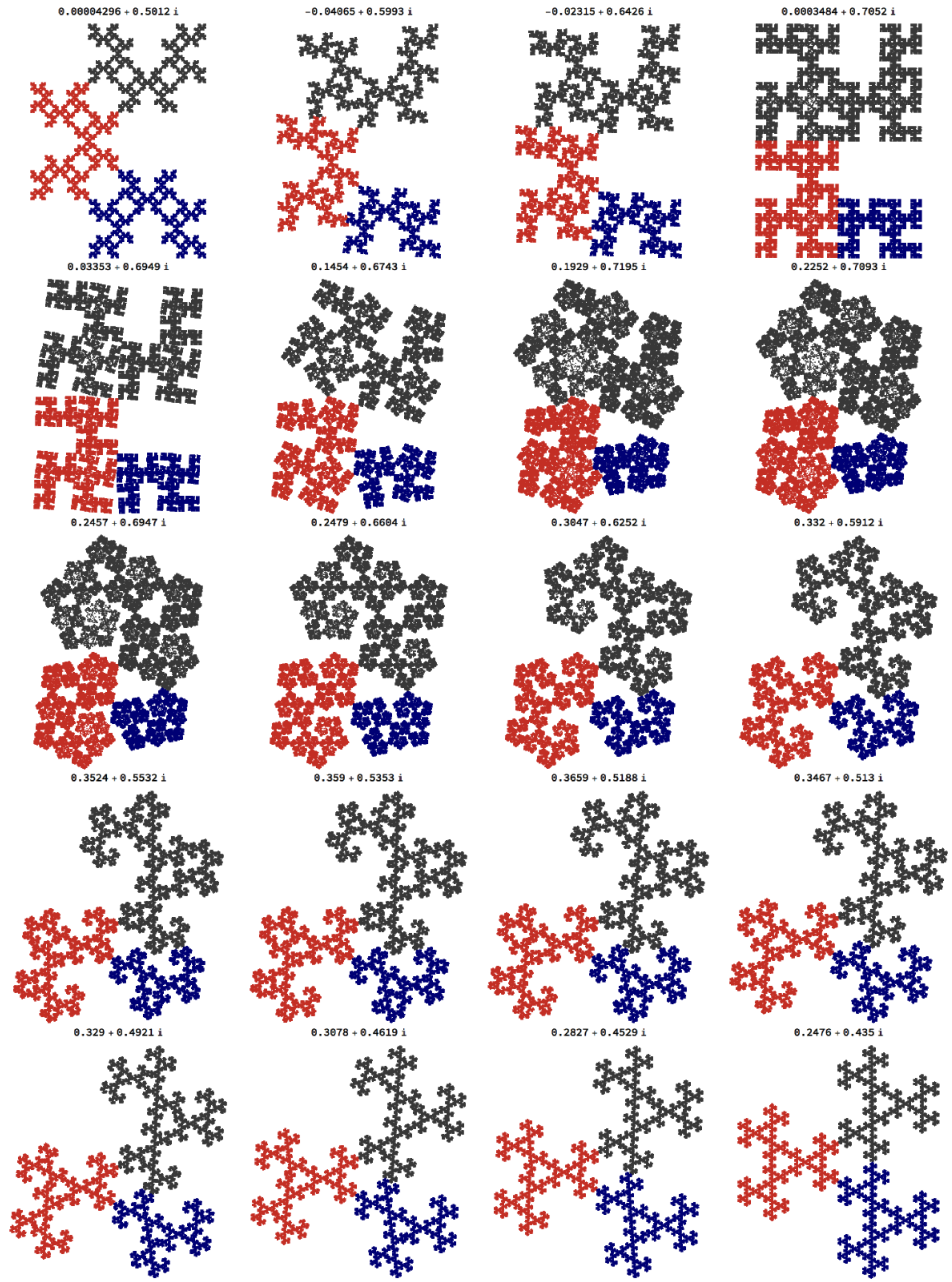
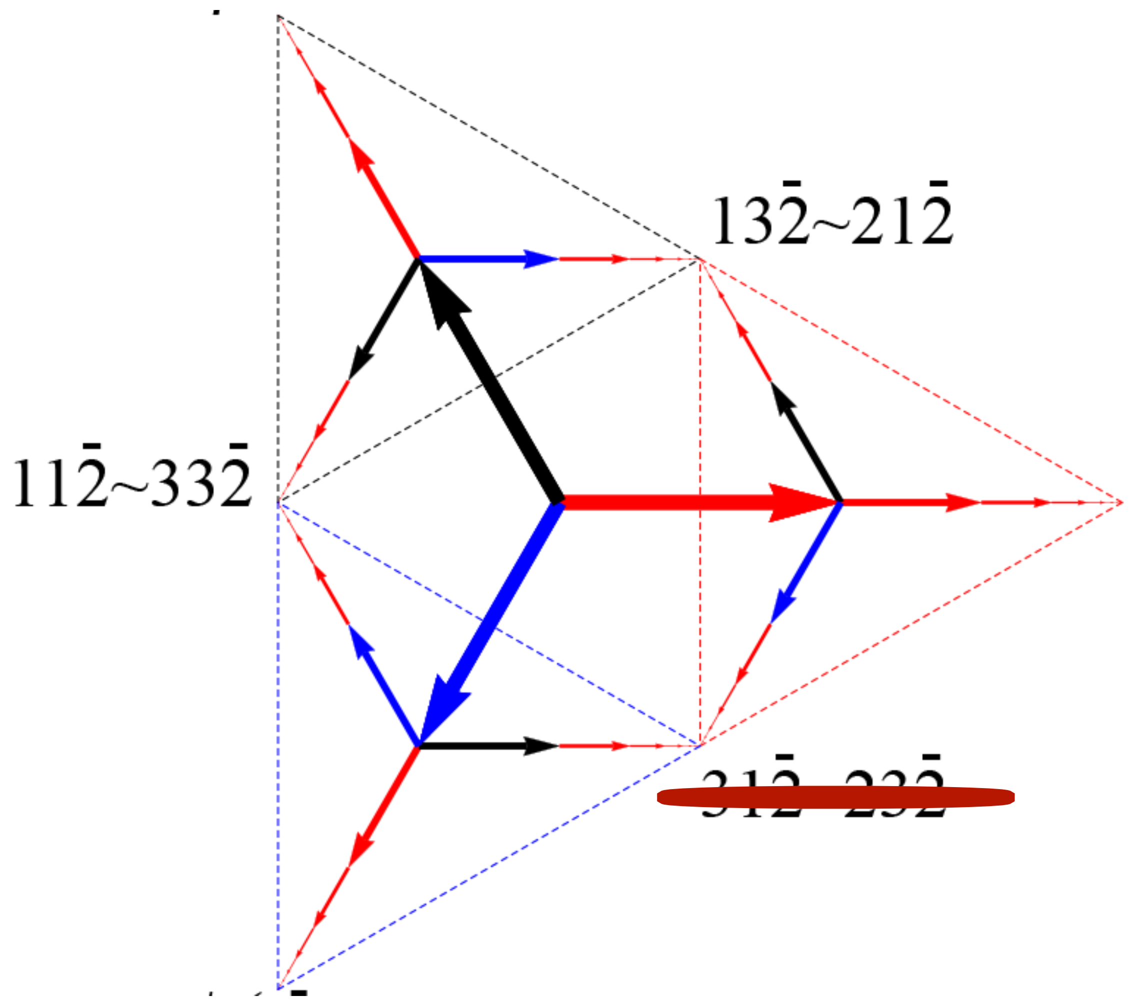
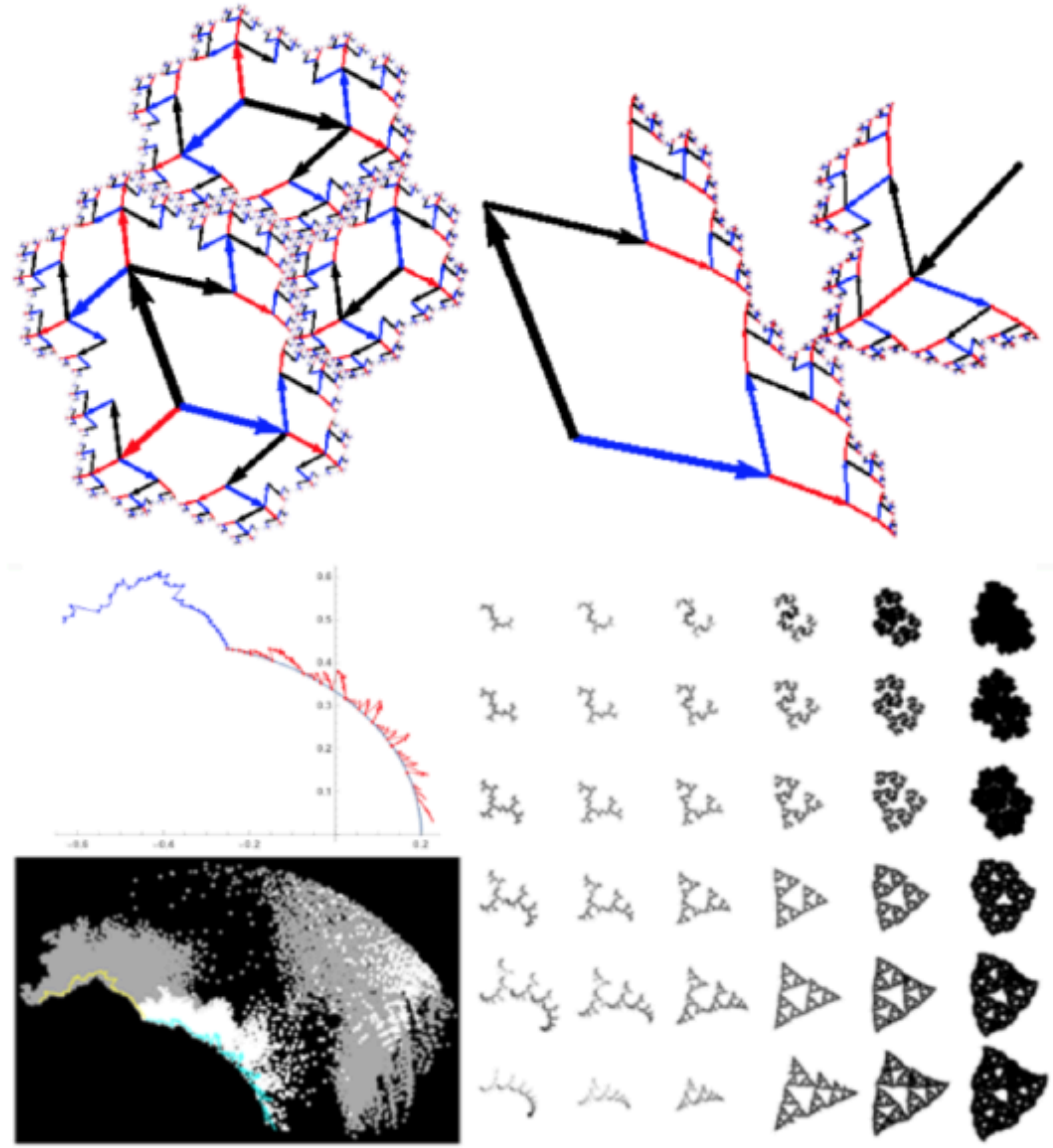

$TS(-\frac{1}{4}+\frac{i\sqrt{3}}{4})$
$TS(-\frac{1}{4}+\frac{13i\sqrt{3}}{56})$
$TS(-\frac{1}{4}+\frac{3i\sqrt{3}}{14})$
$TS(-\frac{1}{4}+\frac{11i\sqrt{3}}{56})$
$TS(-\frac{1}{4}+\frac{5i\sqrt{3}}{28})$
$TS(-\frac{1}{4}+\frac{9i\sqrt{3}}{56})$
$TS(-\frac{\sqrt{3}}{7}+\frac{5i}{14\sqrt{2}})$
$TS(-\frac{\sqrt{3}}{8}+\frac{i\sqrt{5}}{8})$
$TS(-\frac{3\sqrt{3}}{28}+\frac{i\sqrt{71}}{28})$
$TS(-\frac{5\sqrt{3}}{56}+\frac{i\sqrt{317}}{56})$
$TS(-\frac{\sqrt{3}}{14}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{67}{49}-\frac{2\sqrt{3}}{7}\right)})$
$TS(-\frac{3\sqrt{3}}{56}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{473}{392}-\frac{3\sqrt{3}}{14}\right)})$
$TS(-\frac{\sqrt{3}}{28}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{107}{98}-\frac{\sqrt{3}}{7}\right)})$
$TS(-\frac{\sqrt{3}}{56}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{401}{392}-\frac{\sqrt{3}}{14}\right)})$
$TS(\frac{i}{2\sqrt{2}})$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{56}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{28}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{3\sqrt{3}}{56}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{14}\right))$
$TS(\frac{i}{4\left(1-\frac{1}{2\sqrt{2}}-\frac{5\sqrt{3}}{56}\right)})$


0.00004296 + 0.5012 i
-0.04065 + 0.5993 i
-0.02315 + 0.6426 i
0.0003484 + 0.7052 i
0.03353 + 0.6949 i
0.1454 + 0.6743 i
0.1929 + 0.7195 i
0.2252 + 0.7093 i
0.2457 + 0.6947 i
0.2479 + 0.6604 i
0.3047 + 0.6252 i
0.332 + 0.5912 i
0.3524 + 0.5532 i
0.359 + 0.5353 i
0.3659 + 0.5188 i
0.3467 + 0.513 i
0.329 + 0.4921 i
0.3078 + 0.4619 i
0.2827 + 0.4529 i
0.2476 + 0.435 i





Illustrating where the topological changes take place for a family of connected self-similar sets including the classical Rauzy fractal and the Sierpinski triangle. Jan 2019.

IMUB

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Thank you!

"If you wish to advance into the infinite, explore the finite in all directions."

- Goethe.

Bernat Espigulé

Holomorphic Dynamics Group
Núria Fagella & Xavier Jarque

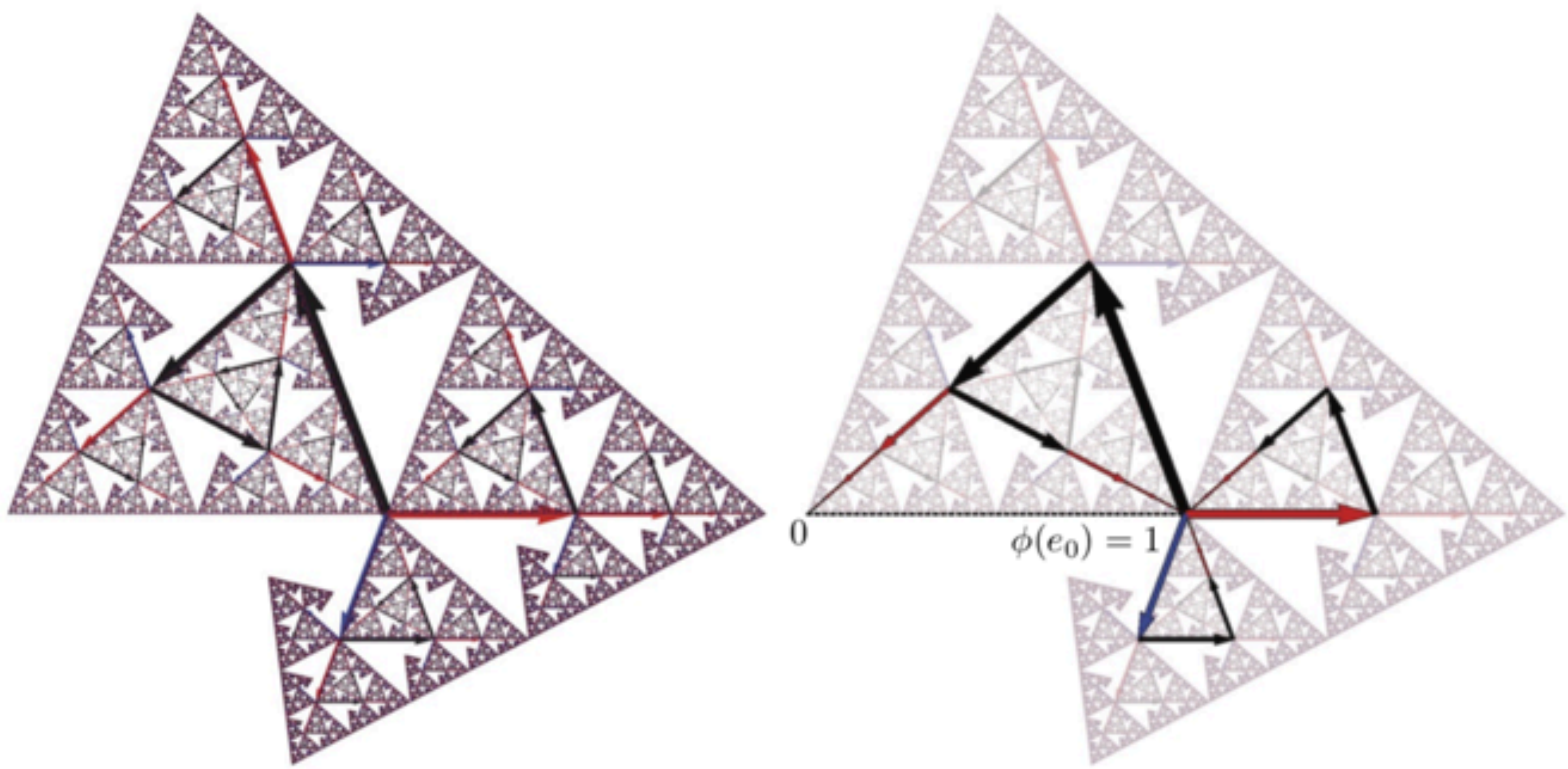


Figure 11: Root-connected ternary tree $T_{\mathcal{A}}(z_0) = T_{\mathcal{A}}((-1+i\sqrt{7})/4) = T\{(-1+i\sqrt{7})/4, 1/2, (-1-i\sqrt{7})/8\}$. Its intersection set is $I_{F_{\mathcal{A}}} = F_{1\mathcal{A}} \cap F_{2\mathcal{A}} \cap F_{2\mathcal{A}} = \{1 = \phi(e_0) = \phi(111\overline{2}) = \phi(211\overline{2}) = \phi(311\overline{2})\}$. The intersection set for this tree is a singleton but the overlap set is not, $I_{F_{\mathcal{A}}} \neq O_{F_{\mathcal{A}}}$.

Theorem 1. **(Tip-to-root connectivity).** *Let $j \in \mathcal{A}$, $w \in \mathcal{A}^\infty$, and $F_\mathcal{A}$ be a tipset of an n-ary complex tree $T_\mathcal{A}$, then the following are equivalent:*

(1) $\phi(jw) = \phi(e_0) = 1$

(2) $\phi(w) = 0$

(3) $\phi(jw) = \phi(kw)$ for any $k \in \mathcal{A}$

(4) $F_\mathcal{A}$ is root-connected

(5) $\phi(v) \in F_\mathcal{A}$ for all $v \in \mathcal{A}^*$

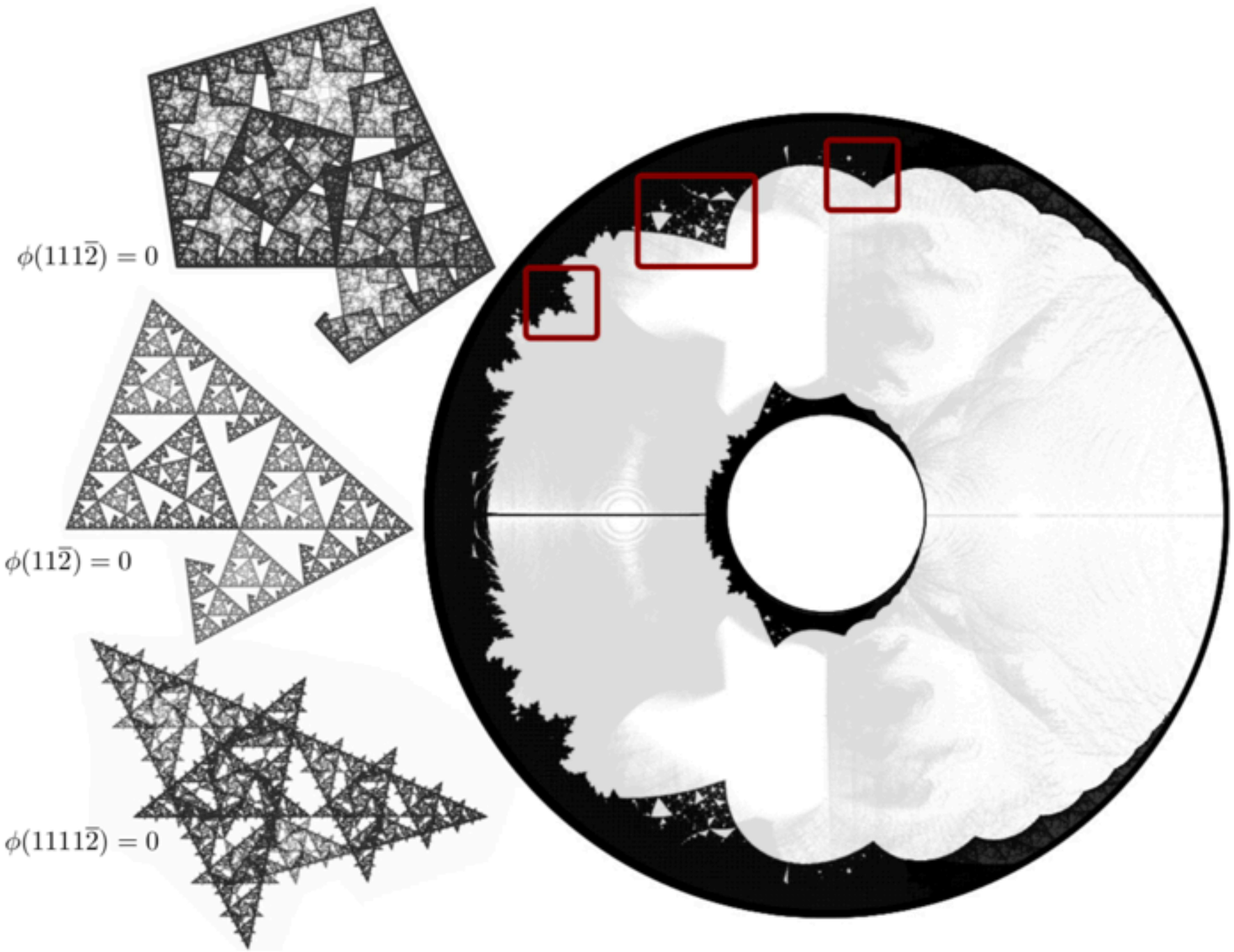
Proof. (1) $\implies$ (2): From (1) we have that $0 = \phi(jw) - \phi(e_0) = 1 + c_j\phi(w) - 1 = c_j\phi(w)$ which implies $\phi(w) = 0$ since $c_j \neq 0$ by definition. (2) $\implies$ (3): Independently of the first letter $k \in \mathcal{A}$ we have $\phi(kw) = 1 + c_k\phi(w) = 1 = \phi(e_0) = \phi(jw)$. (3) $\implies$ (4): For each first-level piece there is at least one tip point that meets the root $\phi(1w) = \phi(2w) = \cdots = \phi(nw) = 1$, hence $I_{F_\mathcal{A}} \neq \emptyset$ and $\phi(e_0) \in I_{F_\mathcal{A}}$ so $F_\mathcal{A}$ is root-connected. (4) $\implies$ (5): If $F_\mathcal{A}$ is root-connected we have that $\phi(e_0) \in F_\mathcal{A}$ hence by self-similarity $f_v(\phi(e_0)) = \phi(v) \in F_{v\mathcal{A}} \subseteq F_\mathcal{A}$, i.e. all the nodes of the tree $T_\mathcal{A}$ are contained in the tipset $F_\mathcal{A}$. (5) $\implies$ (1): We have that $\phi(e_0) \in F_\mathcal{A}$ hence $F_\mathcal{A}$ is root-connected and $\phi(e_0) \in F_j$ so there is at least a tip point with word $jw \in \mathcal{A}^\infty$ such that $\phi(jw) = \phi(e_0)$. □

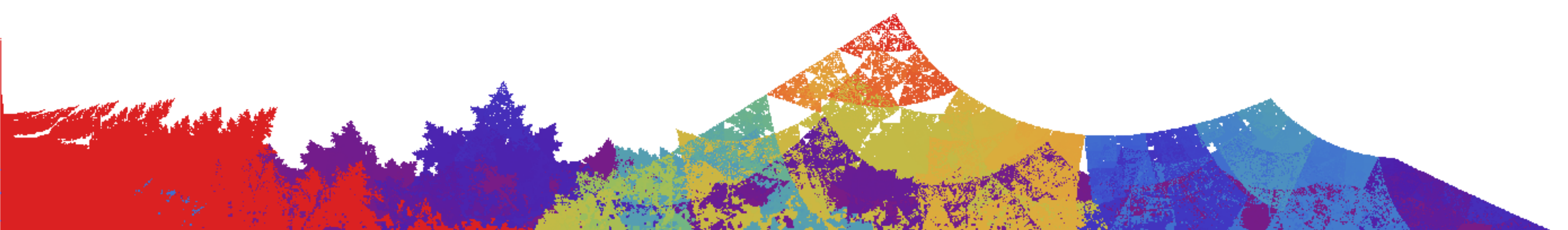
Theorem 2. **(Node-to-root connectivity).** *Let $v \in \mathcal{A}^* \backslash \{e_0\}$, if $\phi(v) = \phi(e_0)$ then $F_\mathcal{A}$ is root-connected.*

Proof. By applying equation (4) to the tip point of $\overline{v} \in \mathcal{A}^\infty$ we have that

$$\begin{aligned}
\phi(\overline{v}) &= \phi(v\overline{v}) \\
&= \phi(v) + v \cdot (\phi(\overline{v}) - 1) \\
&= \phi(e_0) + v \cdot (\phi(\overline{v}) - 1) \\
&= 1 + v \cdot (\phi(\overline{v}) - 1),
\end{aligned}$$

i.e. $\phi(\overline{v}) - 1 = v \cdot (\phi(\overline{v}) - 1)$. So $\phi(\overline{v}) - 1 = 0$ since the complex product $v = v_1 \cdot v_2 \cdot \ldots$ is neither 1 nor 0. Therefore, tip-to-root connectivity $\phi(\overline{v}) = 1 = \phi(e_0)$ takes place and by theorem 1 we have that the tipset $F_\mathcal{A}$ is root-connected. □


$\phi(111\bar{2}) = 0$
$\phi(11\bar{2}) = 0$
$\phi(1111\bar{2}) = 0$



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Complex Trees Research Project

Family of Binary Complex Trees

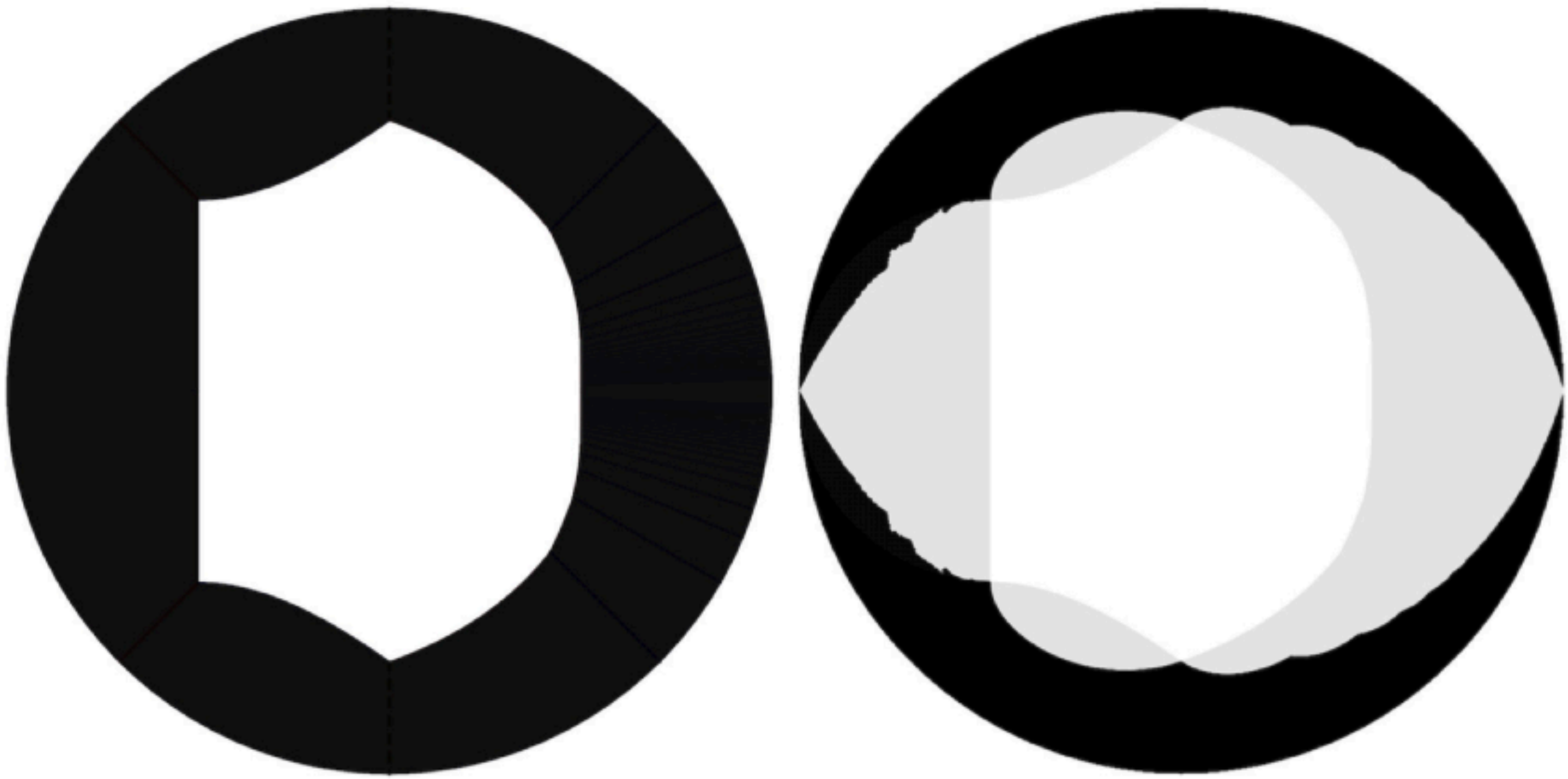
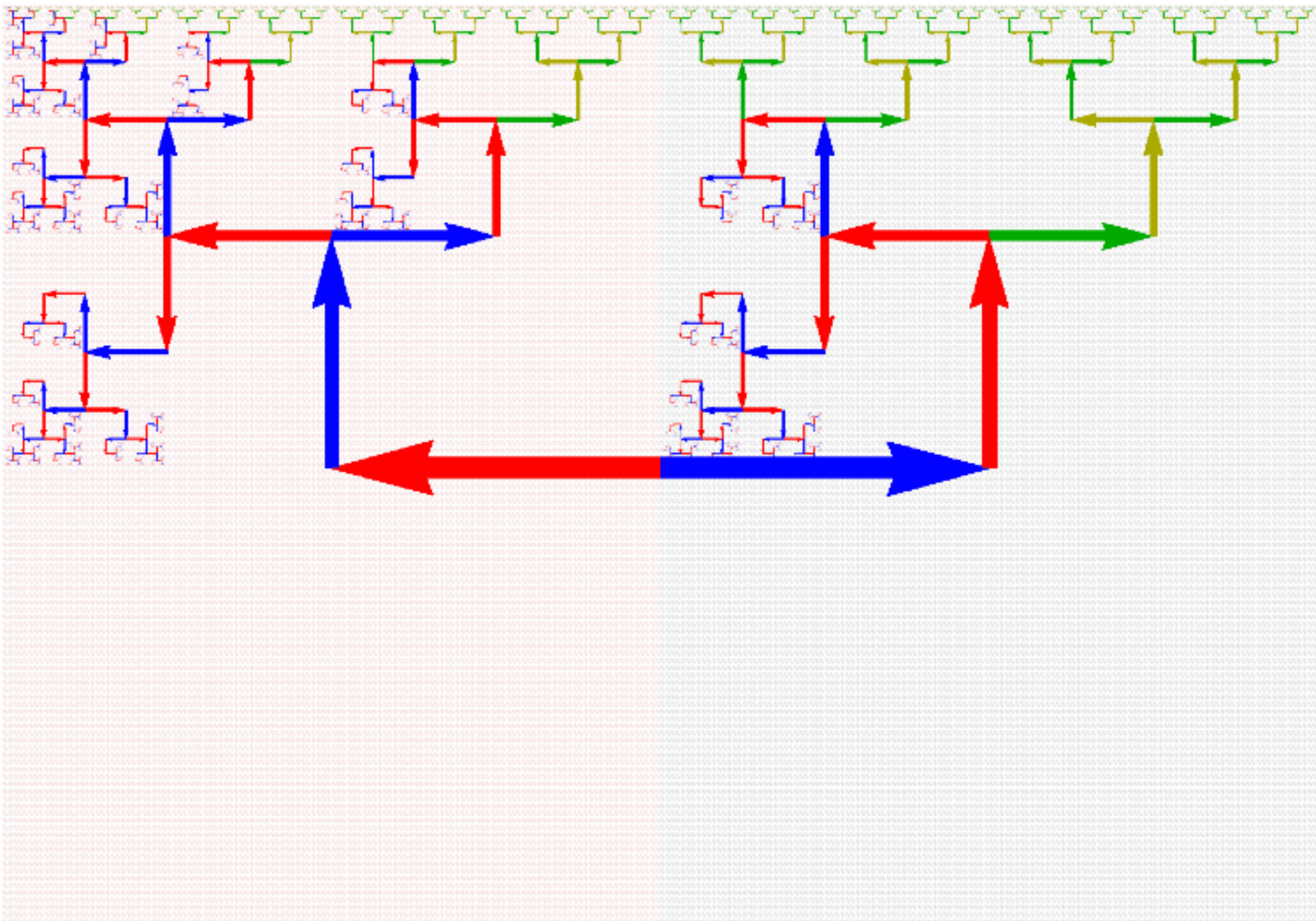


Figure 17: The unstable set $\mathcal{M} = \{c : Q_{\{c,c^*\}} \neq \emptyset\}$, and the root connectivity set $\mathcal{M}_0 = \{c : 0 \in F_{\{c,c^*\}}\}$.



Structural Stability of Connected Self-similar Sets

Bernat Espigulé
ComplexTrees.com
Holomorphic Dynamics Group
Núria Fagella and Xavier Jarque
UNIVERSITAT DE BARCELONA

Binary Complex Tree

$T\{c_1, c_2\} = T\{.3 + i.6, .5 - i.3\}$

nodes

$\phi(1) = 1 + c_1$

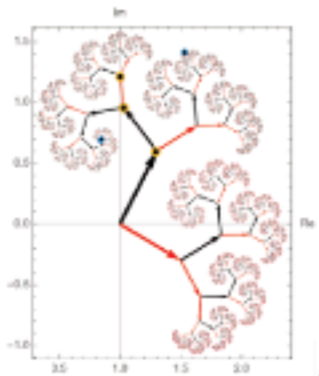
$\phi(11) = 1 + c_1 + c_1^2$

$\phi(112) = 1 + c_1 + c_1^2 + c_1^2 c_2$

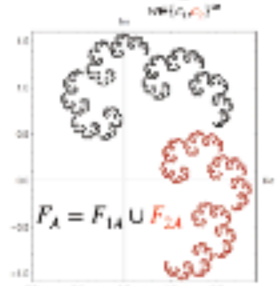
tip points

$\phi(111\ldots) = \phi(\overline{1}) = \sum_{i=0}^{\infty} c_1^i = \frac{1}{1 - c_1}$

$\phi(121212\ldots) = \phi(\overline{12}) = \frac{1 + c_1}{1 - c_1 c_2}$

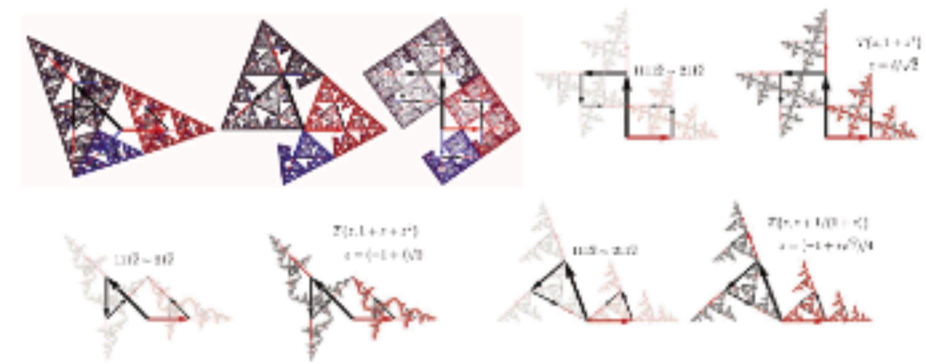


$F_\lambda = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^{\infty}} \phi(w)$

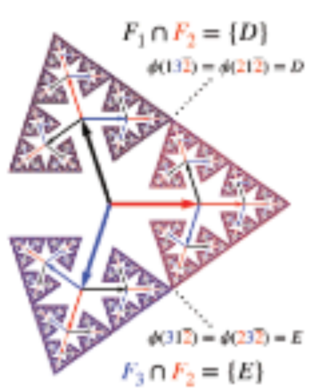


$F_{1\lambda} := f_1(F_\lambda) = 1 + c_1 \cdot F_\lambda$

$F_{2\lambda} := f_2(F_\lambda) = 1 + c_2 \cdot F_\lambda$



Ternary Complex Tree



"tip-to-tip gluing"

$\phi(13\overline{2}) = 1 + c_1 + c_1 c_3 \sum_{k=0}^{\infty} c_2^k = 1 + c_1 + \frac{c_1 c_3}{1 - c_2}$

$\phi(21\overline{2}) = 1 + c_2 + \frac{c_2 c_1}{1 - c_2}$

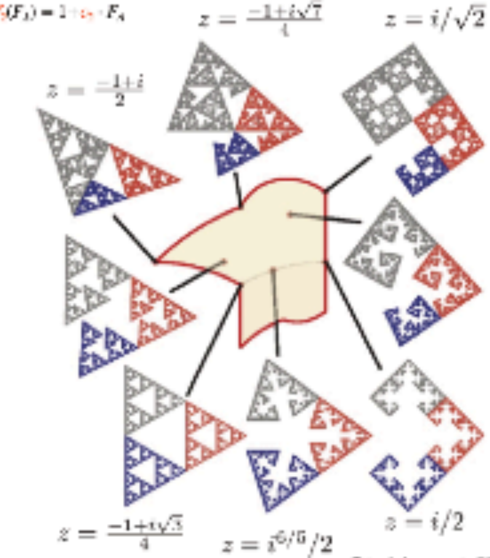
$\phi(23\overline{2}) = 1 + c_2 + \frac{c_2 c_3}{1 - c_2}$

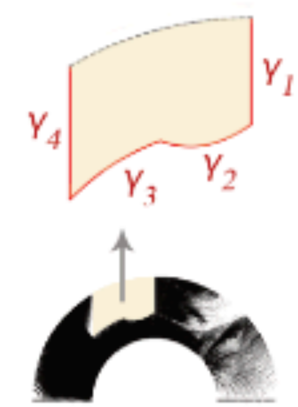
$\phi(31\overline{2}) = 1 + c_3 + \frac{c_3 c_1}{1 - c_2}$

$$\begin{cases} \phi(13\overline{2}) = \phi(21\overline{2}) \\ \phi(23\overline{2}) = \phi(31\overline{2}) \end{cases} \longrightarrow \begin{cases} c_2 = 1/2 \\ c_3 = 1/4c_1 \end{cases}$$

$c_1(z) := z, \quad c_2(z) := 1/2, \quad c_3(z) := 1/4z$

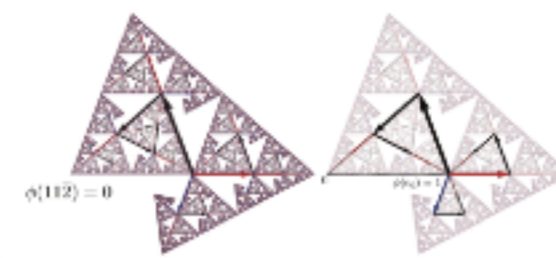
$T_\lambda(z) = T\{c_1(z), c_2(z), c_3(z)\} = T\{z, 1/2, 1/4z\}$

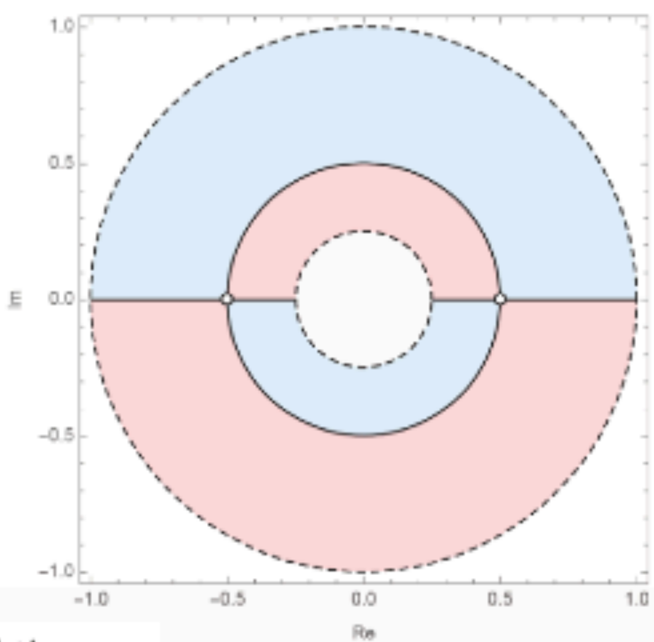




Piecewise smooth boundary

$$\gamma(t) := \begin{cases} -it - \frac{1}{4} & -\frac{\sqrt{3}}{4} \le t \le -\frac{1}{4} \\ t + \frac{i\sqrt{1-8t^2}}{2\sqrt{2}} & -\frac{1}{4} < t \le -\frac{1}{8} \\ t + \frac{i\sqrt{24t^2-4t+1}}{2\sqrt{2}} & -\frac{1}{8} < t < 0 \\ i\left(t + \frac{1}{2\sqrt{2}}\right) & 0 \le t \le \frac{1}{4}\left(2 - \sqrt{2}\right) \end{cases}$$





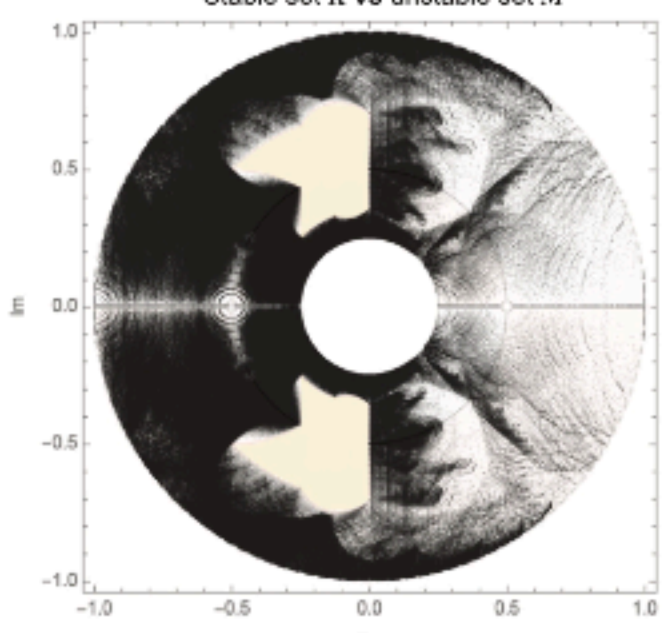
$|c_1(z)| = |z| < 1$

$|c_2(z)| = 1/2 < 1$

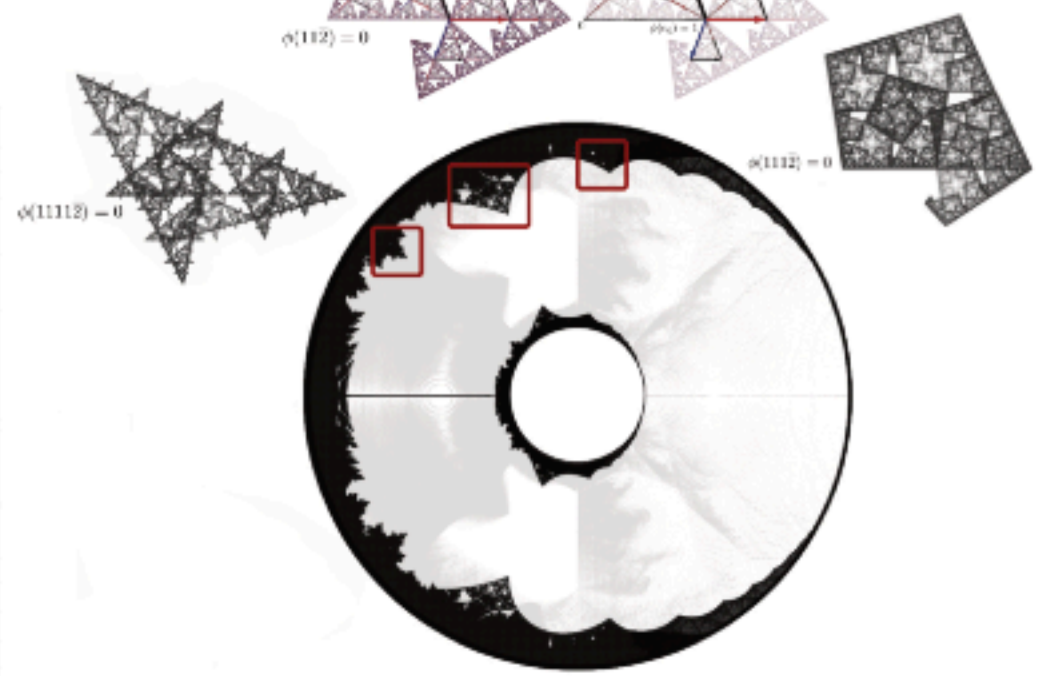
$|c_3(z)| = \frac{1}{4|z|} < 1 \Rightarrow |z| > \frac{1}{4}$

$\mathscr{R} := \{z \in \mathbb{C} : 1/4 < |z| < 1\}$

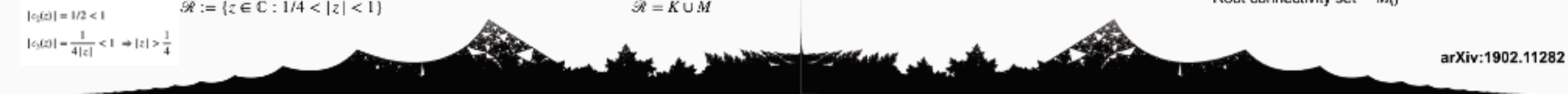
Stable set K vs unstable set M



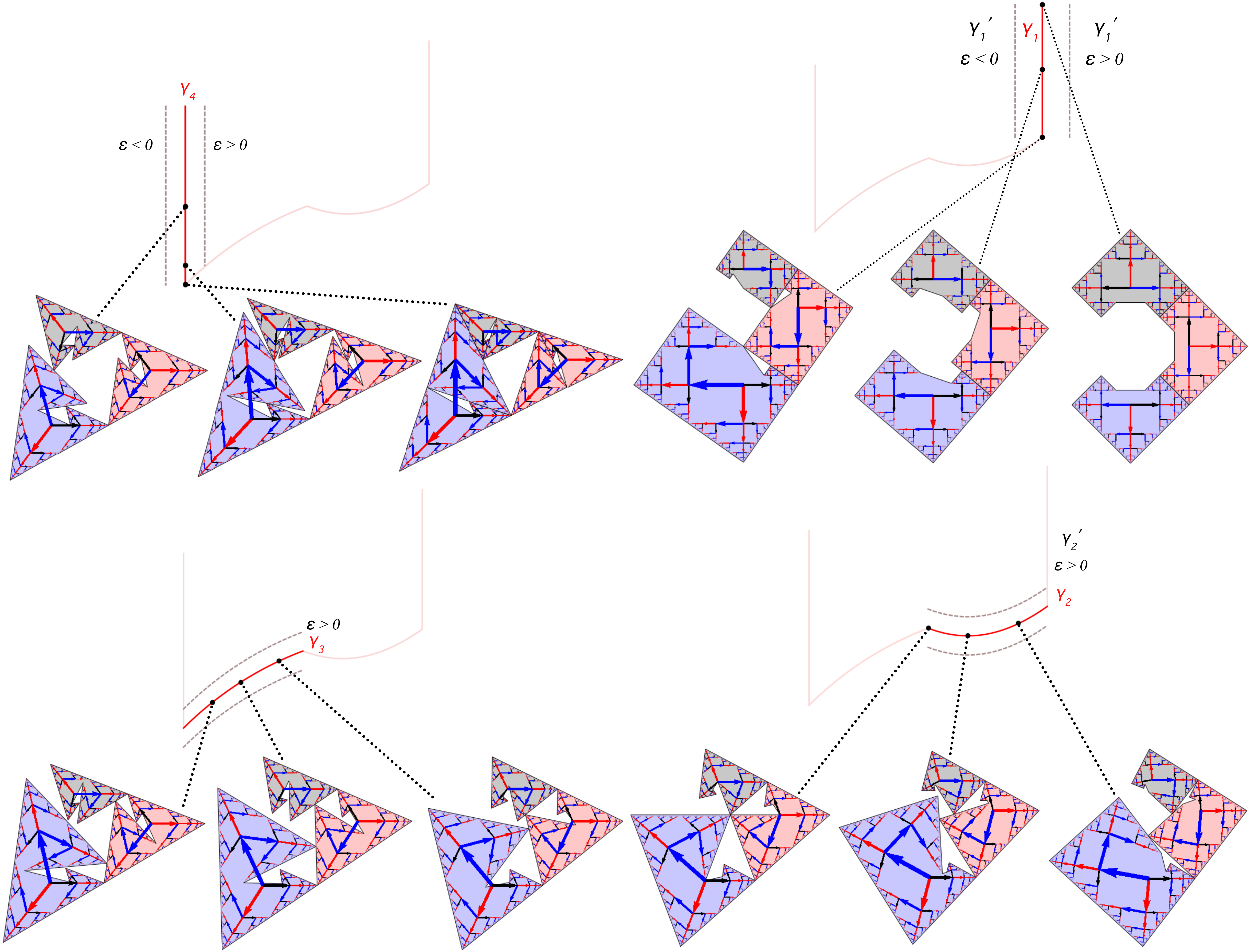
$\mathscr{R} = K \cup M$



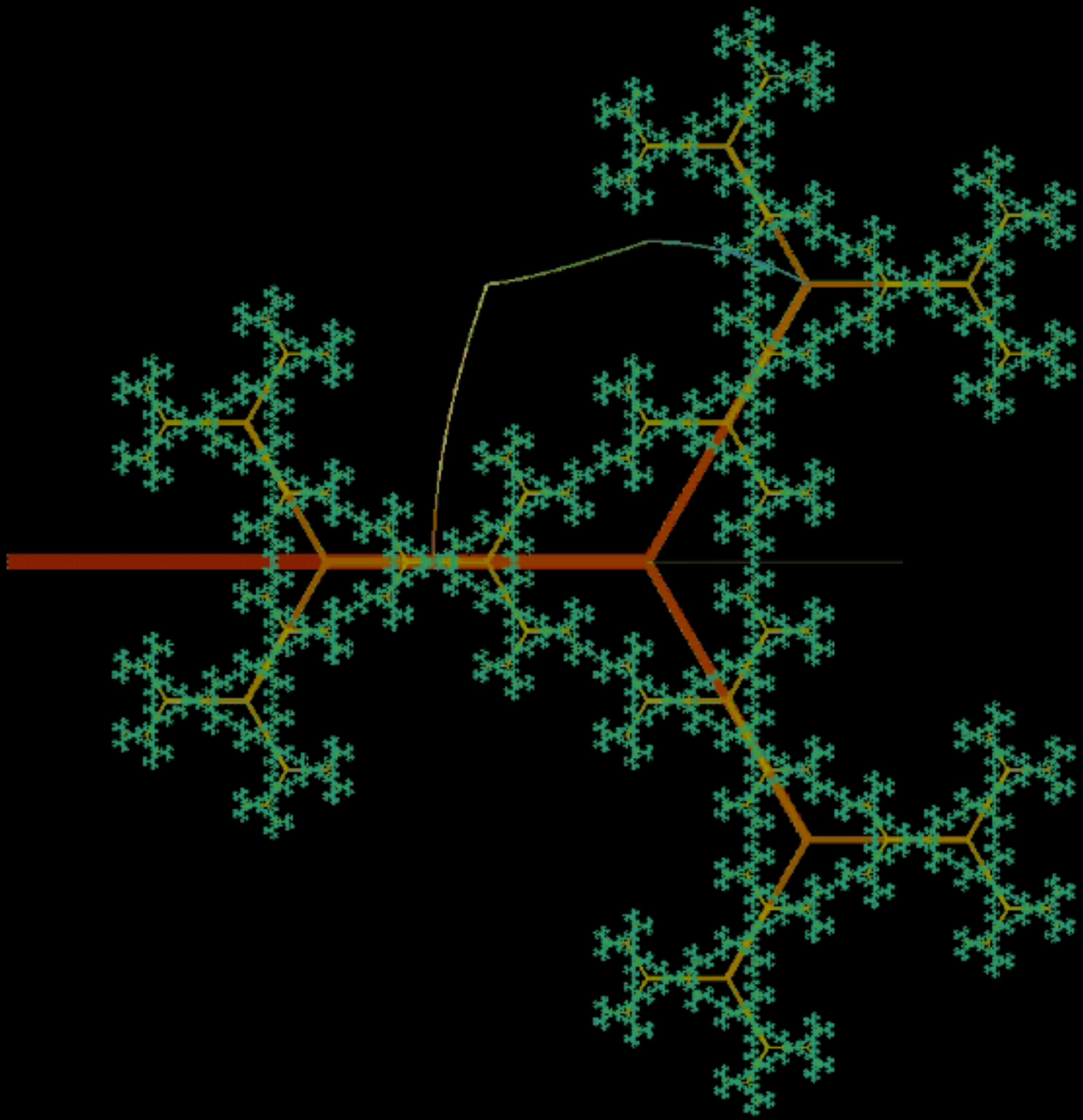
Root connectivity set $\mathcal{M}_0$

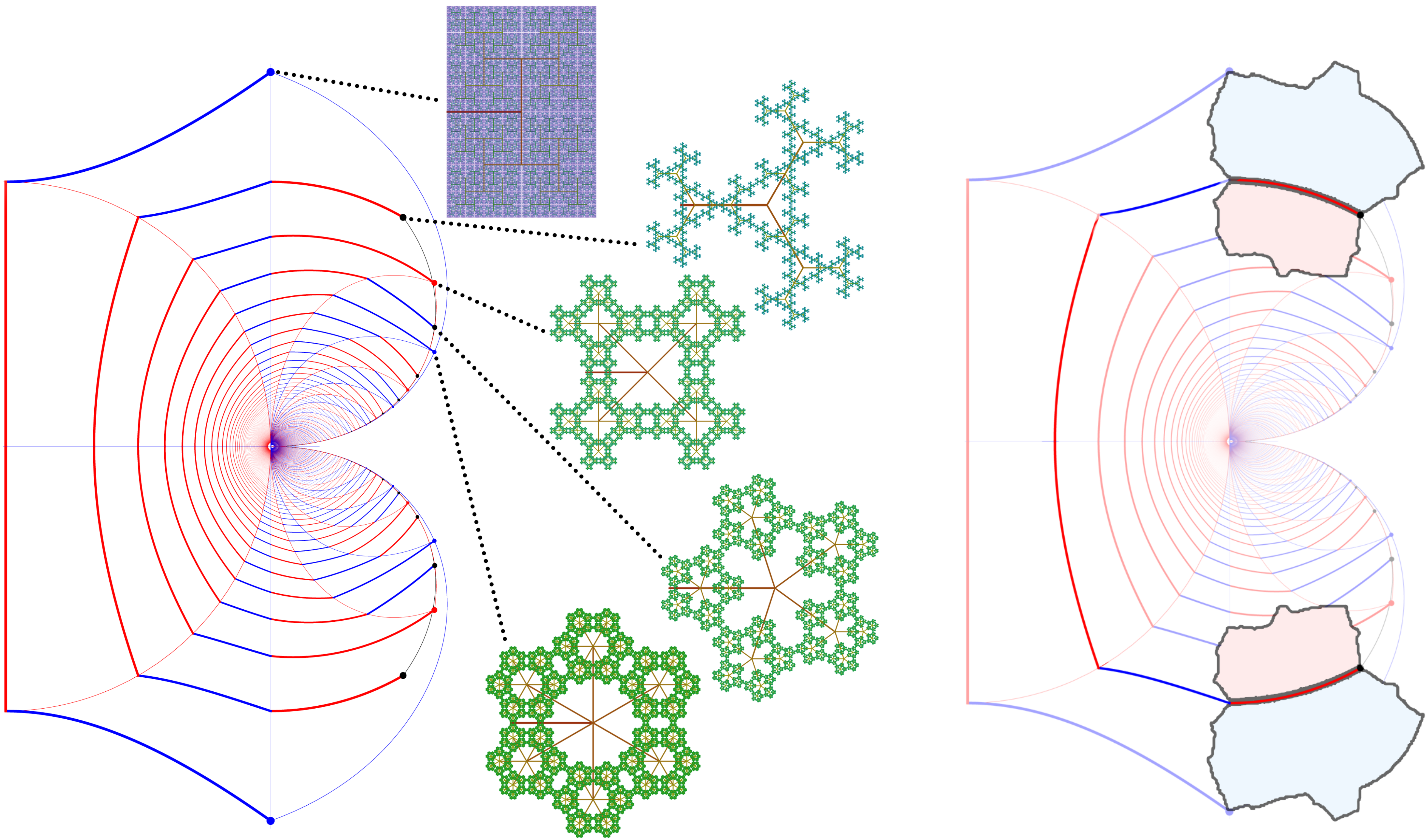


arXiv:1902.11282



γ_4
$\varepsilon < 0$
$\varepsilon > 0$
γ_1'
$\varepsilon < 0$
γ_1
γ_1'
$\varepsilon > 0$
$\varepsilon > 0$
γ_3
γ_2'
$\varepsilon > 0$
γ_2





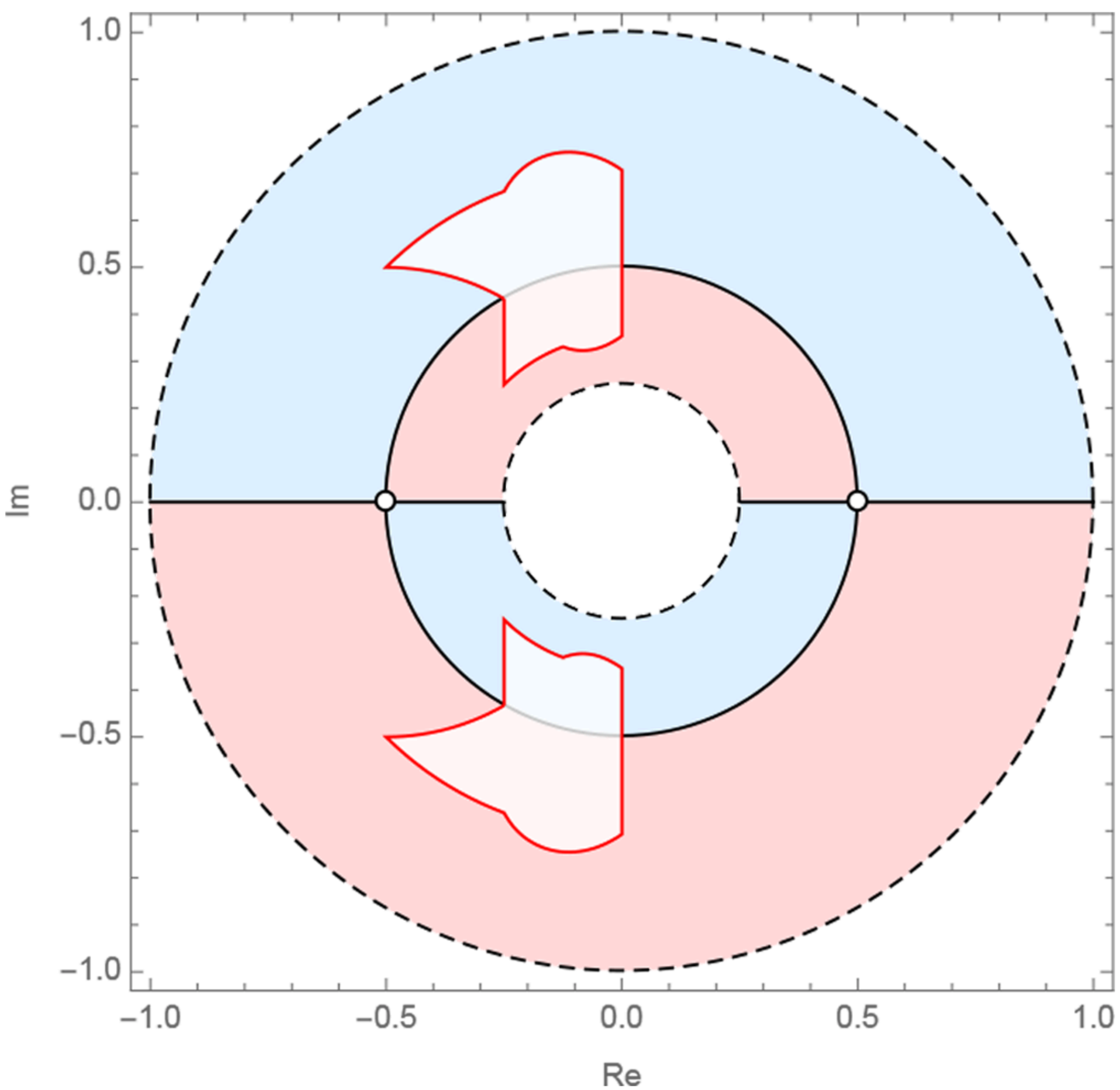
The unstable set M vs the stable set K

$$T_A(z) := T\{c_1(z), c_2(z), \ldots, c_n(z)\} \qquad \mathscr{R} = K \cup M$$

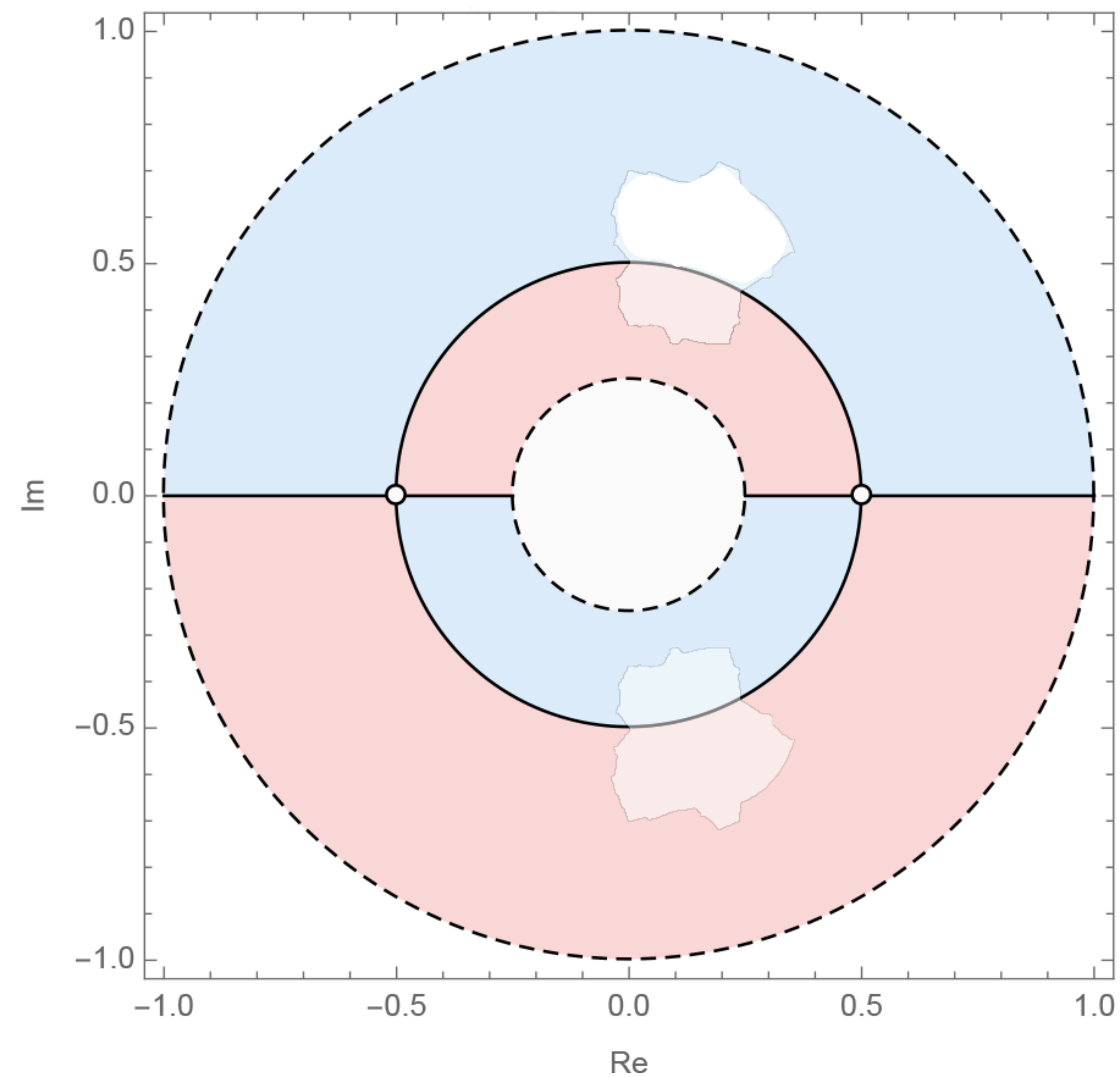
$$K := \{z \in \mathscr{R} : Q_A = Q_A(z)\}$$

$$M := \{z \in \mathscr{R} : Q_A \subsetneq Q_A(z)\}$$

$$Q_A = \{13\overline{2} \sim 21\overline{2},\ 31\overline{2} \sim 23\overline{2}\}$$



$$Q_A = \{12\overline{31} \sim 22\overline{13},\ 32\overline{13} \sim 22\overline{31}\}$$



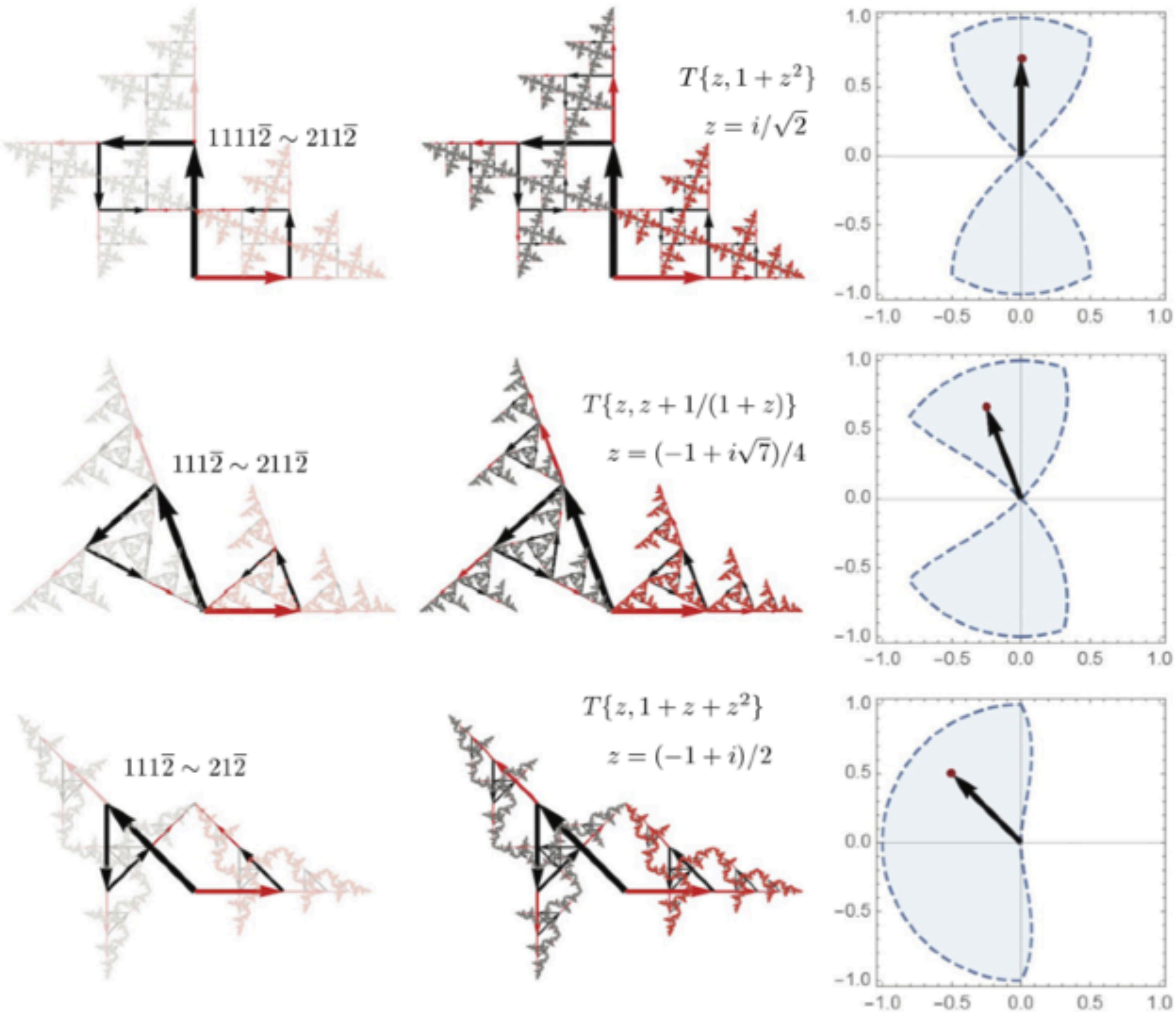


Figure 6: Left column, branch-paths of tip-to-tip equivalence relations $a \sim b$ where $F_{1\mathcal{A}} \cap F_{2\mathcal{A}} = \{\phi(a) = \phi(b)\}$. Central column, binary complex trees $T\{i/\sqrt{2}, 1/2\}$, $T\{(-1+i\sqrt{7})/4, 1/2\}$, and $T\{(-1+i)/2, 1/2\}$. Right column, regions $\mathcal{R} := \{z \in \mathbb{C} : 0 < |z|, |c_2(z)| < 1\}$ where one-parameter families of complex trees $T_{\mathcal{A}}(z) = T\{z, c_2(z)\}$ are defined. From top to bottom the algebraic expression for $c_2(z)$ was obtained in (12), (13), and (14) respectively.

$$\mathcal{M}_2 := \{z \in \mathcal{R} : 1 < |c_1(z)|^2 + |c_2(z)|^2 + \cdots + |c_n(z)|^2\}. \tag{28}$$

See below the regions $\mathcal{M}_2$ for the families of binary trees, $T\{z, 1+z^2\}$, $T\{z, z+1/(1+z)\}$, and $T\{z, 1+z+z^2\}$, obtained in (12), (13), and (14) respectively.

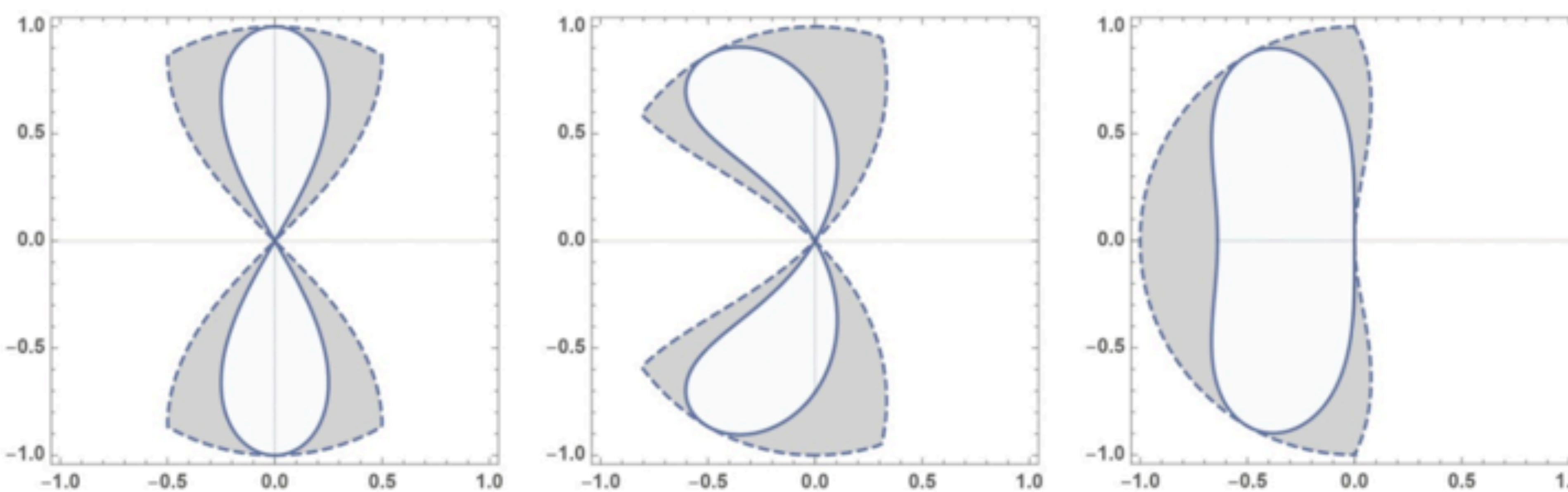
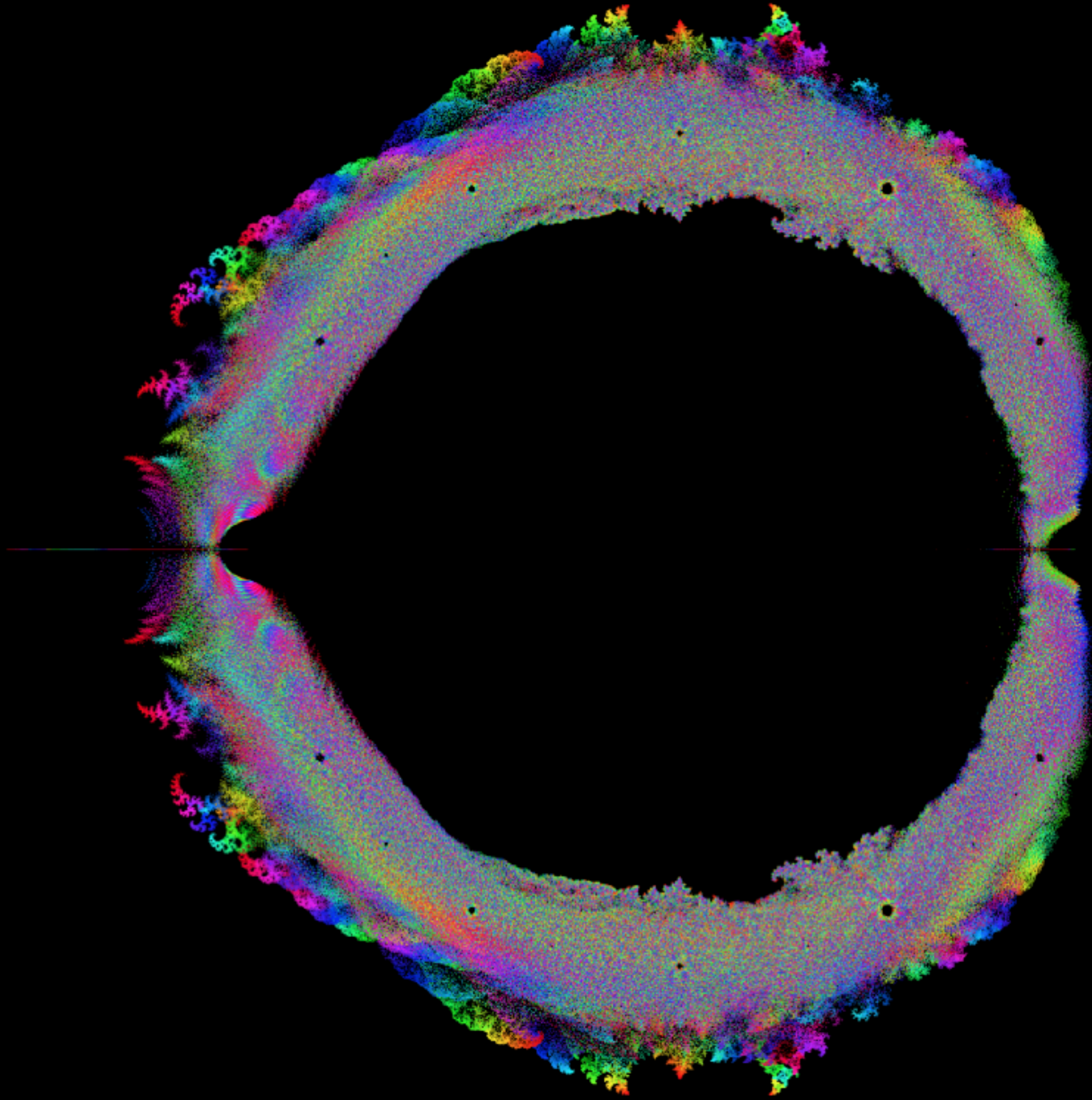
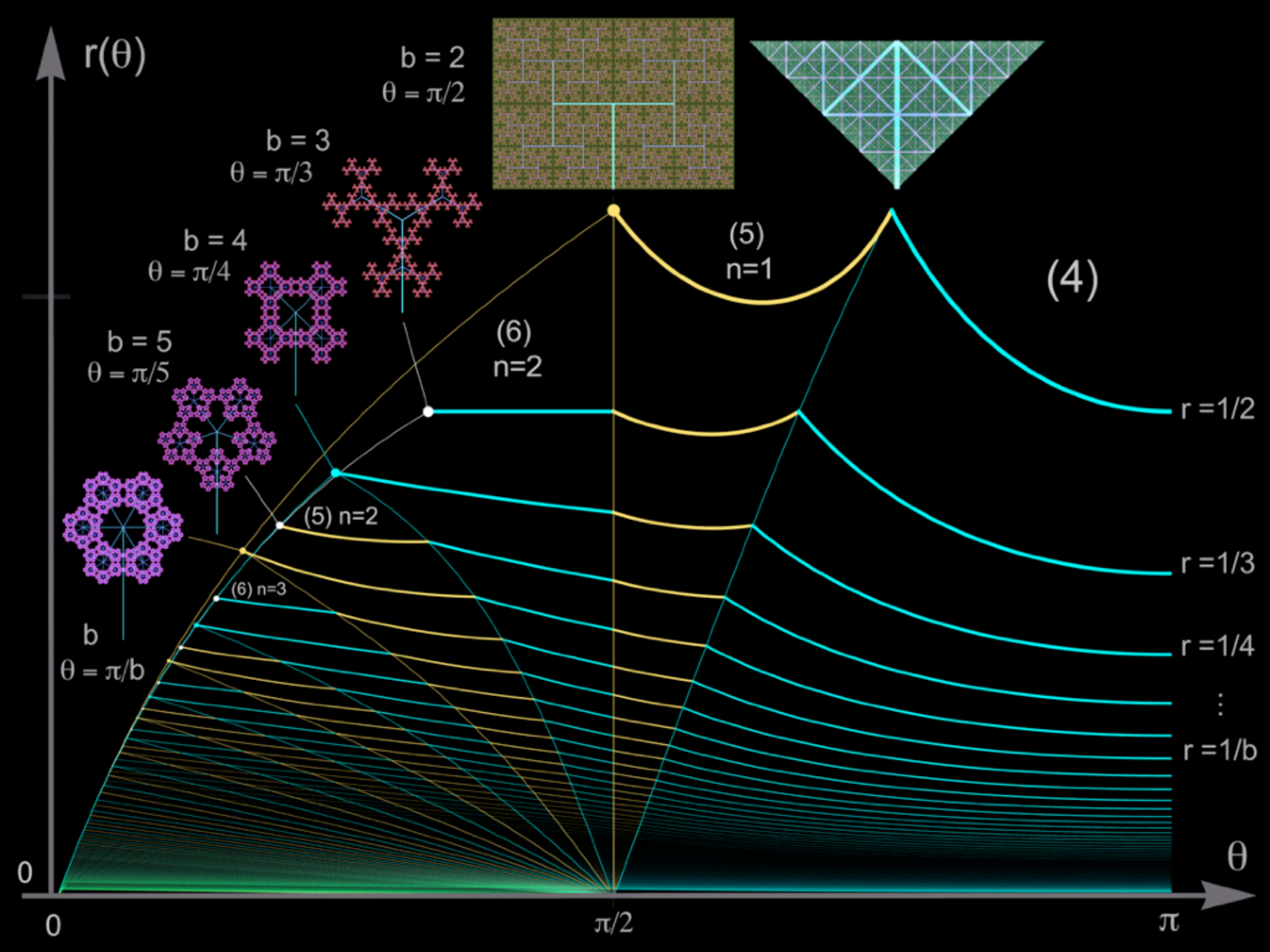


Figure 13: Regions $\mathcal{R}$ shown in figure 6 overlaid with regions $\mathcal{M}_2$ entirely composed of non-p.c.f. unstable trees.





r(θ)
b = 2
θ = π/2
b = 3
θ = π/3
b = 4
θ = π/4
b = 5
θ = π/5
b
θ = π/b
(5)
n=1
(4)
(6)
n=2
(5) n=2
(6) n=3
r =1/2
r =1/3
r =1/4
⋮
r =1/b
θ
0
0
π/2
π