Topics in Complex Dynamics 2019 IMUB, Universitat de Barcelona, March 28th

Families of connected self-similar sets generated by Complex Trees

arXiv:1902.11282

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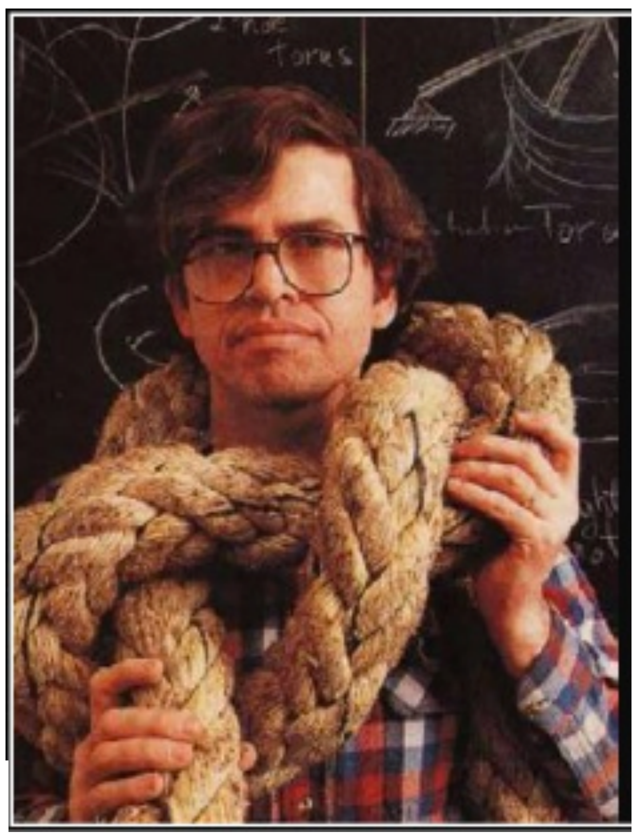
Thurston’s last paper

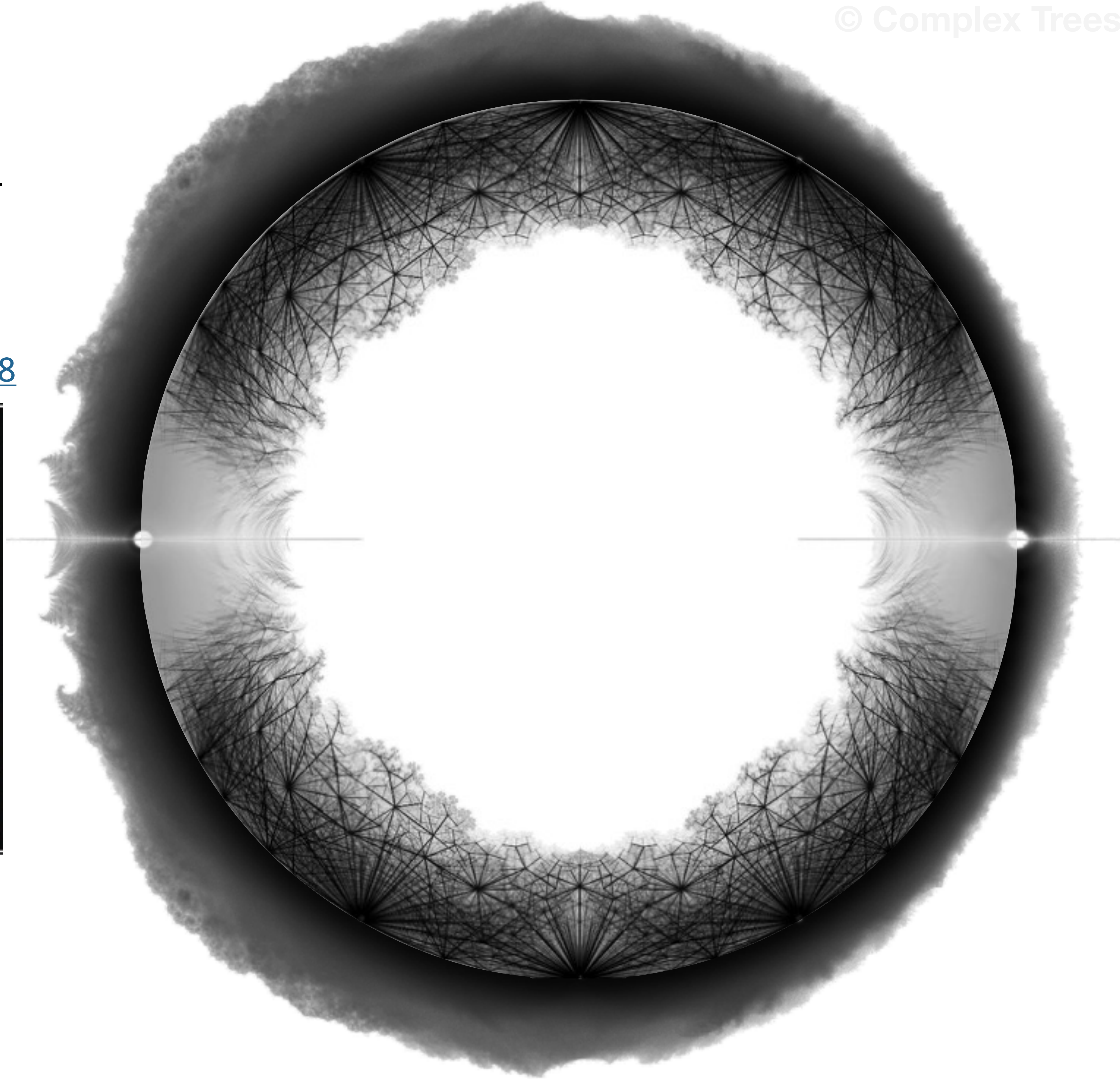
ENTROPY IN DIMENSION ONE

arxiv.org/abs/1402.2008

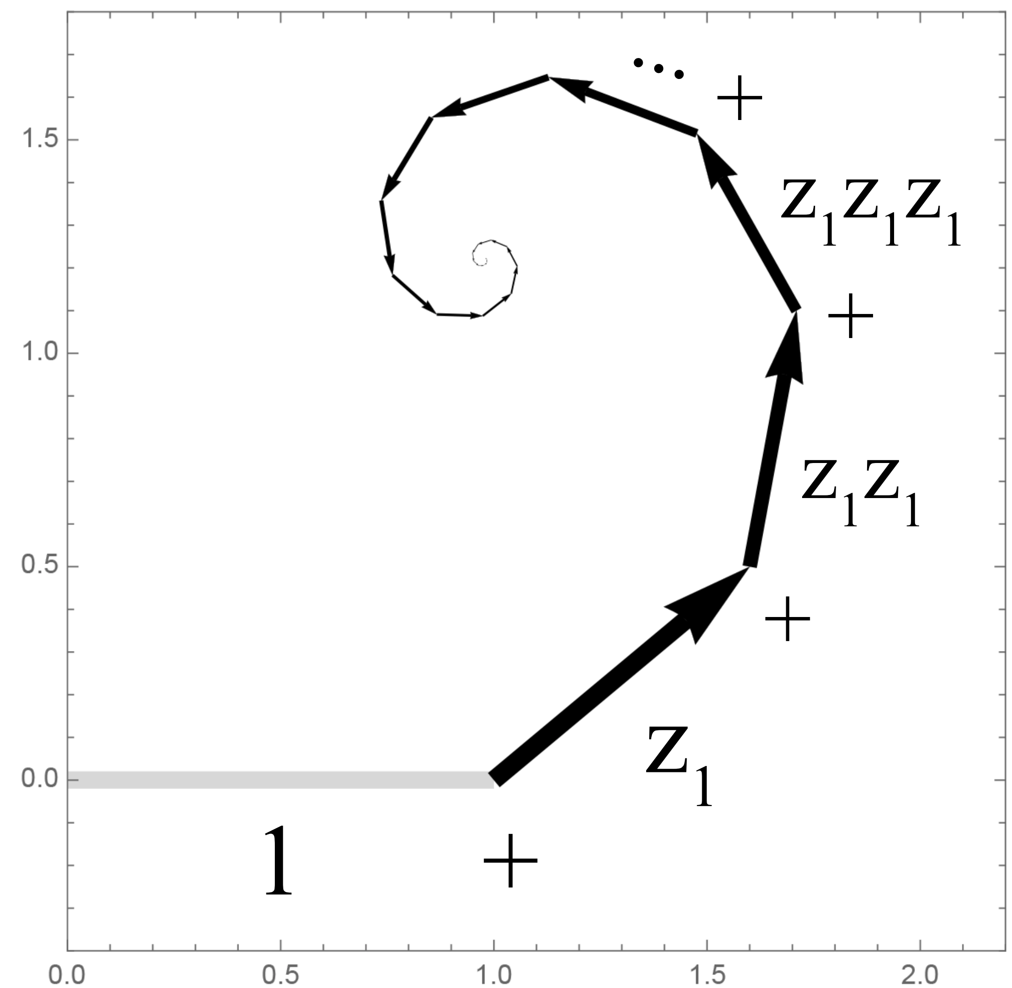


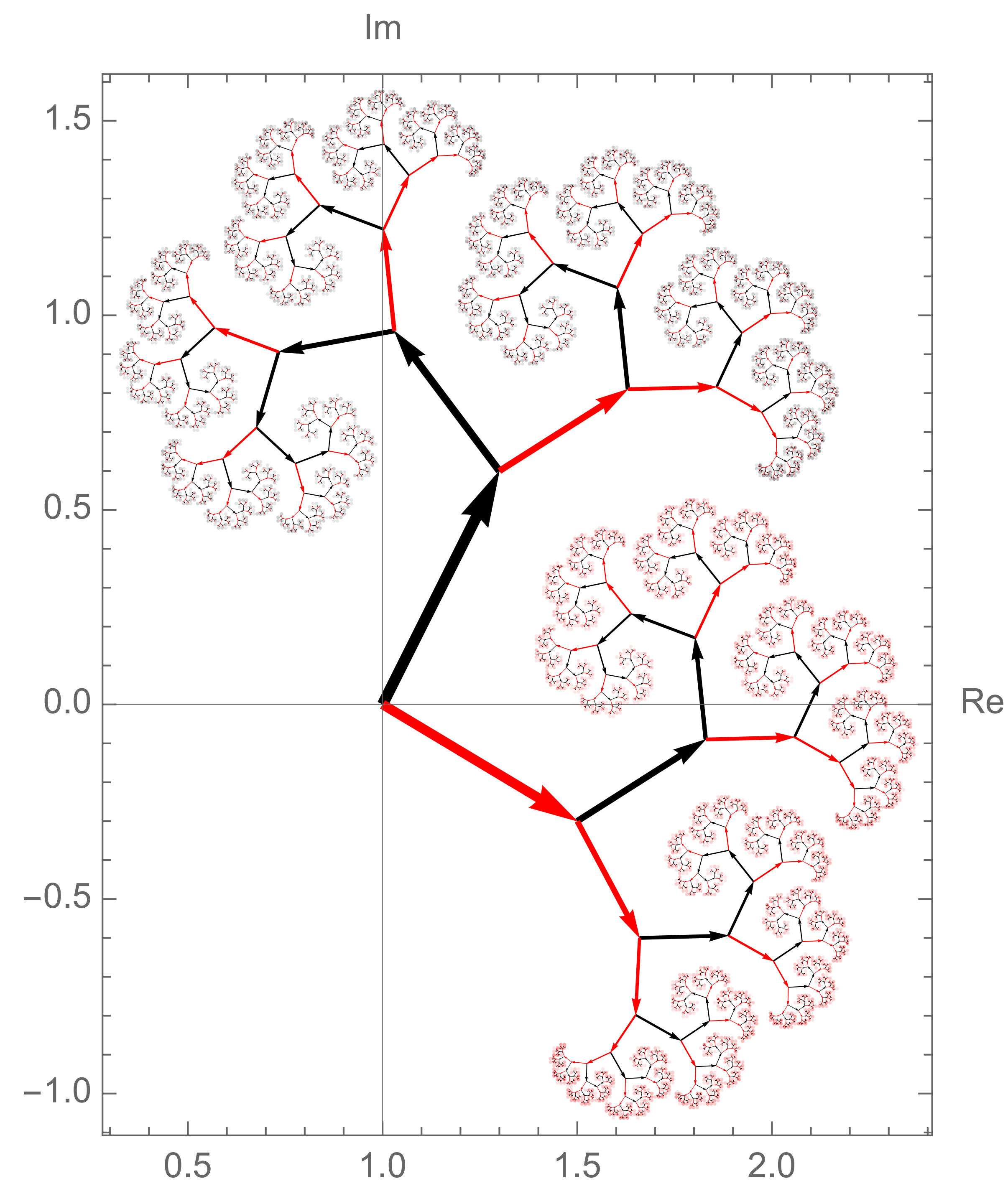
Bill Thurston

(1946 - 2012)



Geometric series ⟶ Complex tree





$T_A = T\{z_1\} \longrightarrow T_A = T\{c_1, c_2\}$

n – ary complex tree $T\{c_1, c_2, \ldots, c_n\}$ with $0 < |c_1|, |c_2|, \ldots, |c_n| < 1$

complex-valued alphabet $A = \{c_1, c_2\}$

letters $w_1, w_2, \ldots, w_m \in A = \{1 \to c_1, 2 \to c_2\}$

word $w = w_1 w_2 \ldots w_m$

$$\phi(w) = 1 + w_1 + w_1 w_2 + \ldots + w_1 w_2 \ldots w_m$$

$$\phi(1) = 1 + c_1$$

$$\phi(11) = 1 + c_1 + c_1^2$$

$$\phi(112) = 1 + c_1 + c_1^2 + c_1^2 c_2$$

nodes

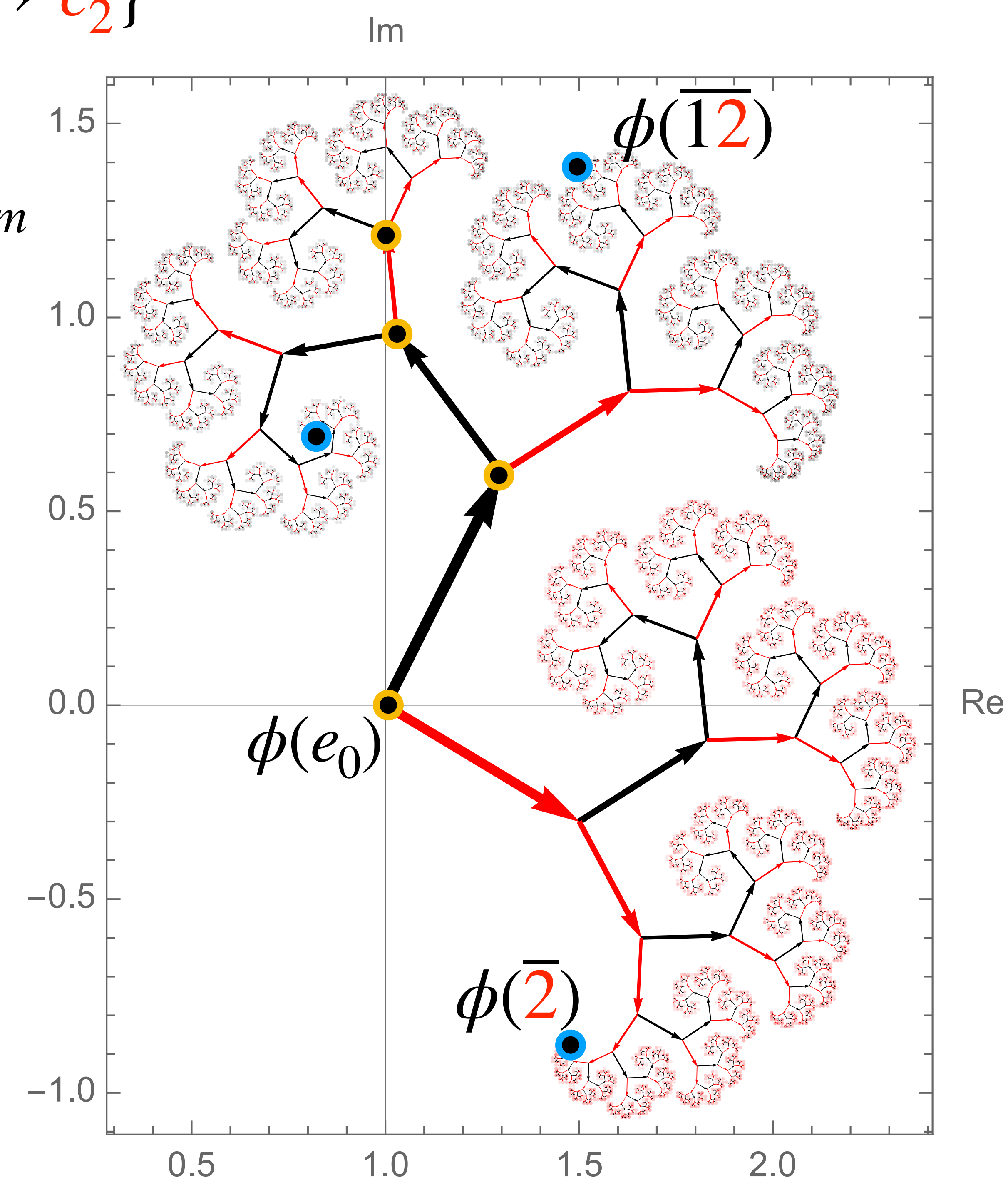
empty string e_0 **root** $\phi(e_0) = 1$

$$w = w_1 w_2 w_3 \ldots w_k \ldots \in \{c_1, c_2\}^\infty = A^\infty$$

tip points $\phi(w)$

$$\phi(111\ldots) = \phi(\overline{1}) = \sum_{k=0}^{\infty} c_1^k = \frac{1}{1 - c_1}$$

$$\phi(121212\ldots) = \phi(\overline{12}) = \frac{1 + c_1}{1 - c_1 c_2}$$

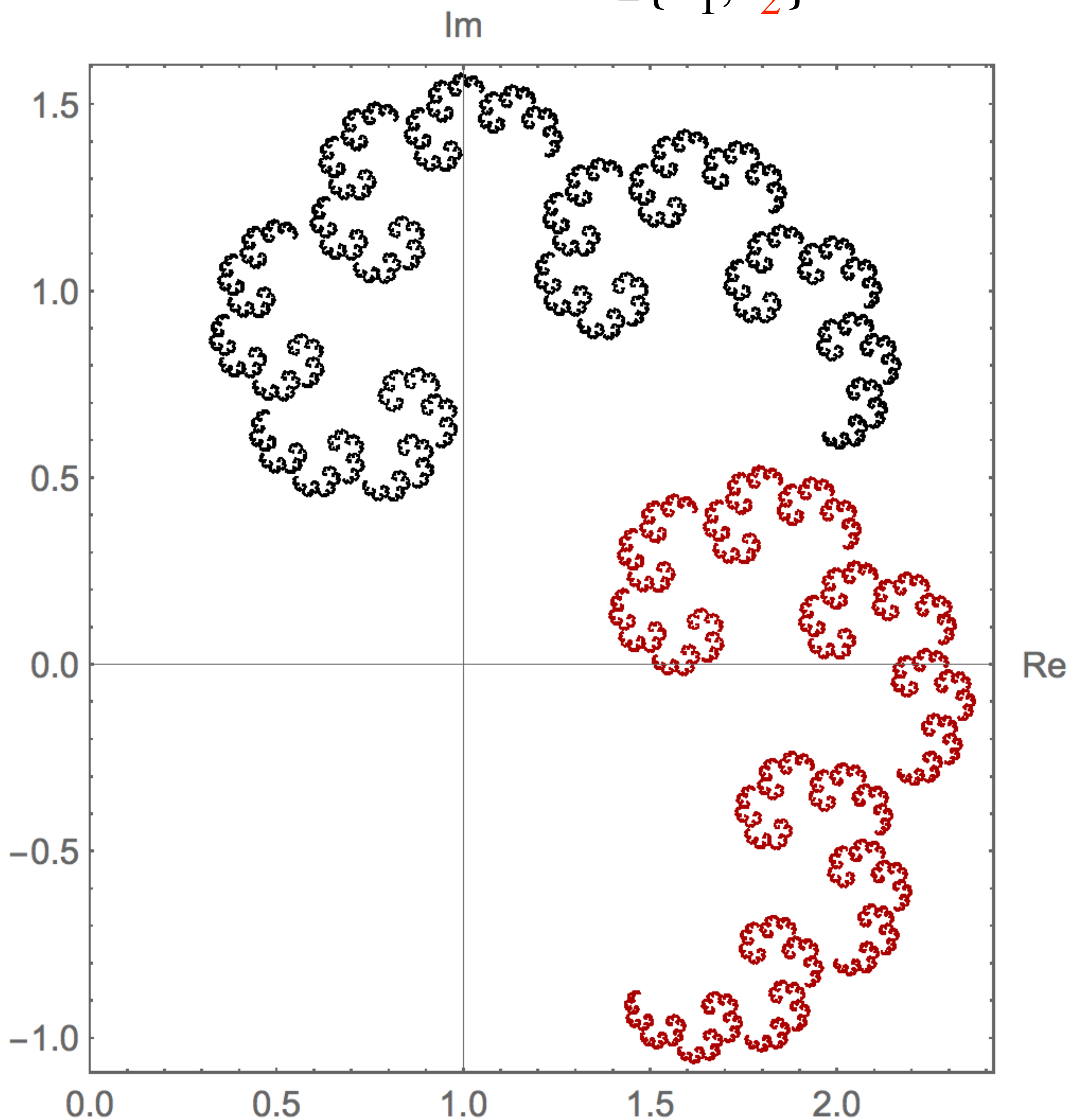


complex tree

$$T_A = T\{c_1, c_2\} = T\{.3 + i.6,\ .5 - i.3\}$$

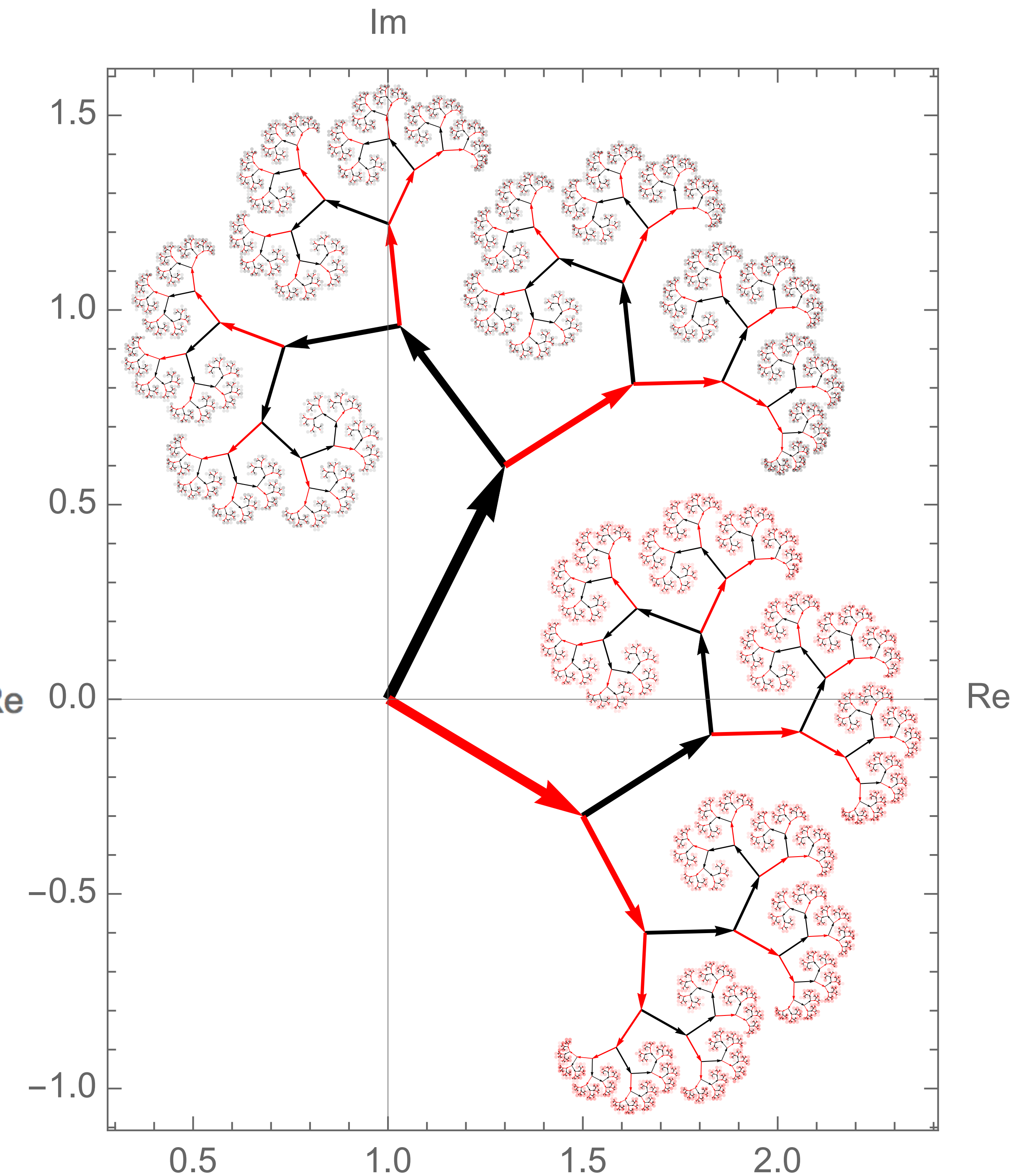
$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \phi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$



tipset

$F\{.3 + i.6,\ .5 - i.3\}$



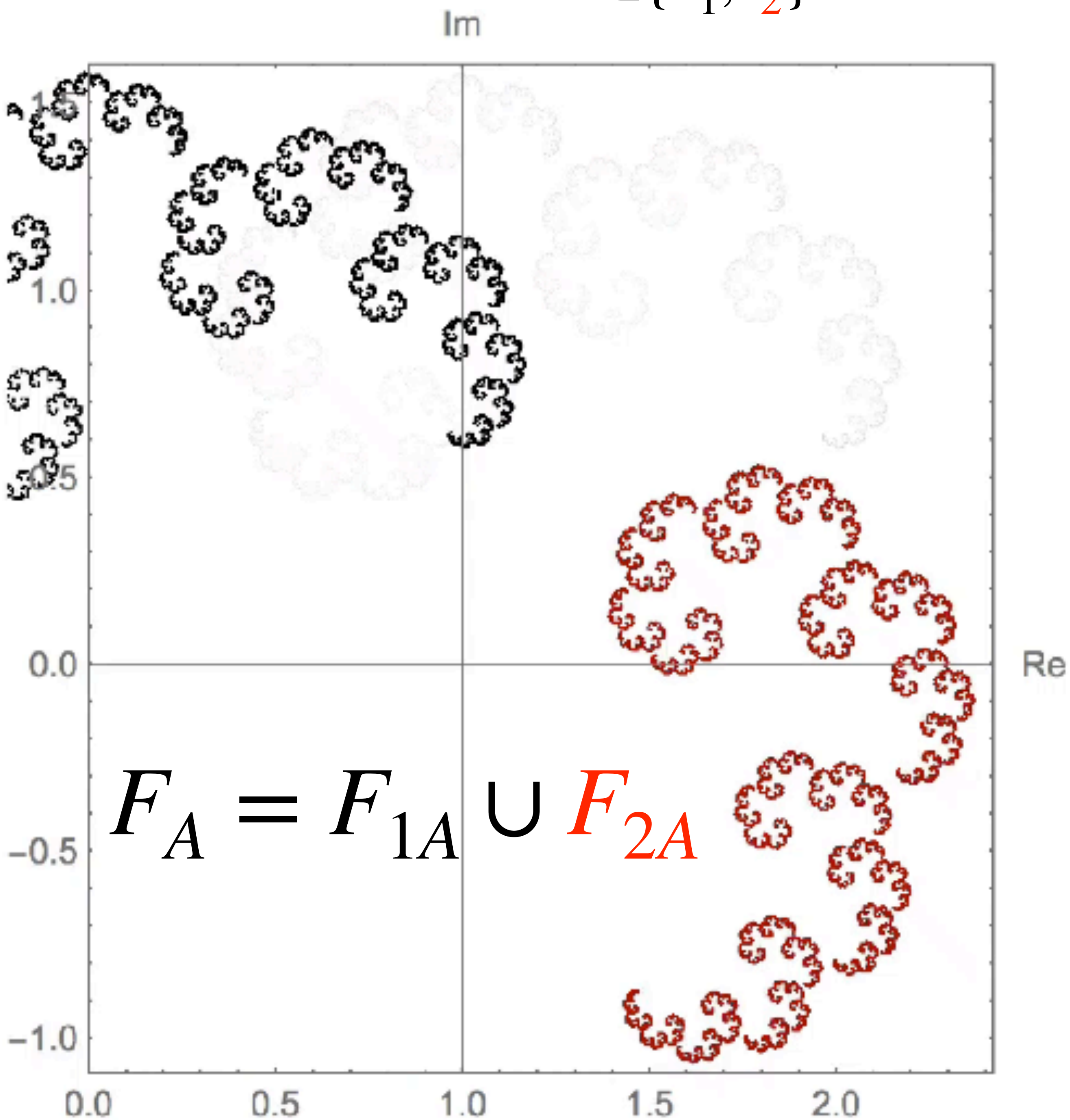
complex tree

$T\{.3 + i.6,\ .5 - i.3\}$

$$F_A = F\{c_1, c_2\} := \bigcup_{w \in \{c_1, c_2\}^\infty} \phi(w)$$

$$\{c_1, c_2\} = \{.3 + i.6,\ .5 - i.3\}$$

$$F_A = f_1(F_A) \cup f_2(F_A)$$



$$F_A = F_{1A} \cup F_{2A}$$

First-level pieces

$$F_{1A} := f_1(F_A) = 1 + c_1 \cdot F_A$$

$$F_{2A} := f_2(F_A) = 1 + c_2 \cdot F_A$$

$$r_2 = |c_2| \approx 0.58$$

$$\theta_2 = \arg(c_2) \approx -31^\circ$$

$$r_1 = |c_1| \approx 0.67$$

$$\theta_1 = \arg(c_1) \approx 63^\circ$$

The tipset is a self-similar set

$$F_{1A} \cap F_{2A} = \varnothing$$

Key idea: “tip-to-tip gluing”

$$\phi(13\bar{2}) = 1 + c_1 + c_1 c_3 \sum_{k=0}^{\infty} c_2^k = 1 + c_1 + \frac{c_1 c_3}{1 - c_2}$$

$$\phi(21\bar{2}) = 1 + c_2 + \frac{c_2 c_1}{1 - c_2}$$

$$\phi(23\bar{2}) = 1 + c_2 + \frac{c_2 c_3}{1 - c_2}$$

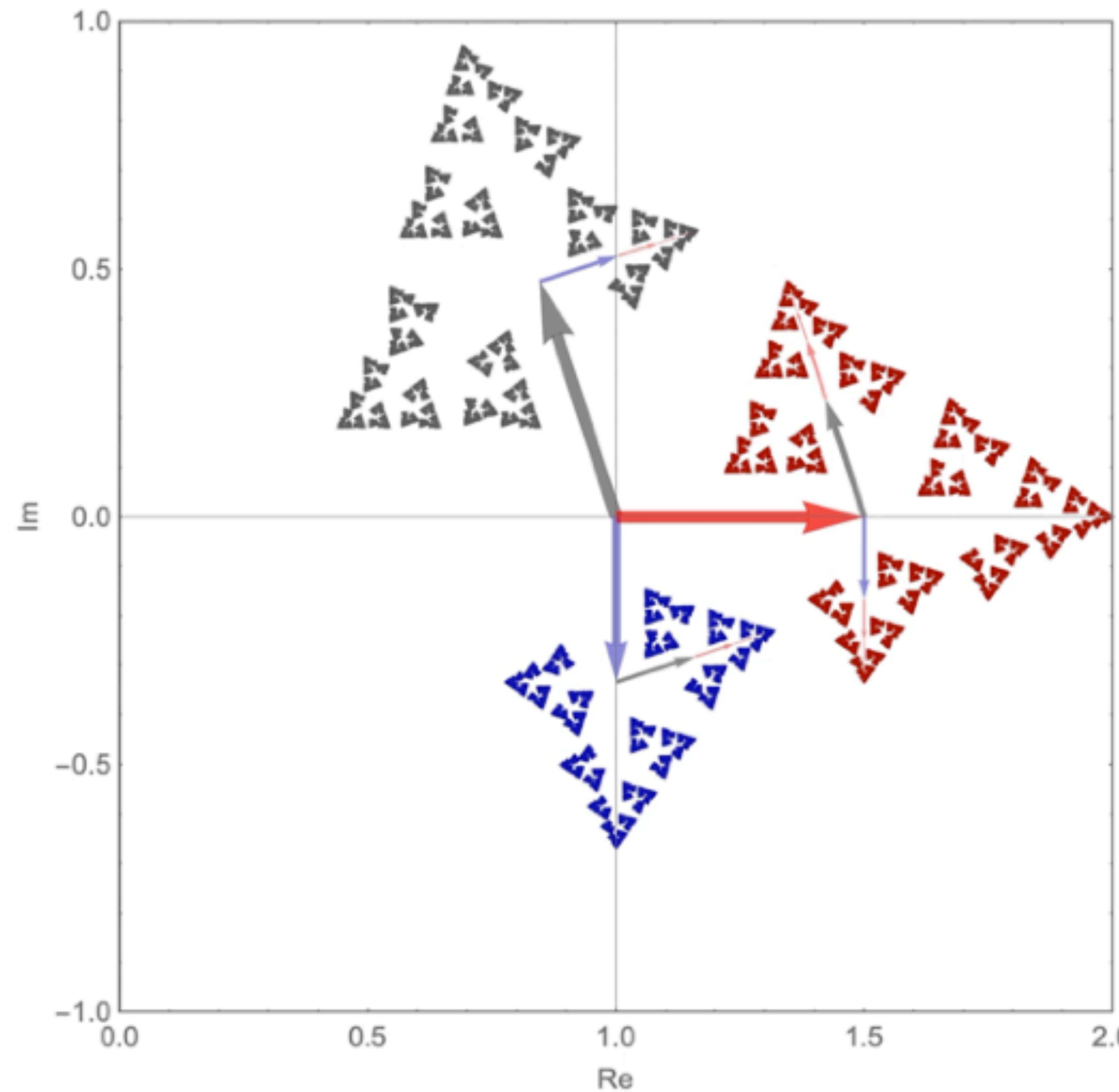
$$\phi(31\bar{2}) = 1 + c_3 + \frac{c_3 c_1}{1 - c_2}$$

$$\begin{cases} \phi(13\bar{2}) = \phi(21\bar{2}) \\ \phi(23\bar{2}) = \phi(31\bar{2}) \end{cases} \Longrightarrow \begin{cases} c_2 = 1/2 \\ c_3 = 1/4c_1 \end{cases}$$

$$c_1(z) := z, \quad c_2(z) := 1/2, \quad c_3(z) := 1/4z$$

$$T_A(z) = T\{c_1(z), c_2(z), c_3(z)\} = T\{z, 1/2, 1/4z\}$$

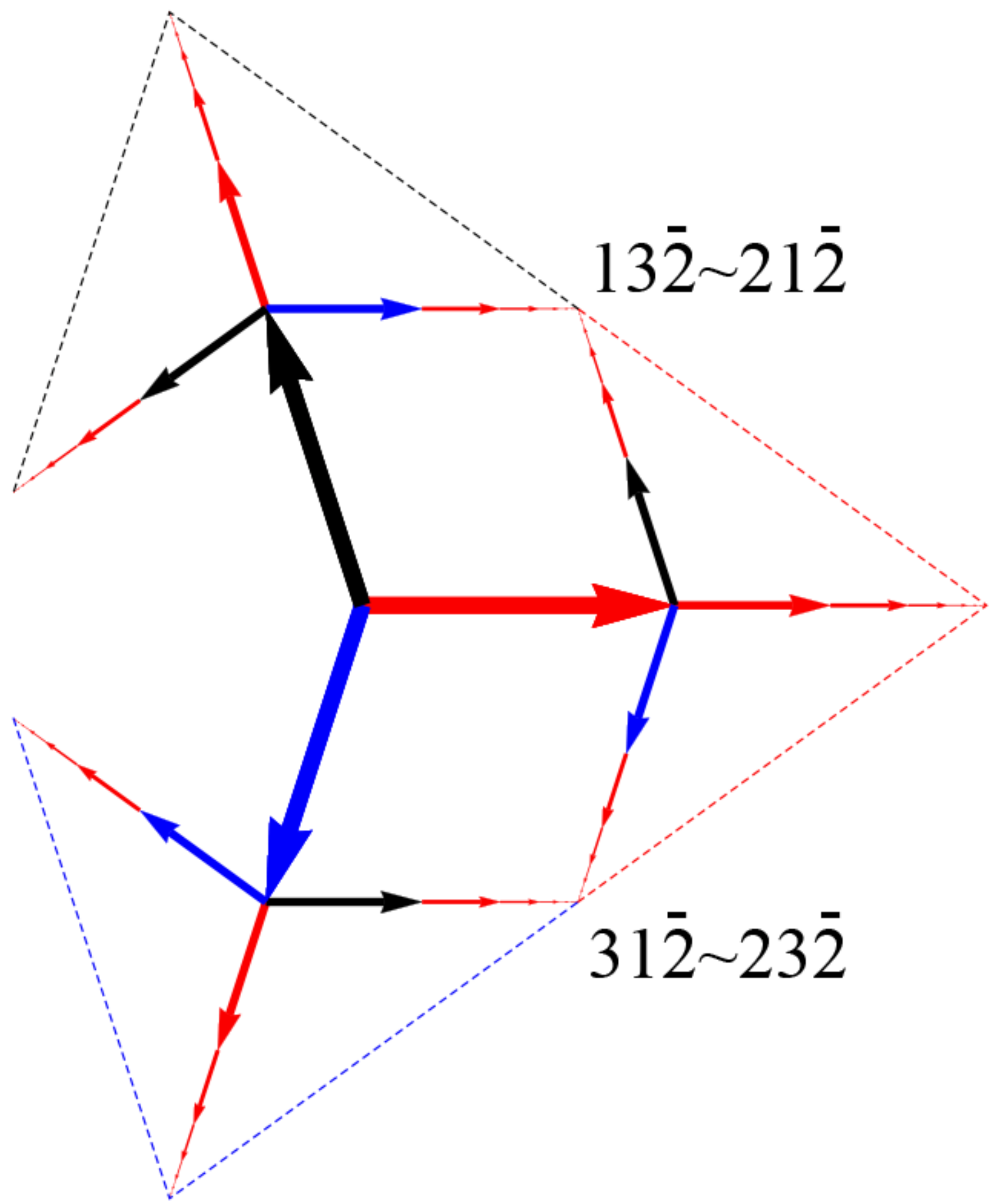
$$T_A(i^{6/5}/2) = T\{\frac{i^{6/5}}{2}, \frac{1}{2}, -\frac{i^{4/5}}{2}\}$$

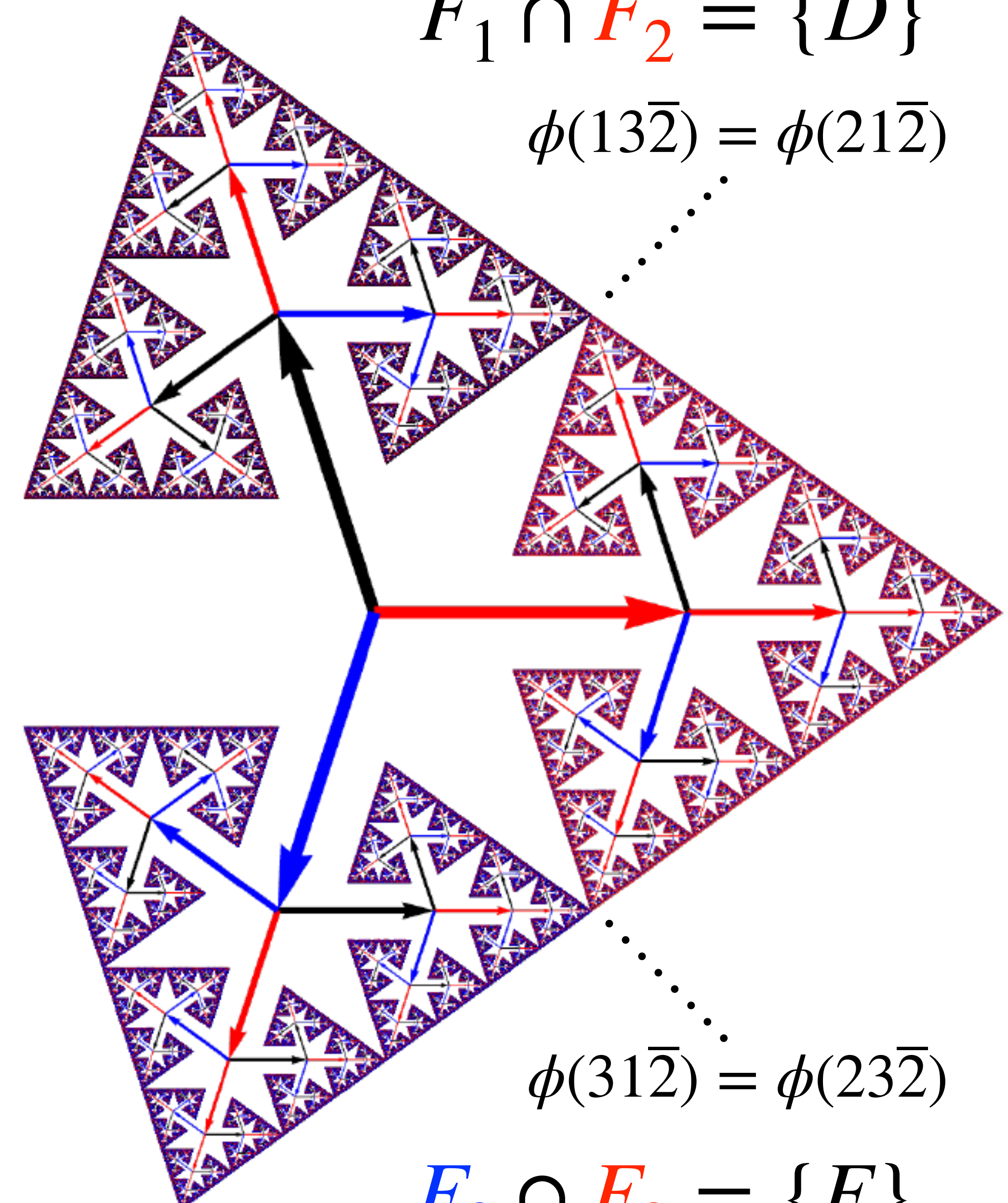


Topological set

$$Q_A(i^{6/5}/2) = \{13\overline{2} \sim 21\overline{2},\ 31\overline{2} \sim 23\overline{2}\}$$

$$T_A(i^{6/5}/2) = T\{\frac{i^{6/5}}{2}, \frac{1}{2}, -\frac{i^{4/5}}{2}\}$$



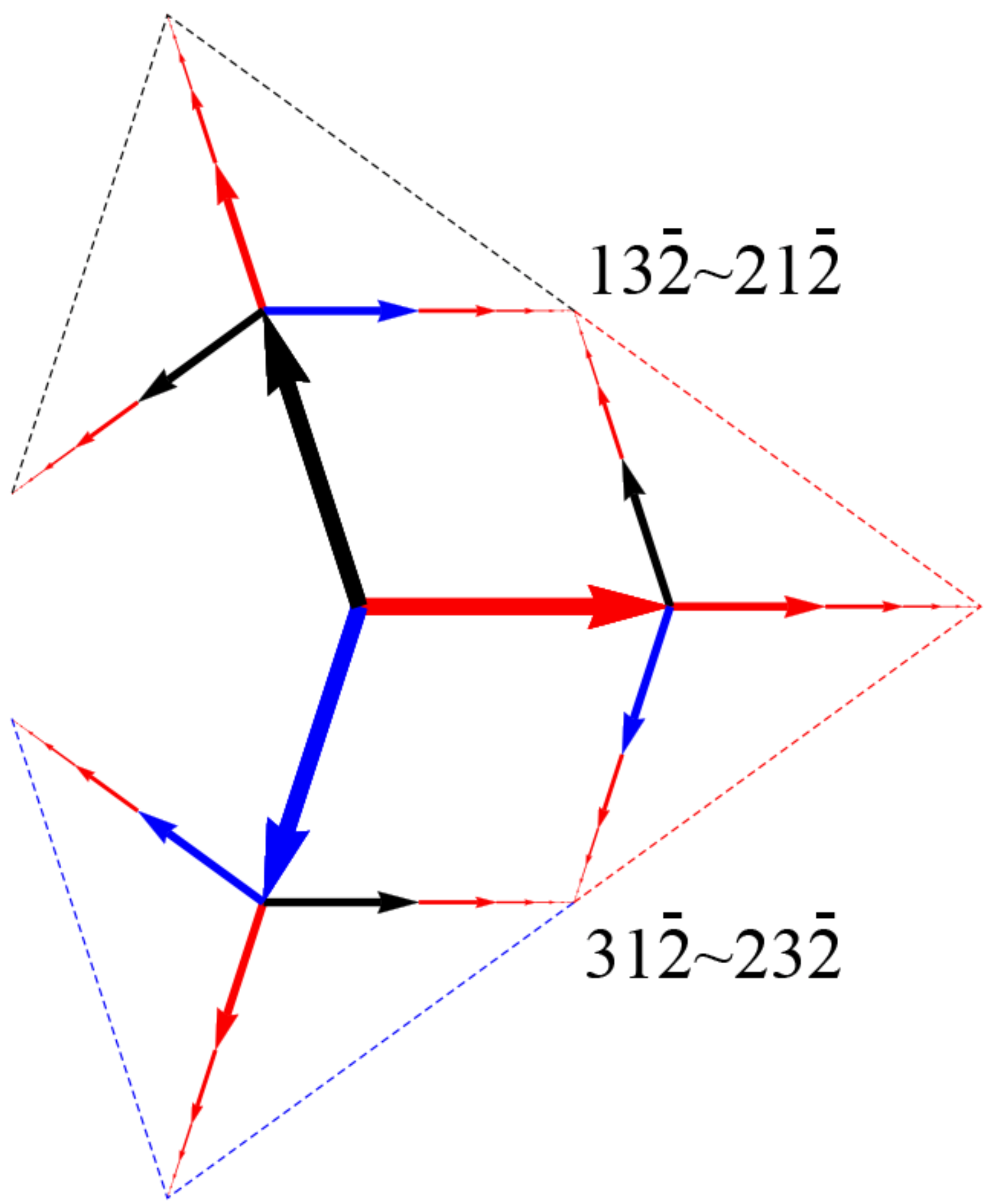


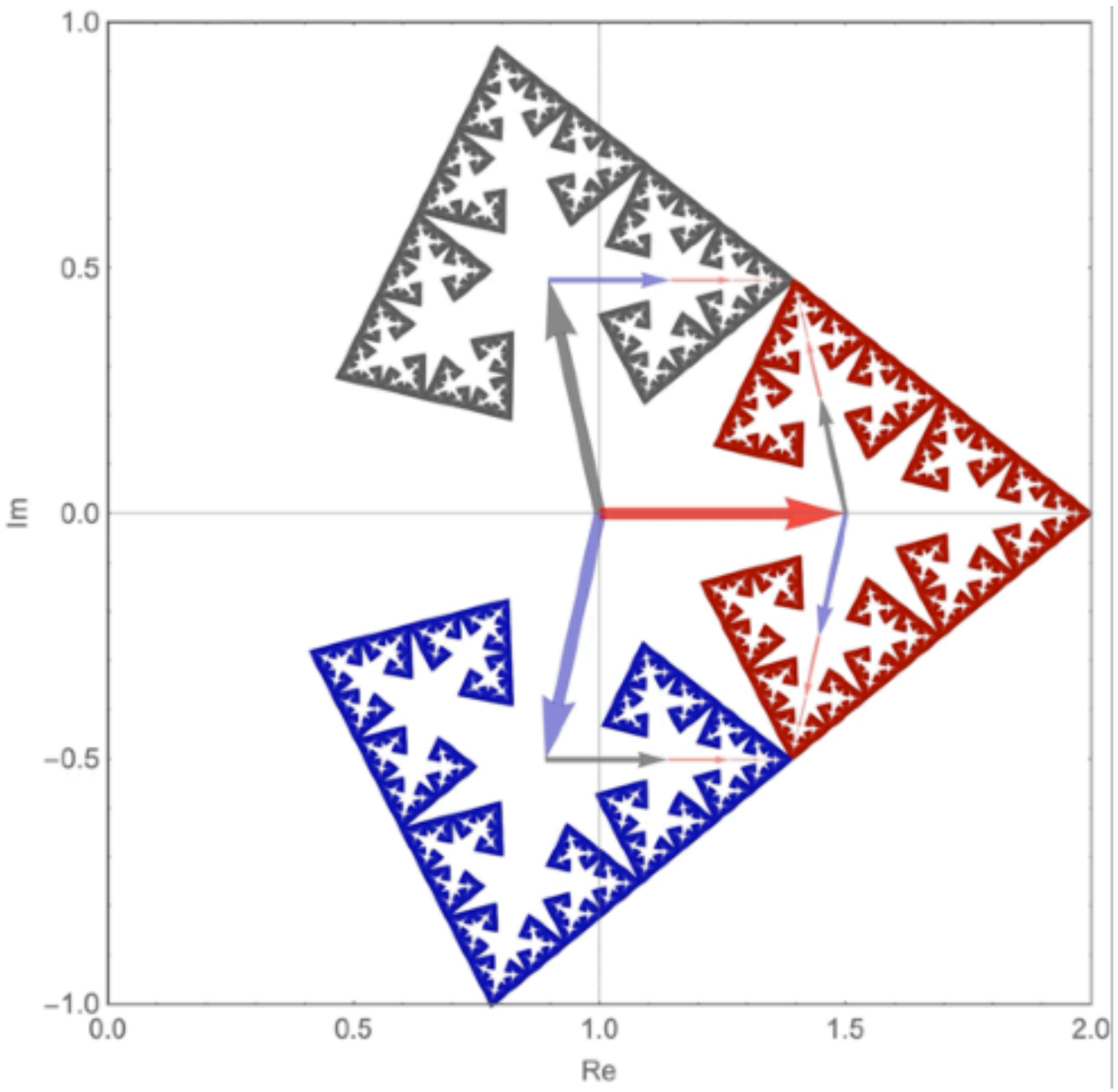
Topological set

Structurally stable tree

$$Q_A(i^{6/5}/2+\epsilon) = \{13\overline{2} \sim 21\overline{2},\ 31\overline{2} \sim 23\overline{2}\}$$

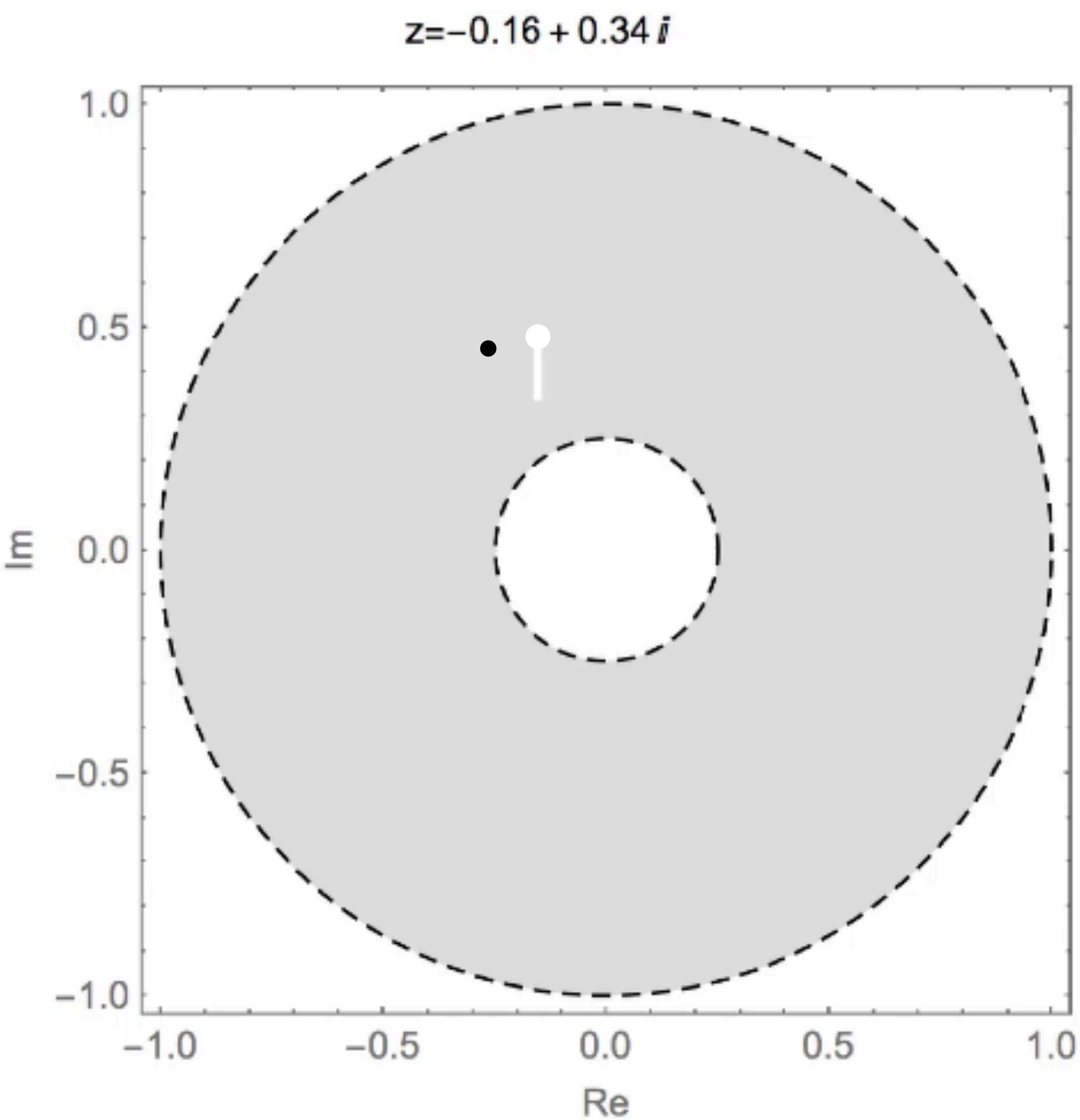
$$T_A(i^{6/5}/2+\epsilon) = T\{i^{6/5}/2+\epsilon, \frac{1}{2}, c_3(i^{6/5}/2+\epsilon)\}$$

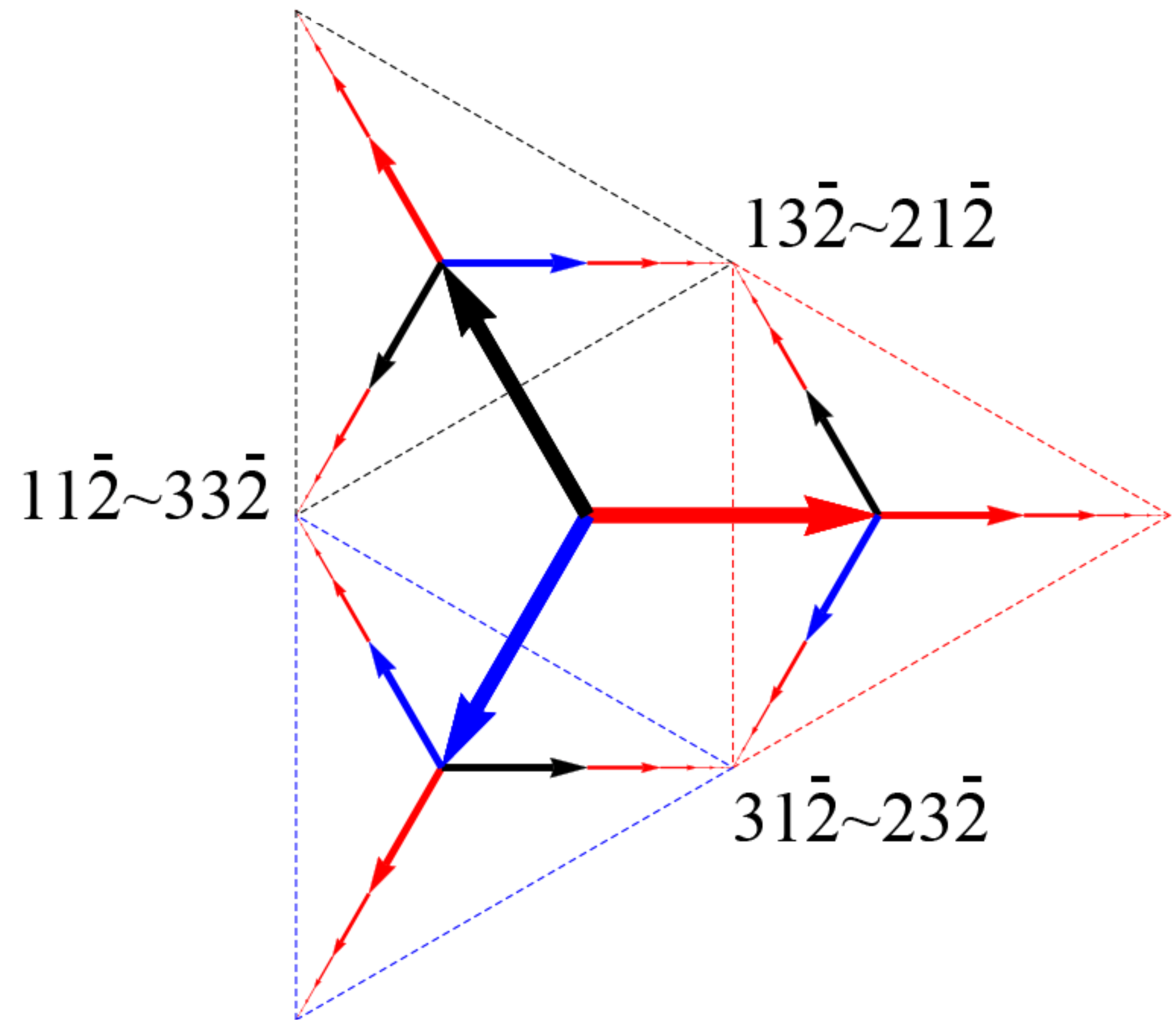




A family of tipset connected trees $T_A(z) := T\{z, \frac{1}{2}, \frac{1}{4z}\}$

$$Q_A(-1/4 + i\sqrt{3}/4) = \{13\bar{2} \sim 21\bar{2},\ 31\bar{2} \sim 23\bar{2},\ \underline{11\bar{2} \sim 33\bar{2}}\}$$





$\mathscr{R} := \{z \in \mathbb{C} : 1/4 < |z| < 1\}$