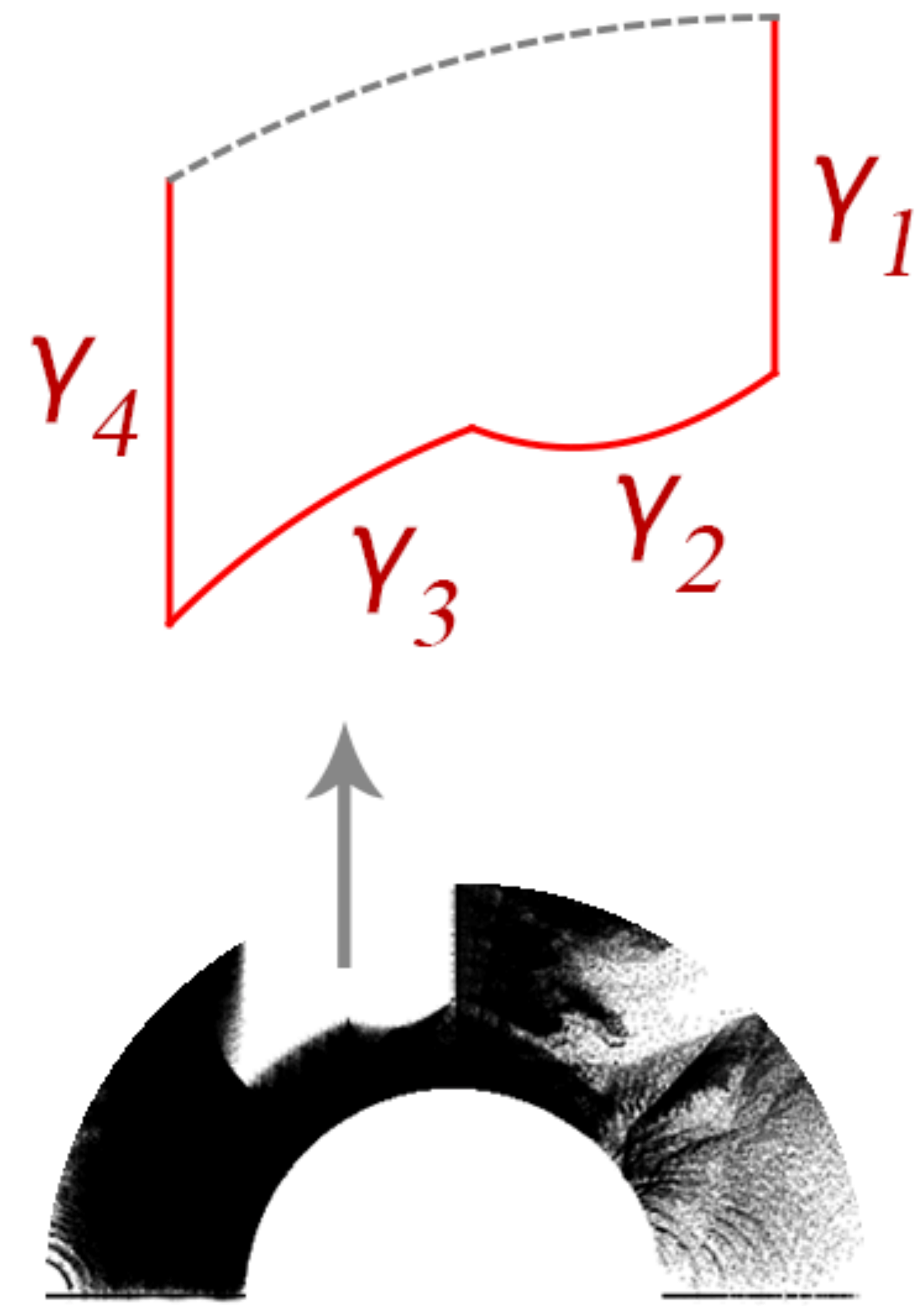
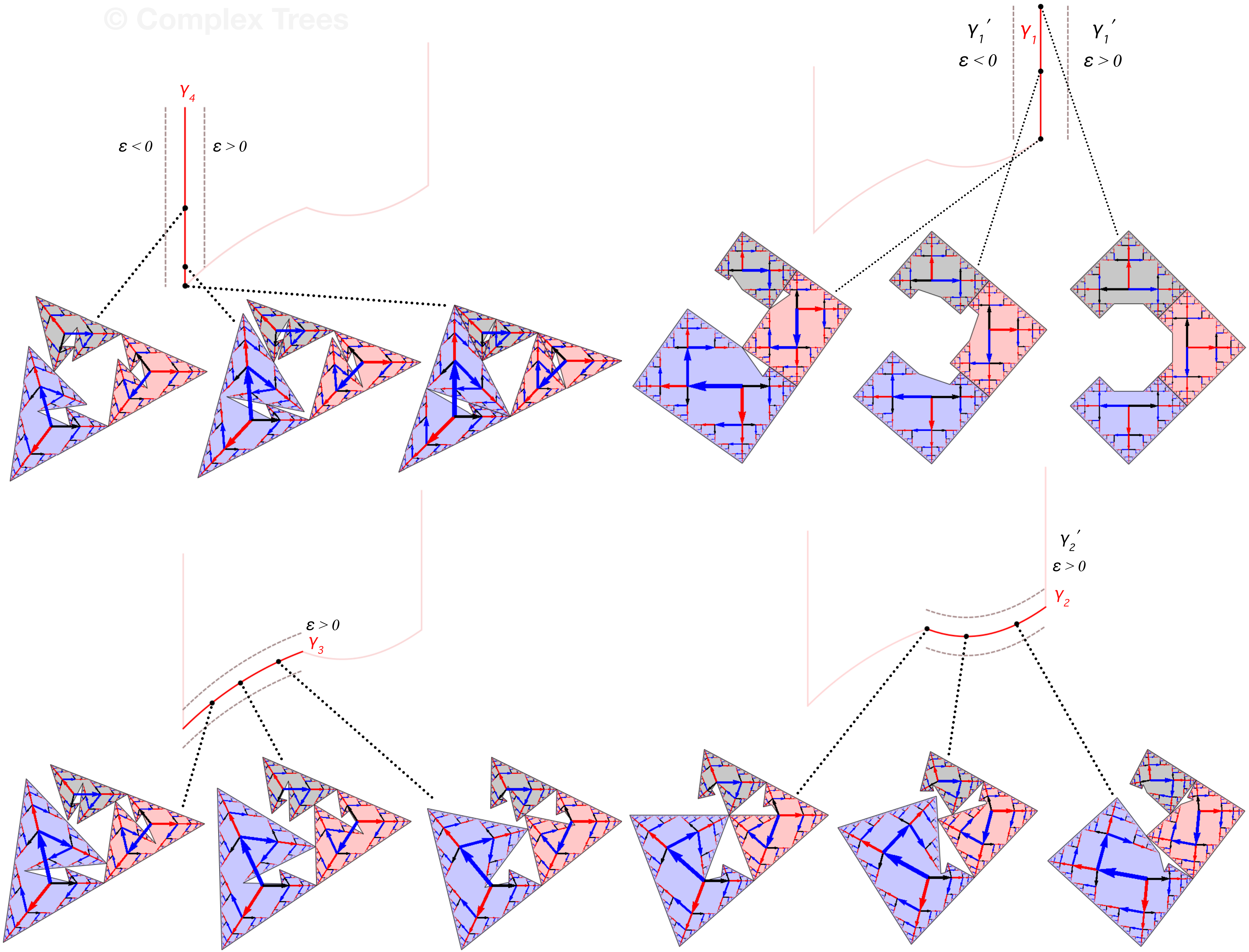
$$K \cup K^* \subseteq \{z \in \mathscr{R} : Q_A(z) = \{13\overline{2} \sim 21\overline{2},\ 31\overline{2} \sim 23\overline{2}\}\} \qquad T_A(z) := T\{z, \frac{1}{2}, \frac{1}{4z}\}$$

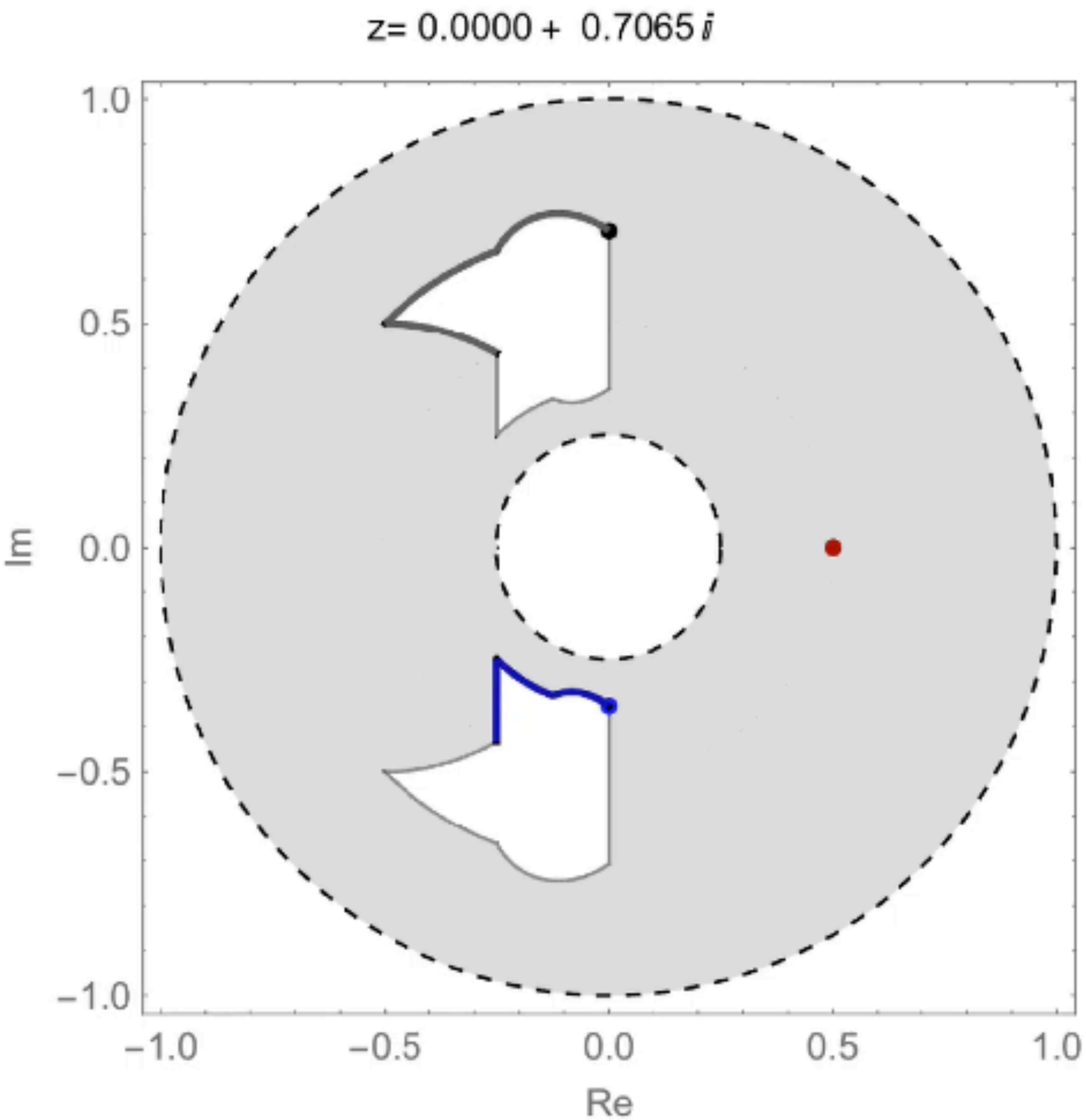
$$\gamma(t) := \begin{cases} -it - \frac{1}{4} & -\frac{\sqrt{3}}{4} \leq t \leq -\frac{1}{4} \\ t + \frac{i\sqrt{1-8t^2}}{2\sqrt{2}} & -\frac{1}{4} < t \leq -\frac{1}{8} \\ t + \frac{i\sqrt{24t^2+4t+1}}{2\sqrt{2}} & -\frac{1}{8} < t < 0 \\ i\left(t + \frac{1}{2\sqrt{2}}\right) & 0 \leq t \leq \frac{1}{4}\left(2-\sqrt{2}\right) \end{cases}$$

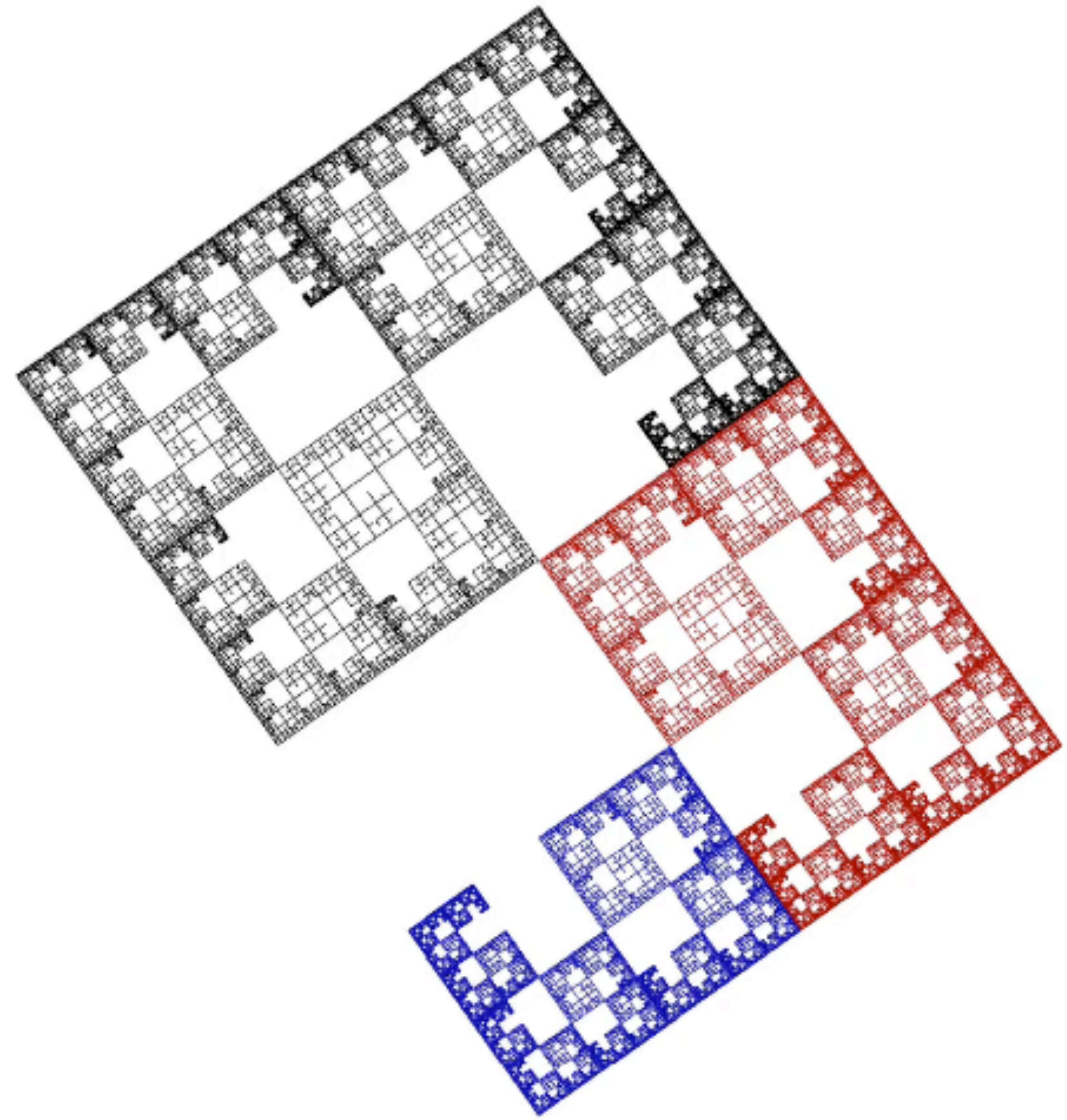



γ_1'
$\varepsilon < 0$
γ_1
γ_1'
$\varepsilon > 0$
γ_4
$\varepsilon < 0$
$\varepsilon > 0$
γ_2'
$\varepsilon > 0$
γ_2
$\varepsilon > 0$
γ_3

A family of tipset connected trees $T_A(z) := T\{z, \frac{1}{2}, \frac{1}{4z}\}$

$$Q_A(z) \supset \{13\overline{2} \sim 21\overline{2},\ 31\overline{2} \sim 23\overline{2}\}$$





$$\mathscr{R} := \{z \in \mathbb{C} : 1/4 < |z| < 1\}$$

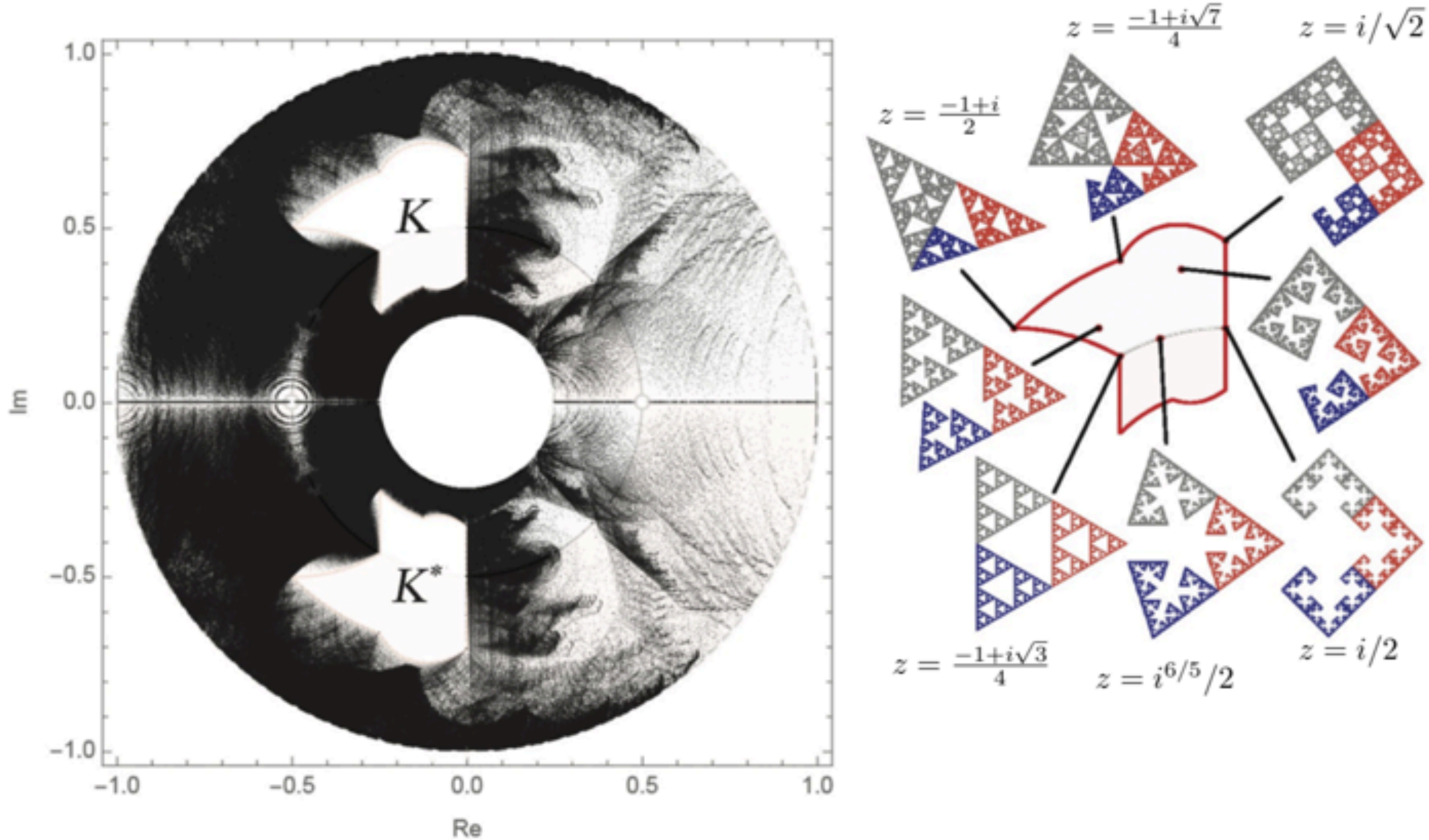


Figure 9: The unstable set $\mathcal{M}$ for $T\{z, 1/2, 1/4z\}$ with a pair of open regions K and K^* contained in the stable set $\mathcal{K}$. On the right, examples of tipsets $F_{\mathcal{A}}(z)$, five of which have their parameters z taken from the boundary of $\mathcal{M}$.

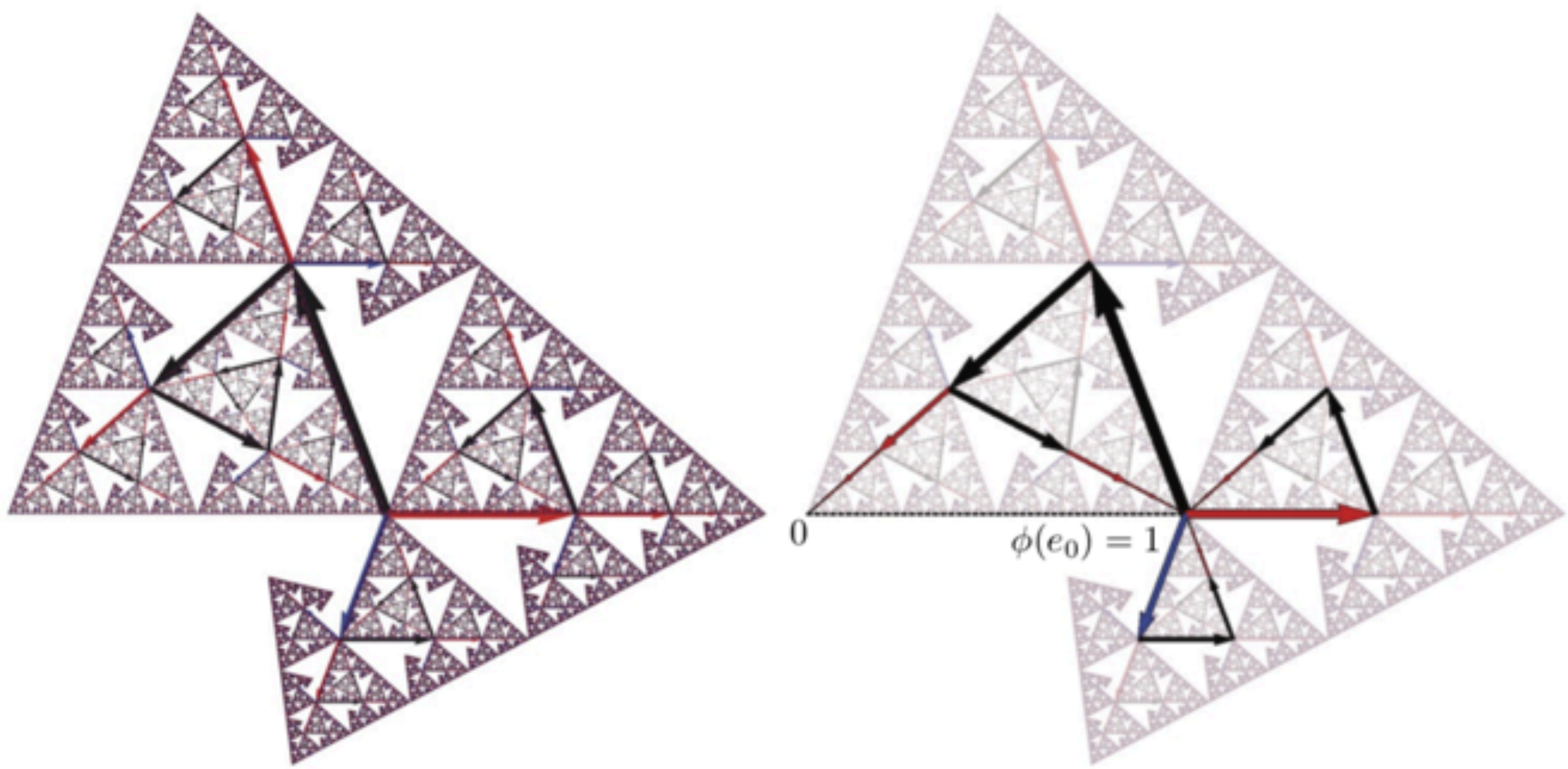


Figure 11: Root-connected ternary tree $T_{\mathcal{A}}(z_0) = T_{\mathcal{A}}((-1+i\sqrt{7})/4) = T\{(-1+i\sqrt{7})/4, 1/2, (-1-i\sqrt{7})/8\}$. Its intersection set is $I_{F_{\mathcal{A}}} = F_{1\mathcal{A}} \cap F_{2\mathcal{A}} \cap F_{2\mathcal{A}} = \{1 = \phi(e_0) = \phi(111\bar{2}) = \phi(211\bar{2}) = \phi(311\bar{2})\}$. The intersection set for this tree is a singleton but the overlap set is not, $I_{F_{\mathcal{A}}} \neq O_{F_{\mathcal{A}}}$.

Theorem 1. **(Tip-to-root connectivity).** *Let $j \in \mathcal{A}$, $w \in \mathcal{A}^\infty$, and $F_\mathcal{A}$ be a tipset of an n-ary complex tree $T_\mathcal{A}$, then the following are equivalent:*

(1) $\phi(jw) = \phi(e_0) = 1$

(2) $\phi(w) = 0$

(3) $\phi(jw) = \phi(kw)$ for any $k \in \mathcal{A}$

(4) $F_\mathcal{A}$ is root-connected

(5) $\phi(v) \in F_\mathcal{A}$ for all $v \in \mathcal{A}^*$

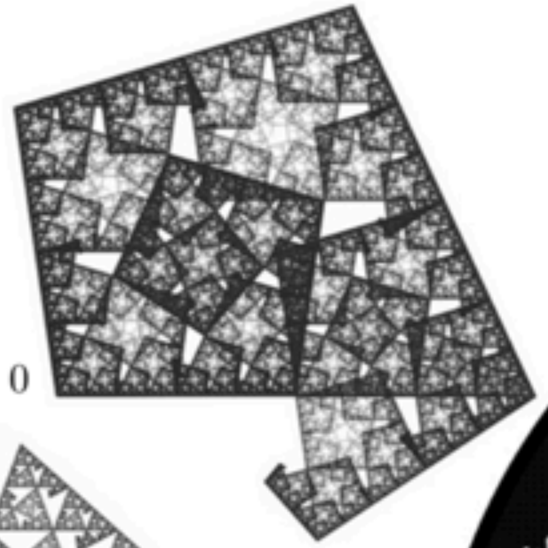
Proof. (1) $\implies$ (2): From (1) we have that $0 = \phi(jw) - \phi(e_0) = 1 + c_j\phi(w) - 1 = c_j\phi(w)$ which implies $\phi(w) = 0$ since $c_j \neq 0$ by definition. (2) $\implies$ (3): Independently of the first letter $k \in \mathcal{A}$ we have $\phi(kw) = 1 + c_k\phi(w) = 1 = \phi(e_0) = \phi(jw)$. (3) $\implies$ (4): For each first-level piece there is at least one tip point that meets the root $\phi(1w) = \phi(2w) = \cdots = \phi(nw) = 1$, hence $I_{F_\mathcal{A}} \neq \emptyset$ and $\phi(e_0) \in I_{F_\mathcal{A}}$ so $F_\mathcal{A}$ is root-connected. (4) $\implies$ (5): If $F_\mathcal{A}$ is root-connected we have that $\phi(e_0) \in F_\mathcal{A}$ hence by self-similarity $f_v(\phi(e_0)) = \phi(v) \in F_{v\mathcal{A}} \subseteq F_\mathcal{A}$, i.e. all the nodes of the tree $T_\mathcal{A}$ are contained in the tipset $F_\mathcal{A}$. (5) $\implies$ (1): We have that $\phi(e_0) \in F_\mathcal{A}$ hence $F_\mathcal{A}$ is root-connected and $\phi(e_0) \in F_j$ so there is at least a tip point with word $jw \in \mathcal{A}^\infty$ such that $\phi(jw) = \phi(e_0)$. $\square$

Theorem 2. **(Node-to-root connectivity).** *Let $v \in \mathcal{A}^* \setminus \{e_0\}$, if $\phi(v) = \phi(e_0)$ then $F_\mathcal{A}$ is root-connected.*

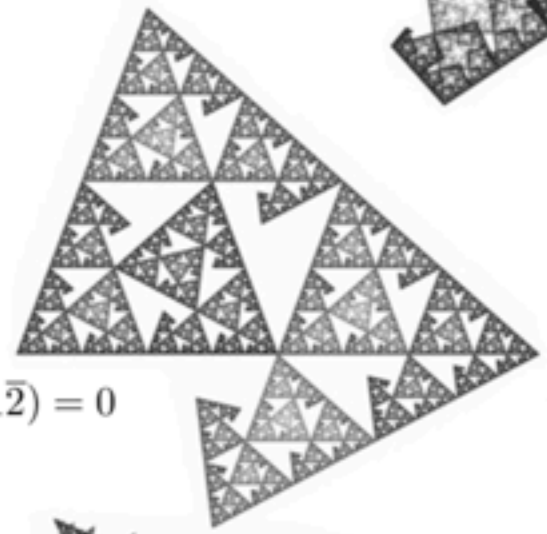
Proof. By applying equation (4) to the tip point of $\overline{v} \in \mathcal{A}^\infty$ we have that

$$\begin{aligned}
\phi(\overline{v}) &= \phi(v\overline{v}) \\
&= \phi(v) + v \cdot (\phi(\overline{v}) - 1) \\
&= \phi(e_0) + v \cdot (\phi(\overline{v}) - 1) \\
&= 1 + v \cdot (\phi(\overline{v}) - 1),
\end{aligned}$$

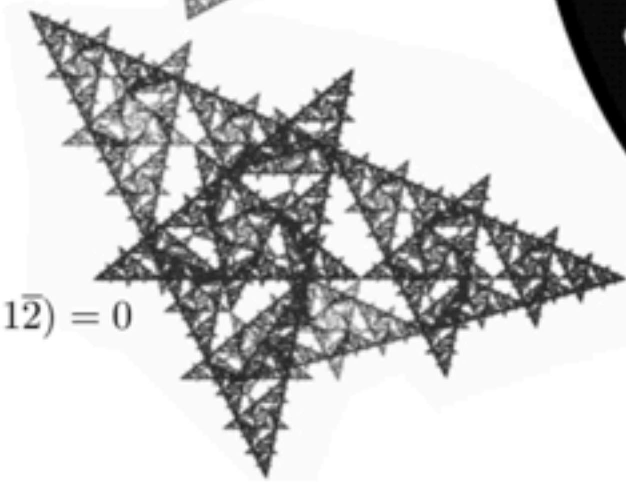
i.e. $\phi(\overline{v}) - 1 = v \cdot (\phi(\overline{v}) - 1)$. So $\phi(\overline{v}) - 1 = 0$ since the complex product $v = v_1 \cdot v_2 \cdot \ldots$ is neither 1 nor 0. Therefore, tip-to-root connectivity $\phi(\overline{v}) = 1 = \phi(e_0)$ takes place and by theorem 1 we have that the tipset $F_\mathcal{A}$ is root-connected. $\square$



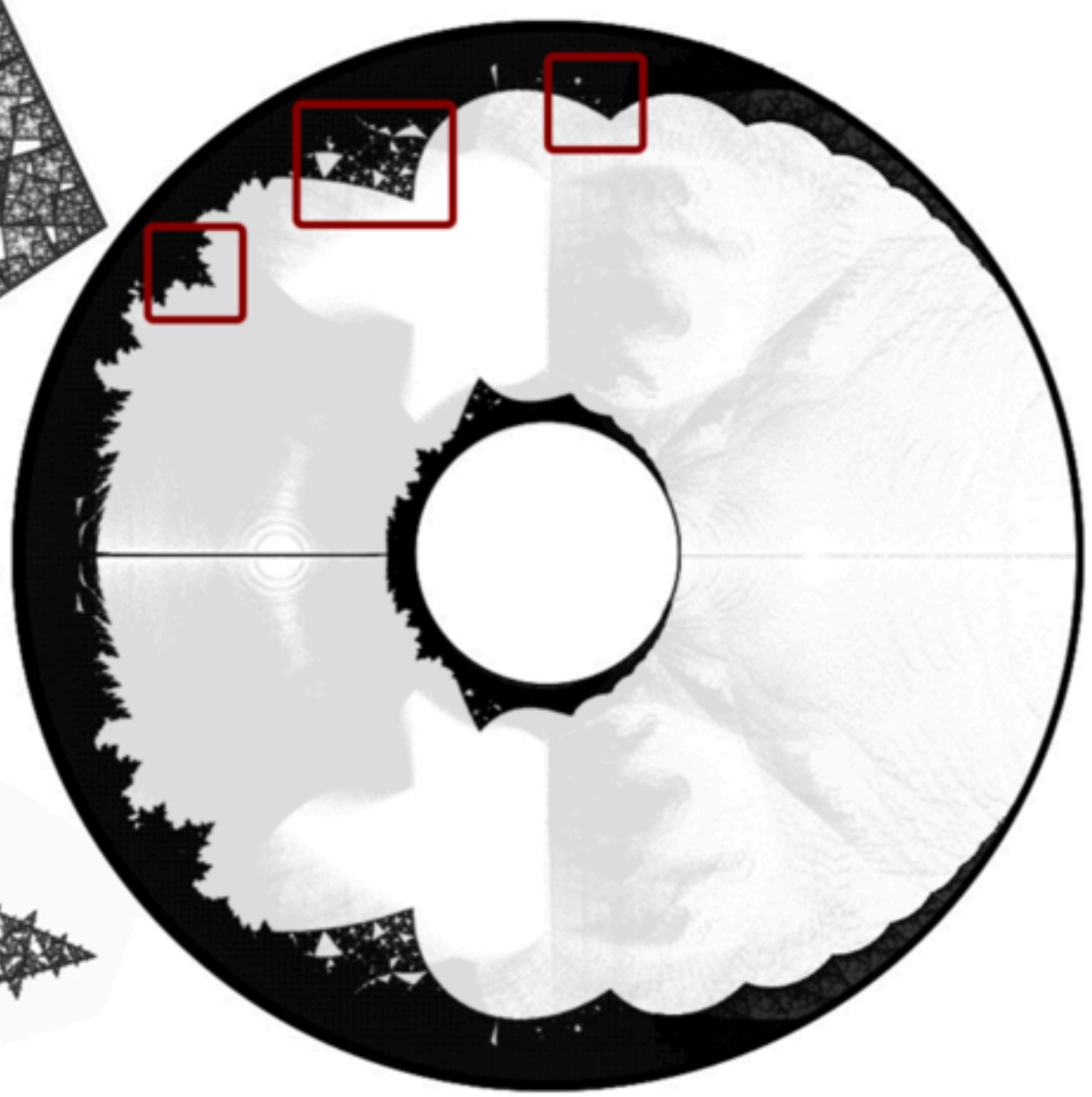
$\phi(111\bar{2}) = 0$



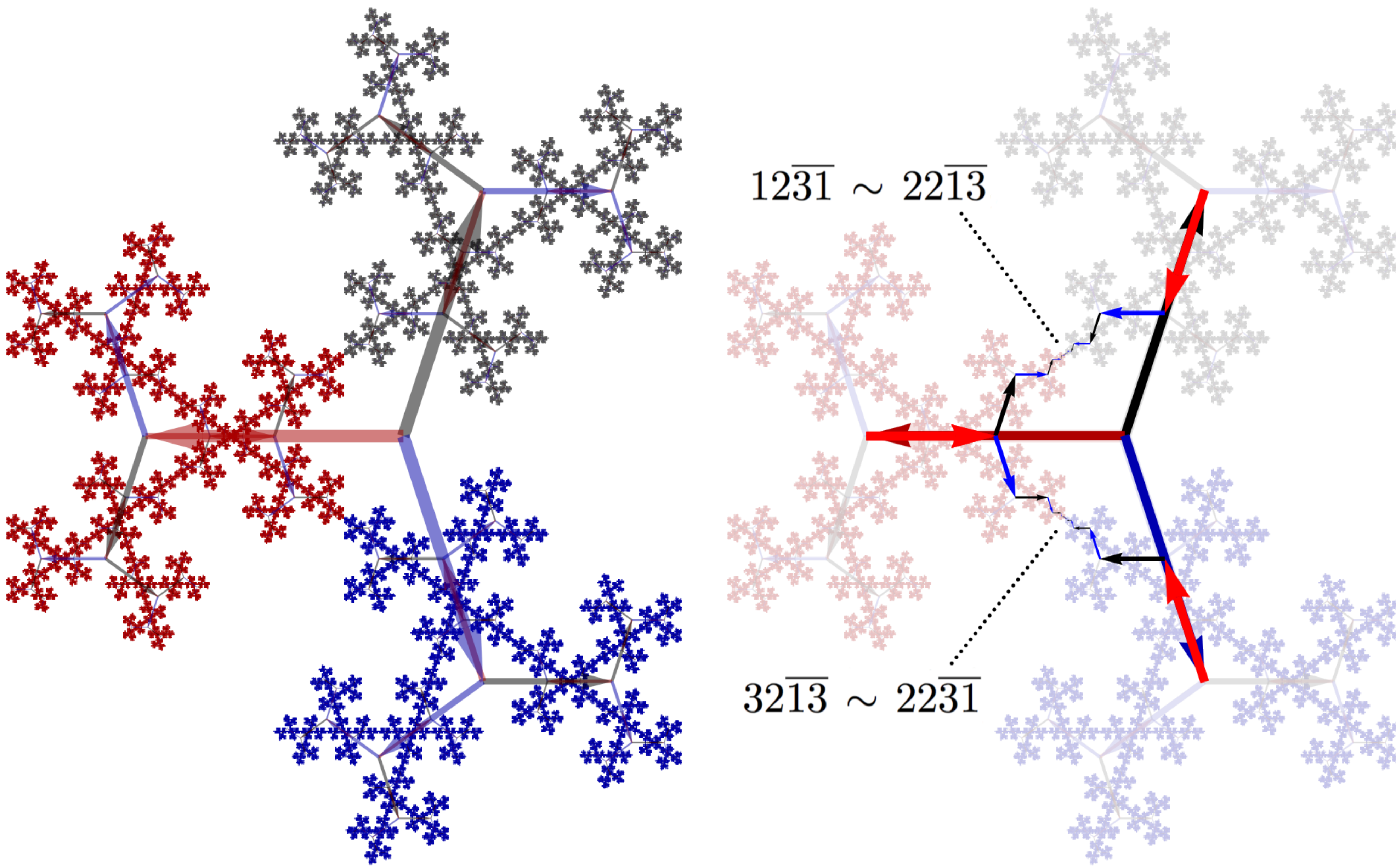
$\phi(11\bar{2}) = 0$



$\phi(1111\bar{2}) = 0$



A family of tipset connected trees $T_A(z) := T\{z, -\frac{1}{2}, \frac{1}{4z}\}$



$$\begin{cases} \phi(12\overline{31}) = \phi(22\overline{13}) \\ \phi(32\overline{13}) = \phi(22\overline{31}) \end{cases} = \begin{cases} c_1 + c_1c_2(\dfrac{1+c_3}{1-c_1c_3}) = c_2 + c_2^2(\dfrac{1+c_1}{1-c_3c_1}) \\ c_3 + c_3c_2(\dfrac{1+c_1}{1-c_1c_3}) = c_2 + c_2^2(\dfrac{1+c_3}{1-c_3c_1}) \end{cases} \Longrightarrow \begin{cases} c_2 = -1/2 \\ c_3 = 1/4c_1 \end{cases}$$

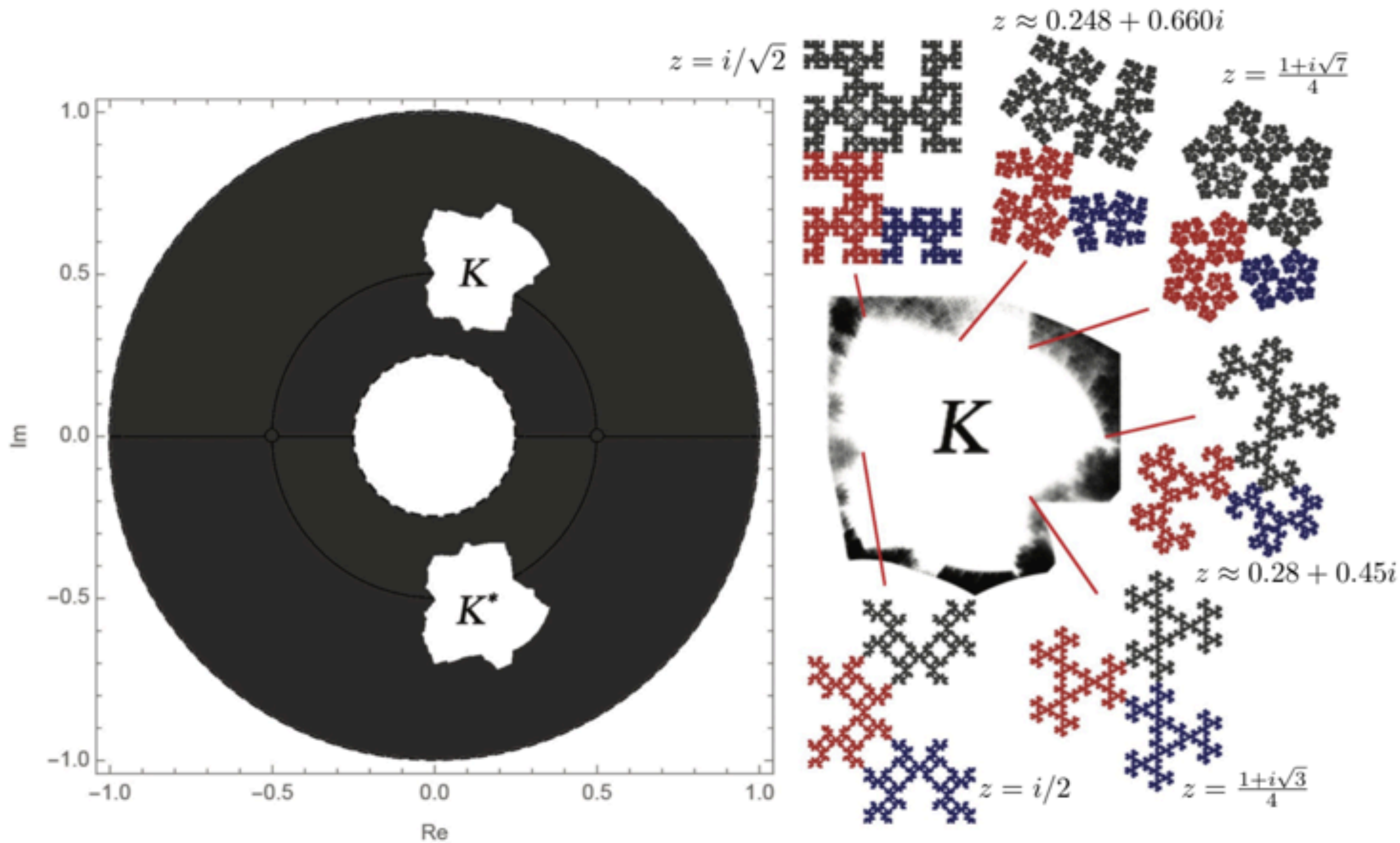
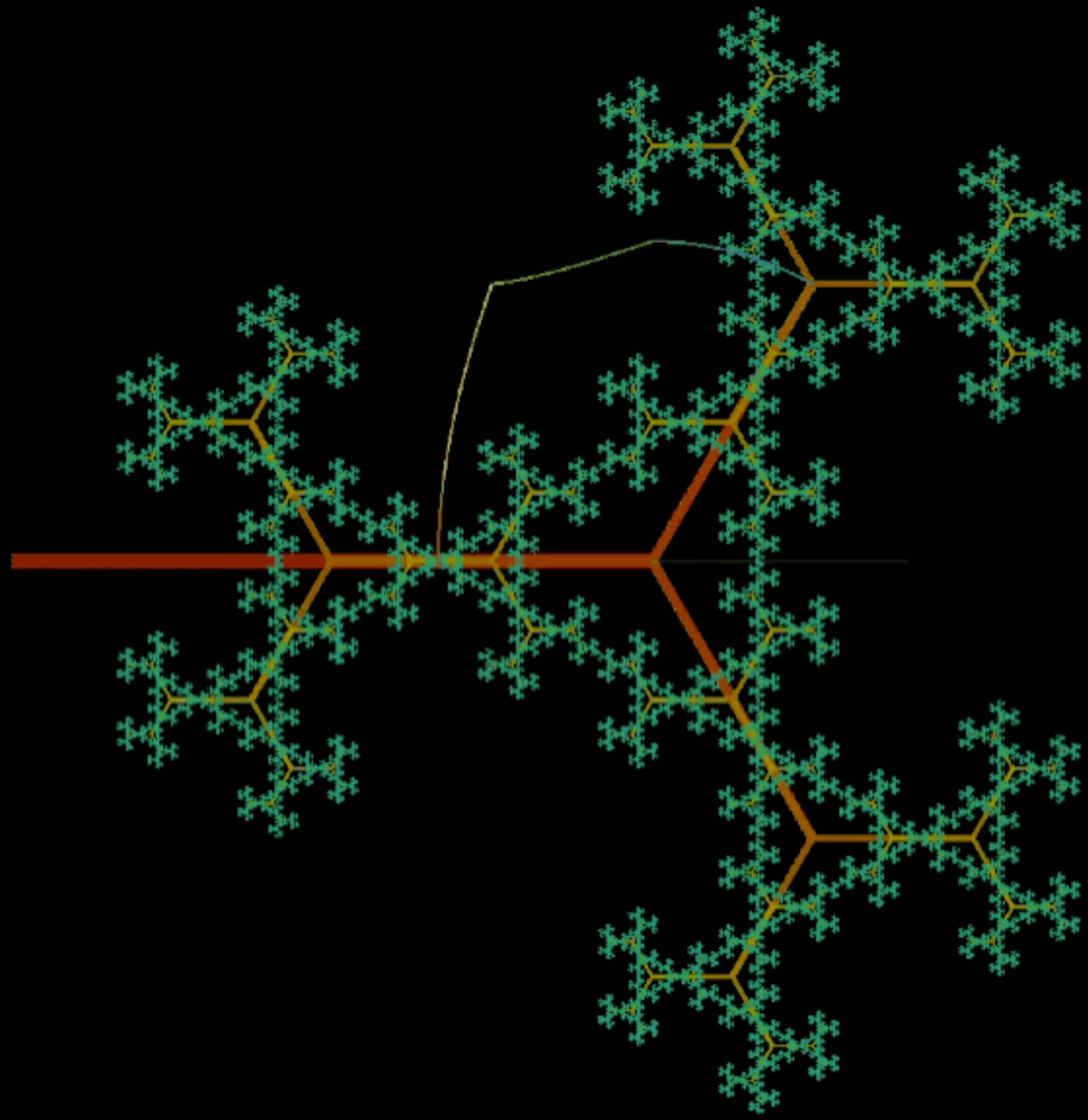
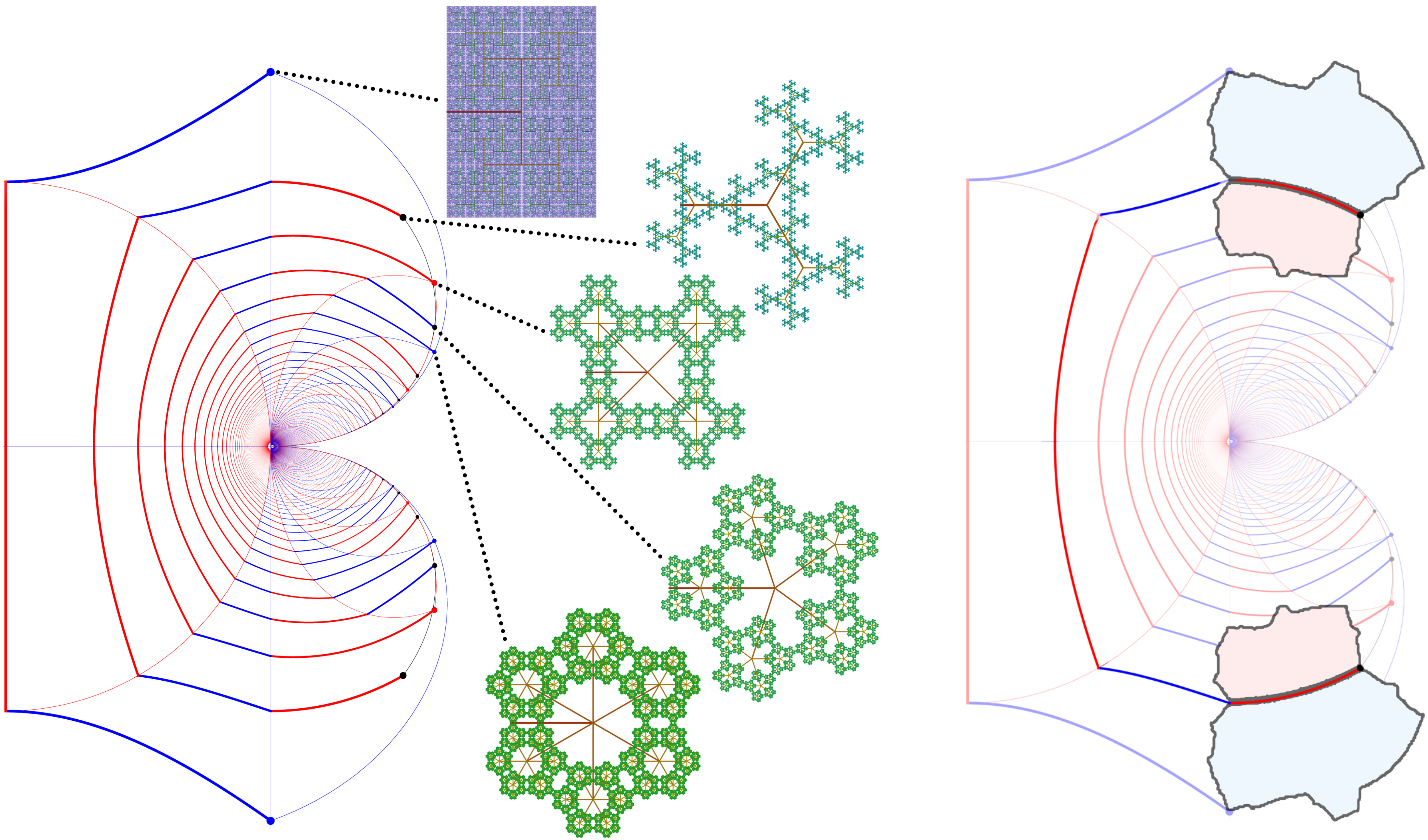


Figure 10: The unstable set $\mathcal{M}$ for $T\{z, -1/2, 1/4z\}$ with six examples of tipsets $F_{\mathcal{A}}(z)$ found in $\partial\mathcal{M}$.





Bernat Espigule. Generalized self-contacting symmetric fractal trees. *Journal Symmetry*, 24(1-4):320–338, 2013.

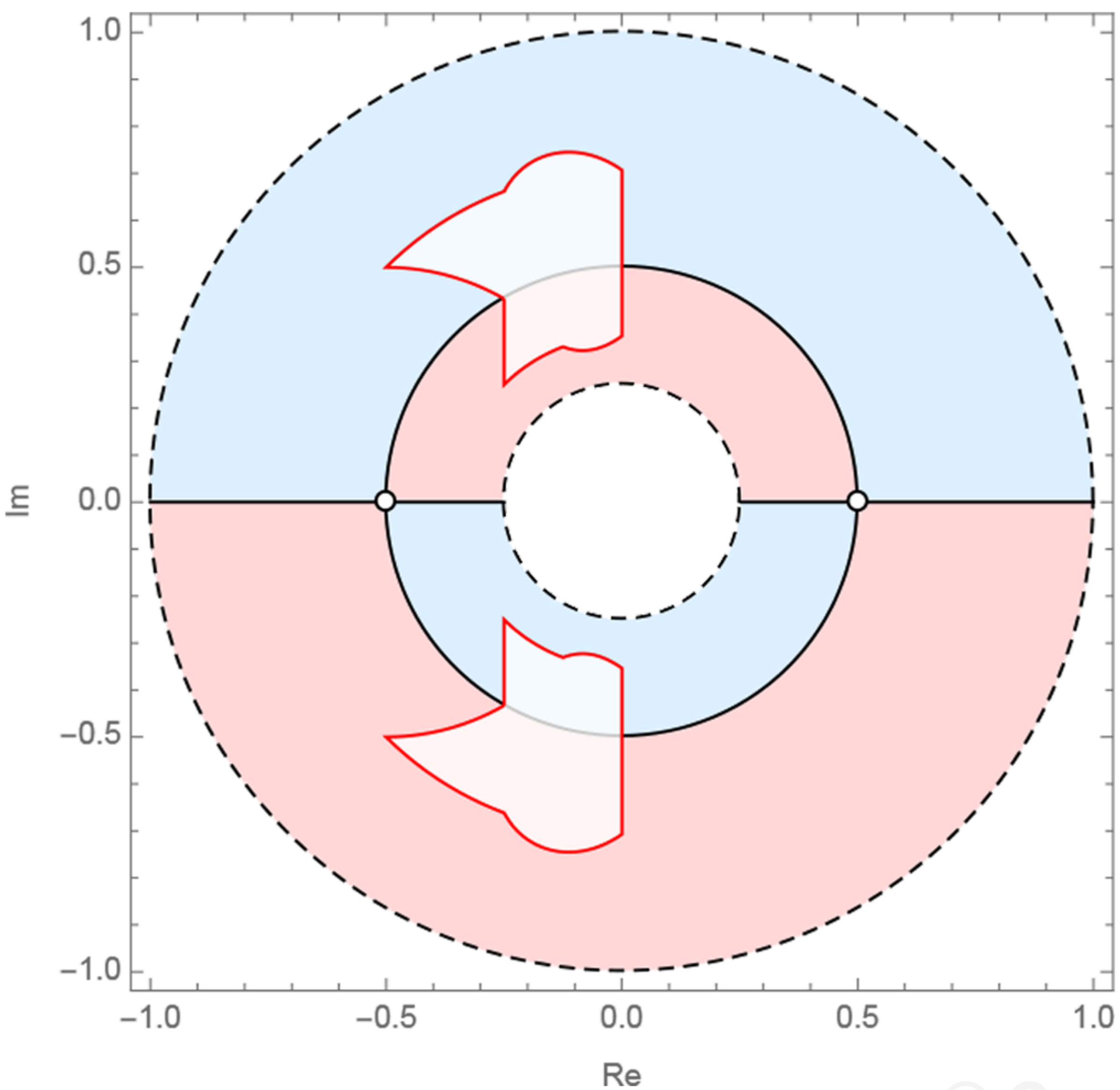
The unstable set M vs the stable set K

$$T_A(z) := T\{c_1(z), c_2(z), \ldots, c_n(z)\} \qquad \mathscr{R} = K \cup M$$

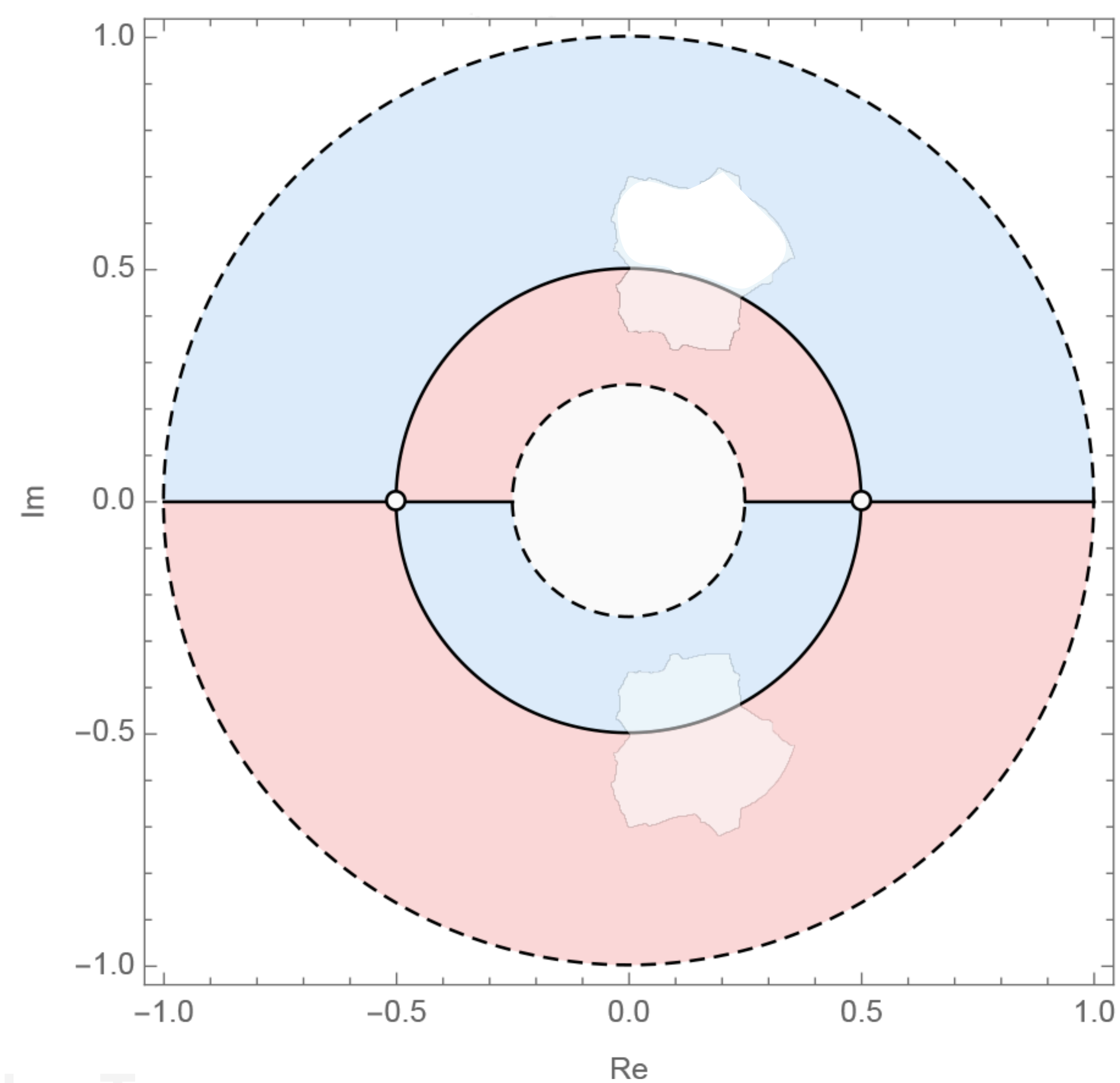
$$K := \{z \in \mathscr{R} : Q_A = Q_A(z)\}$$

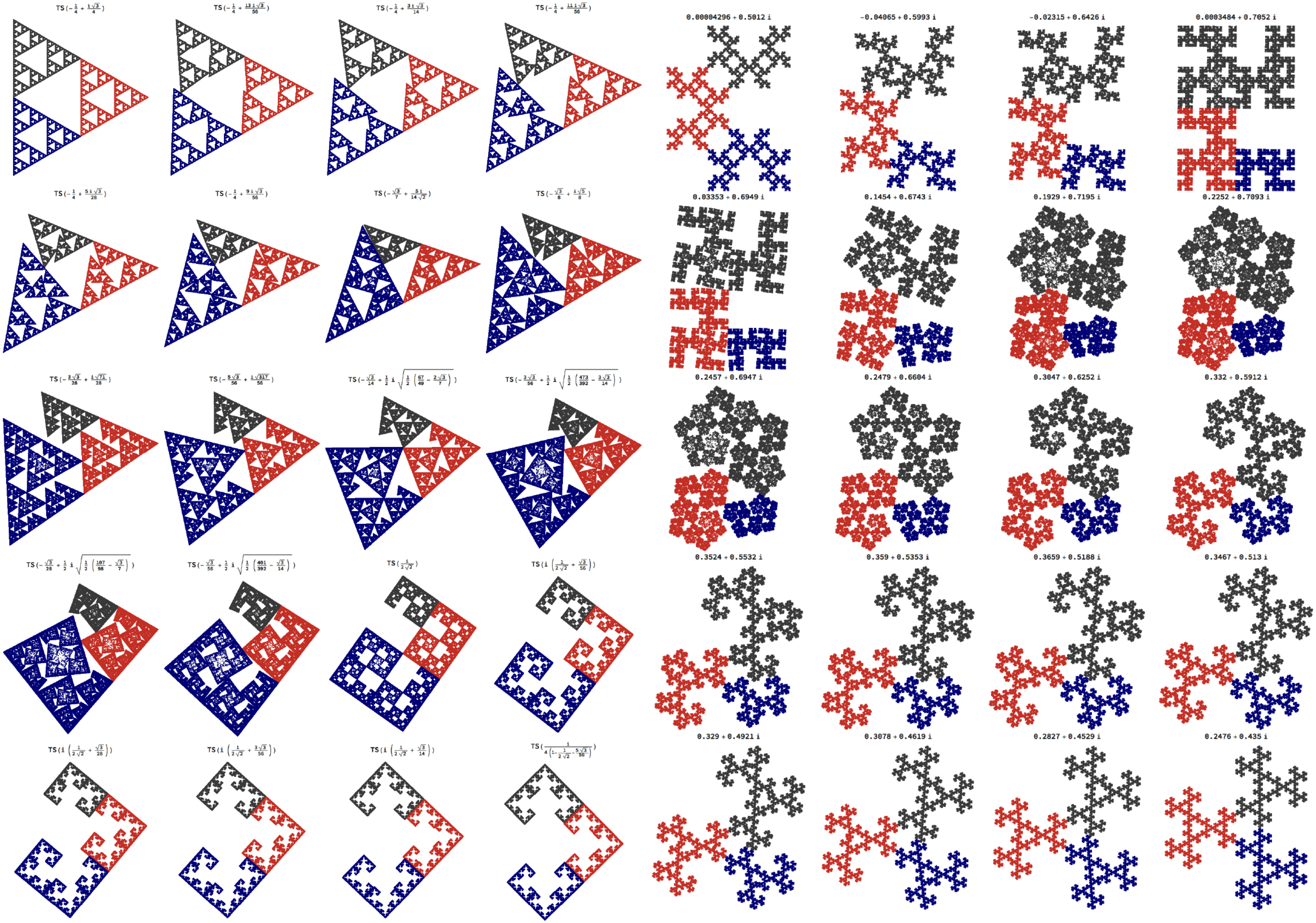
$$M := \{z \in \mathscr{R} : Q_A \subsetneq Q_A(z)\}$$

$$Q_A = \{13\bar{2} \sim 21\bar{2},\ 31\bar{2} \sim 23\bar{2}\}$$



$$Q_A = \{12\overline{31} \sim 22\overline{13},\ 32\overline{13} \sim 22\overline{31}\}$$





$TS(-\frac{1}{4}+\frac{i\sqrt{3}}{4})$
$TS(-\frac{1}{4}+\frac{13i\sqrt{3}}{56})$
$TS(-\frac{1}{4}+\frac{3i\sqrt{3}}{14})$
$TS(-\frac{1}{4}+\frac{11i\sqrt{3}}{56})$
$TS(-\frac{1}{4}+\frac{5i\sqrt{3}}{28})$
$TS(-\frac{1}{4}+\frac{9i\sqrt{3}}{56})$
$TS(-\frac{\sqrt{3}}{7}+\frac{5i}{14\sqrt{2}})$
$TS(-\frac{\sqrt{3}}{8}+\frac{i\sqrt{5}}{8})$
$TS(-\frac{3\sqrt{3}}{28}+\frac{i\sqrt{71}}{28})$
$TS(-\frac{5\sqrt{3}}{56}+\frac{i\sqrt{317}}{56})$
$TS(-\frac{\sqrt{3}}{14}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{67}{49}-\frac{2\sqrt{3}}{7}\right)})$
$TS(-\frac{3\sqrt{3}}{56}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{473}{392}-\frac{3\sqrt{3}}{14}\right)})$
$TS(-\frac{\sqrt{3}}{28}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{107}{98}-\frac{\sqrt{3}}{7}\right)})$
$TS(-\frac{\sqrt{3}}{56}+\frac{1}{2}i\sqrt{\frac{1}{2}\left(\frac{401}{392}-\frac{\sqrt{3}}{14}\right)})$
$TS(\frac{i}{2\sqrt{2}})$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{56}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{28}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{3\sqrt{3}}{56}\right))$
$TS(i\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{14}\right))$
$TS(\frac{i}{4\left(1-\frac{1}{2\sqrt{2}}-\frac{5\sqrt{3}}{56}\right)})$
0.00004296 + 0.5012 i
-0.04065 + 0.5993 i
-0.02315 + 0.6426 i
0.0003484 + 0.7052 i
0.03353 + 0.6949 i
0.1454 + 0.6743 i
0.1929 + 0.7195 i
0.2252 + 0.7093 i
0.2457 + 0.6947 i
0.2479 + 0.6604 i
0.3047 + 0.6252 i
0.332 + 0.5912 i
0.3524 + 0.5532 i
0.359 + 0.5353 i
0.3659 + 0.5188 i
0.3467 + 0.513 i
0.329 + 0.4921 i
0.3078 + 0.4619 i
0.2827 + 0.4529 i
0.2476 + 0.435 i

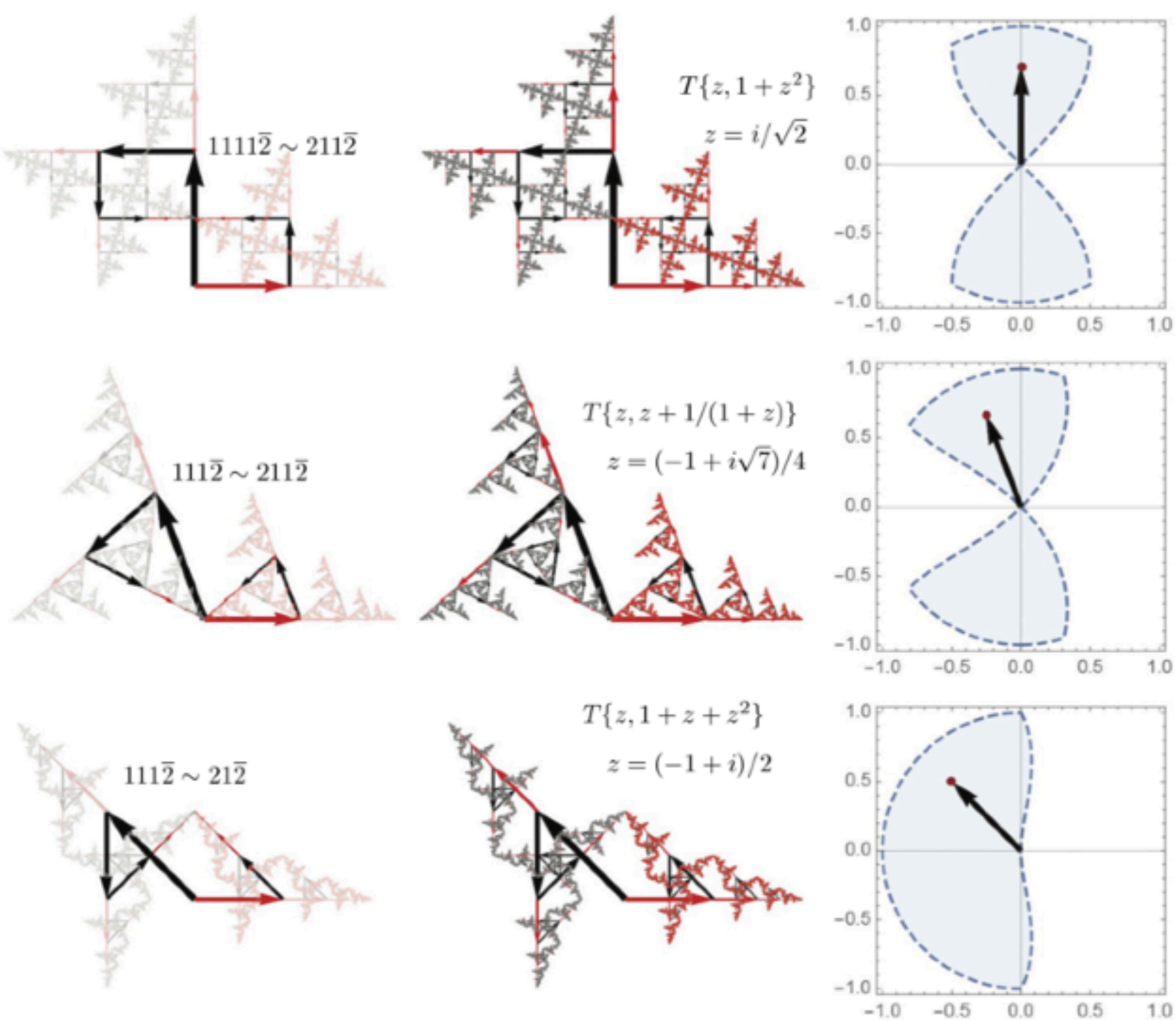


Figure 6: Left column, branch-paths of tip-to-tip equivalence relations $a \sim b$ where $F_{1\mathcal{A}} \cap F_{2\mathcal{A}} = \{\phi(a) = \phi(b)\}$. Central column, binary complex trees $T\{i/\sqrt{2}, 1/2\}$, $T\{(-1+i\sqrt{7})/4, 1/2\}$, and $T\{(-1+i)/2, 1/2\}$. Right column, regions $\mathcal{R} := \{z \in \mathbb{C} : 0 < |z|, |c_2(z)| < 1\}$ where one-parameter families of complex trees $T_{\mathcal{A}}(z) = T\{z, c_2(z)\}$ are defined. From top to bottom the algebraic expression for $c_2(z)$ was obtained in (12), (13), and (14) respectively.

$$\mathcal{M}_2 := \{z \in \mathcal{R} : 1 < |c_1(z)|^2 + |c_2(z)|^2 + \cdots + |c_n(z)|^2\}. \tag{28}$$

See below the regions $\mathcal{M}_2$ for the families of binary trees, $T\{z, 1+z^2\}$, $T\{z, z+1/(1+z)\}$, and $T\{z, 1+z+z^2\}$, obtained in (12), (13), and (14) respectively.

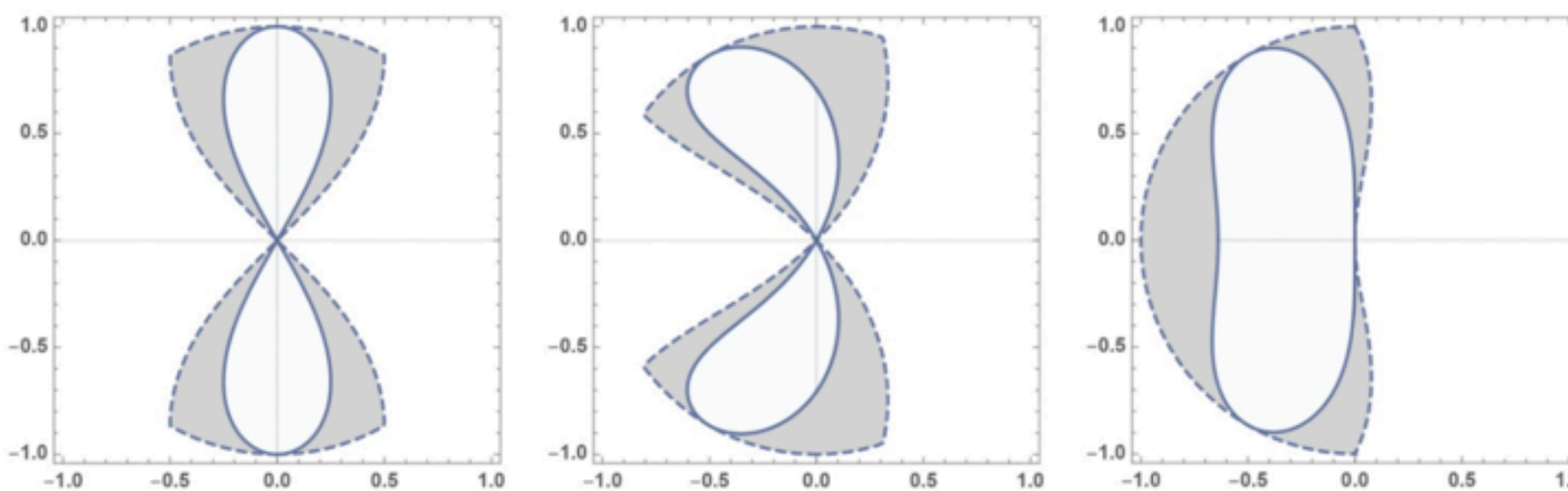


Figure 13: Regions $\mathcal{R}$ shown in figure 6 overlaid with regions $\mathcal{M}_2$ entirely composed of non-p.c.f. unstable trees.

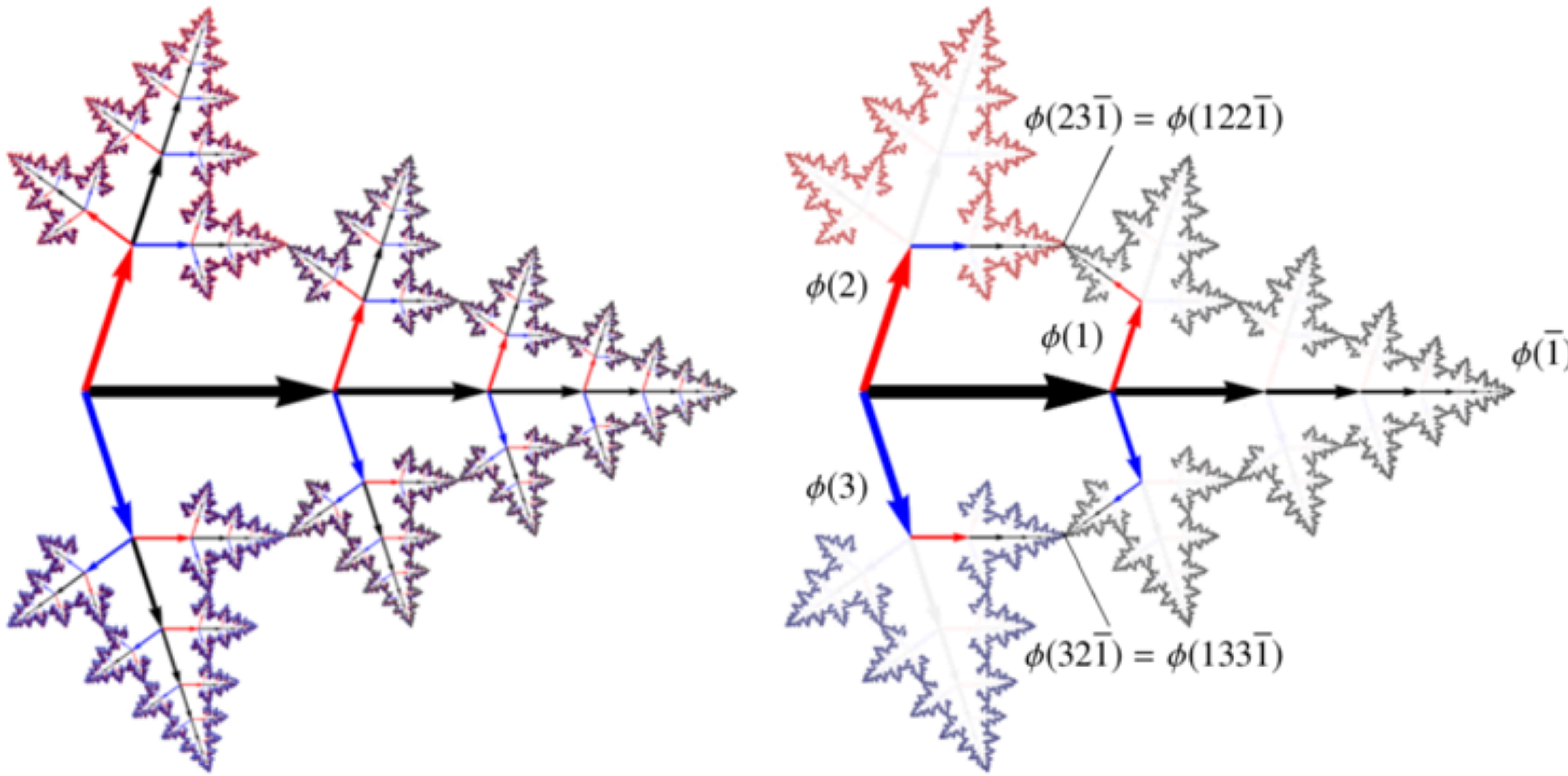
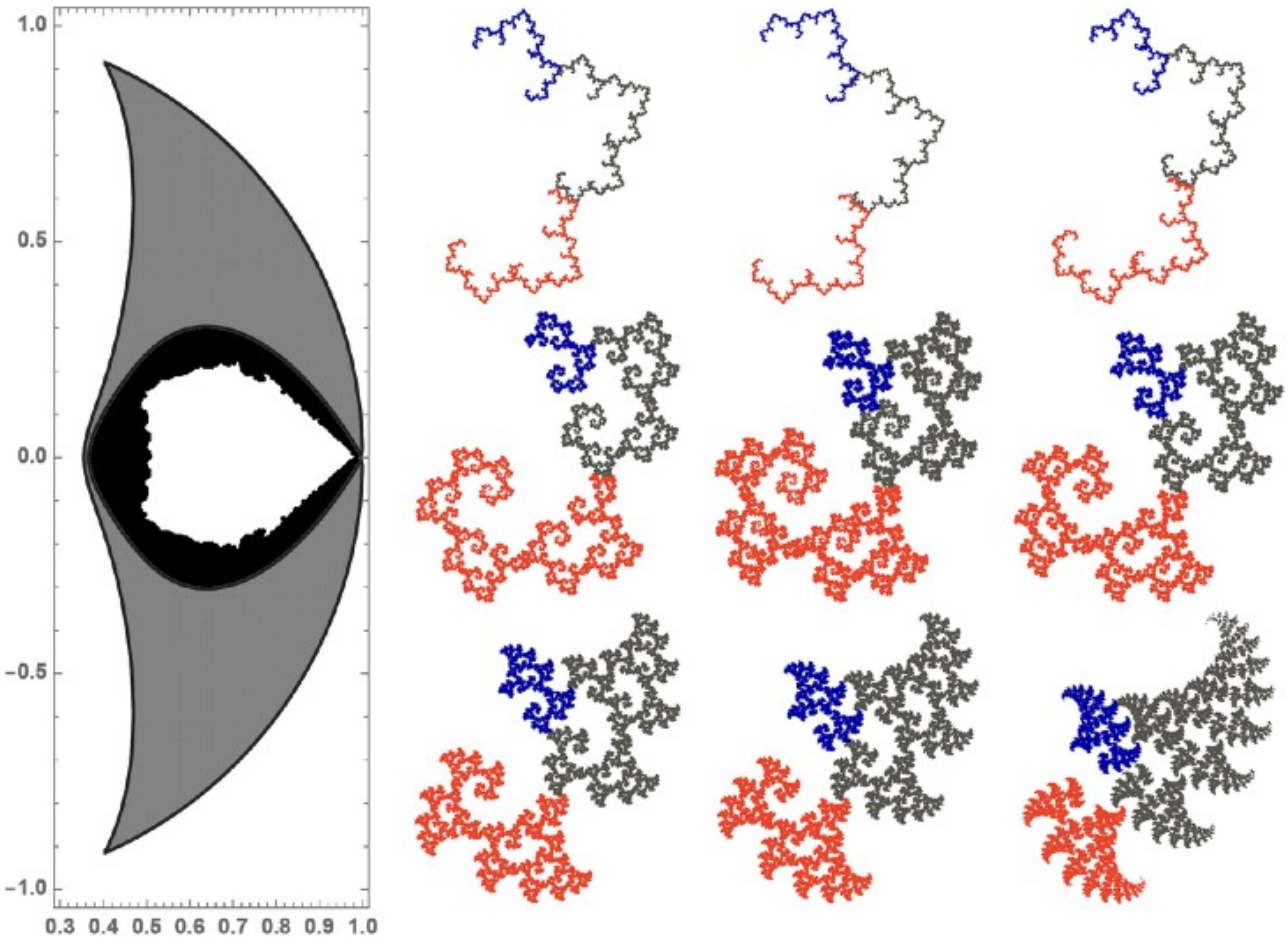


Figure 1: *Ternary complex teee $T\{c_1, c_2, c_3\}$ with pieces scaling as $|c_1| = 1/\tau$ and $|c_2| = |c_3| = 1/\tau^2$, where τ is the golden ratio $\tau = 1.6180\ldots$. The tree is mirror-symmetric since $c_1 = 1/\tau$ and $c_2 = c_3^*$.*

$$\begin{cases} \phi(23\bar{1}) = \phi(122\bar{1}) \\ \phi(32\bar{1}) = \phi(133\bar{1}) \end{cases} \Rightarrow \begin{cases} c_2(z) := \frac{1-z-z^2+z^3+\sqrt{1-2z-z^2-z^4+2z^5+z^6}}{2(z+z^2)} \\ c_3(z) := 2 + 1/z + z - c_2(z) \end{cases} \Rightarrow \quad T_{\mathcal{A}}(z) = T\{z, c_2(z), c_3(z)\}.$$


1.0
0.5
0.0
−0.5
−1.0
0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

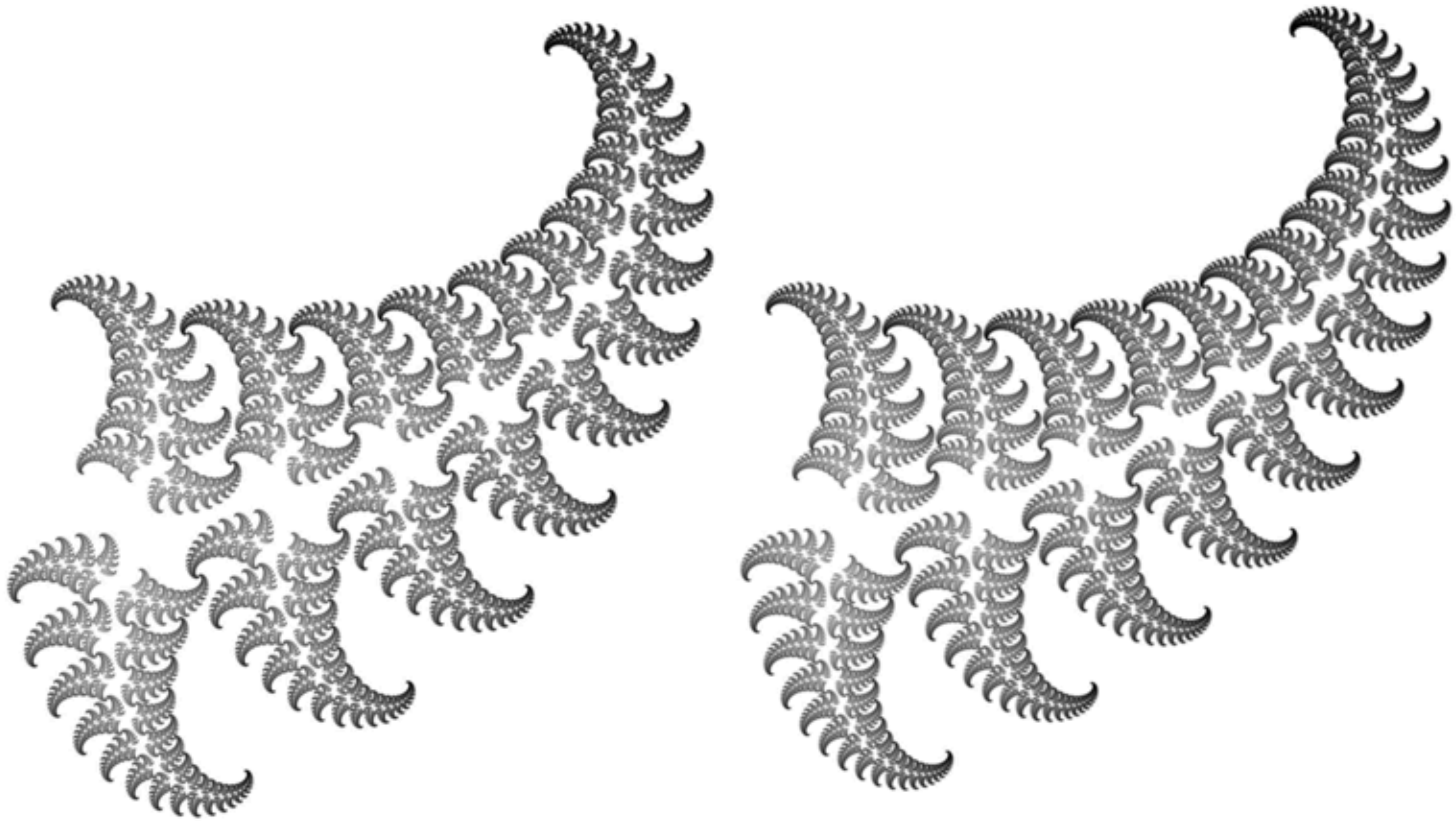


Figure 4: *Tipsets* $F_{\mathcal{A}}(z)$ *and* $F_{\mathcal{A}}(z')$ *where* $z \approx 0.814 + i0.146$ *is a root of* $1 - z^2 - z^3 - z^4 + z^5 + z^6 + z^7$ *and* $z' \approx 0.843 + i0.126$ *is a root of* $1 - z^2 - z^4 - z^5 + z^6 + z^7 + z^9$. *Both points lie in the boundary* $\partial\mathcal{M}$ *with extra tip-to-tip equivalence relations* $21113\overline{1} \sim 12\overline{1}$ *and* $211113\overline{1} \sim 12\overline{1}$.

Related Work: Family of Binary Complex Trees

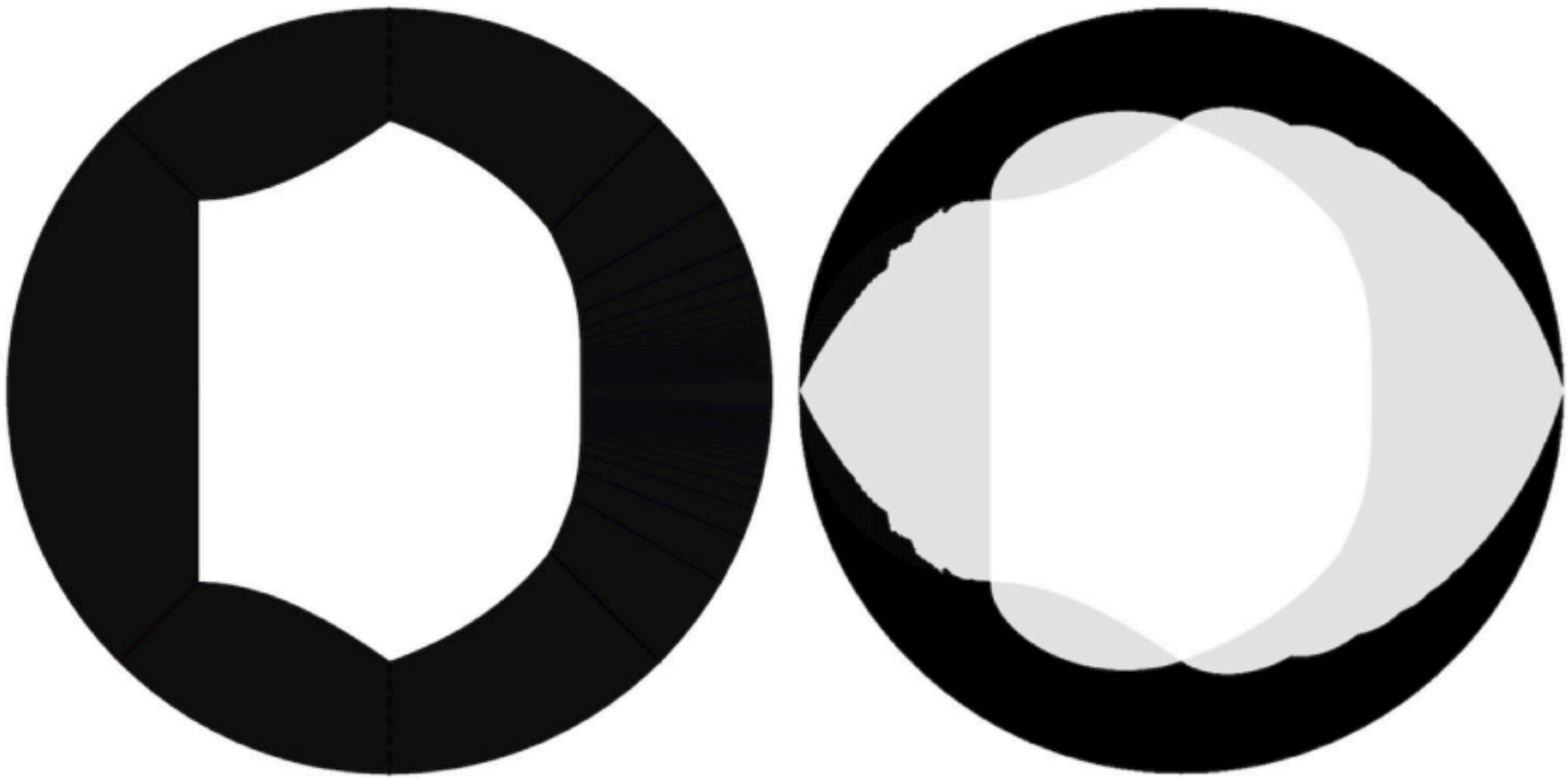
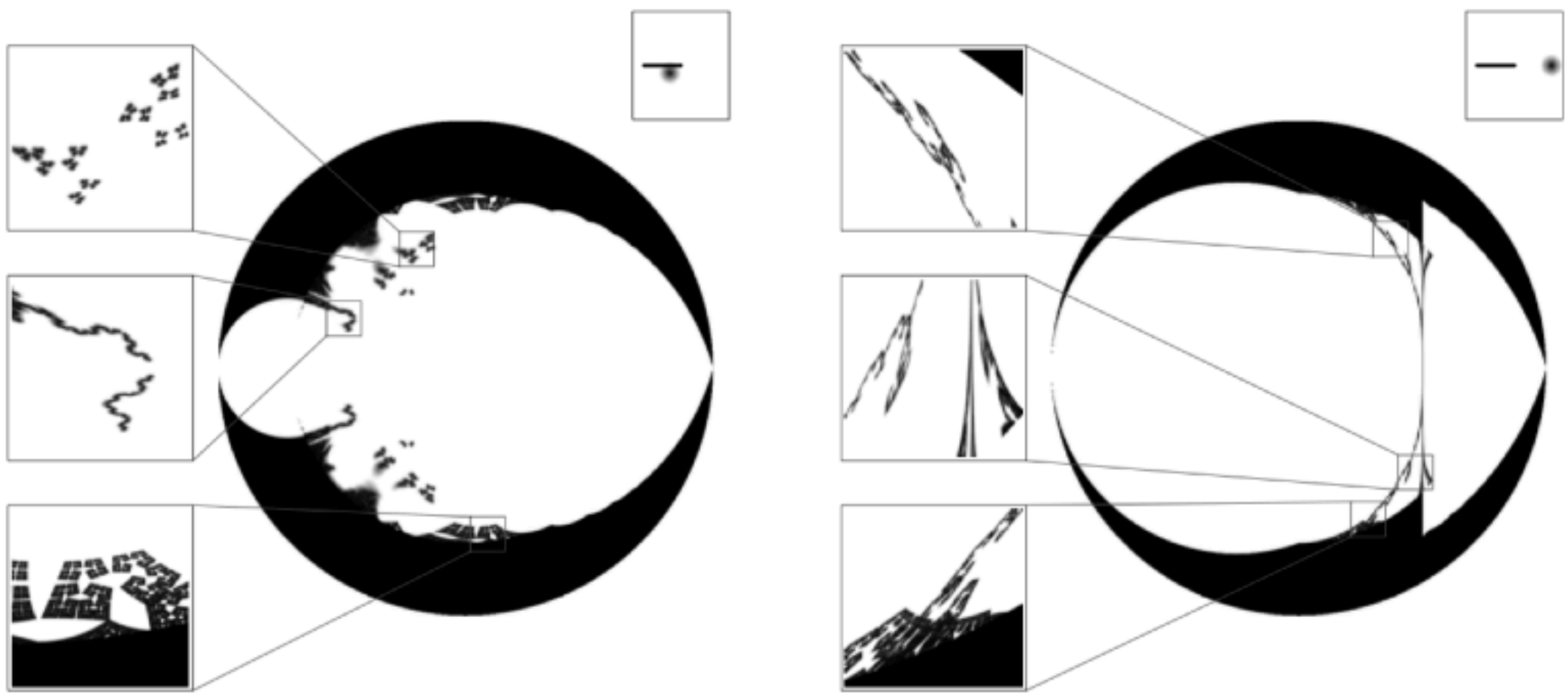
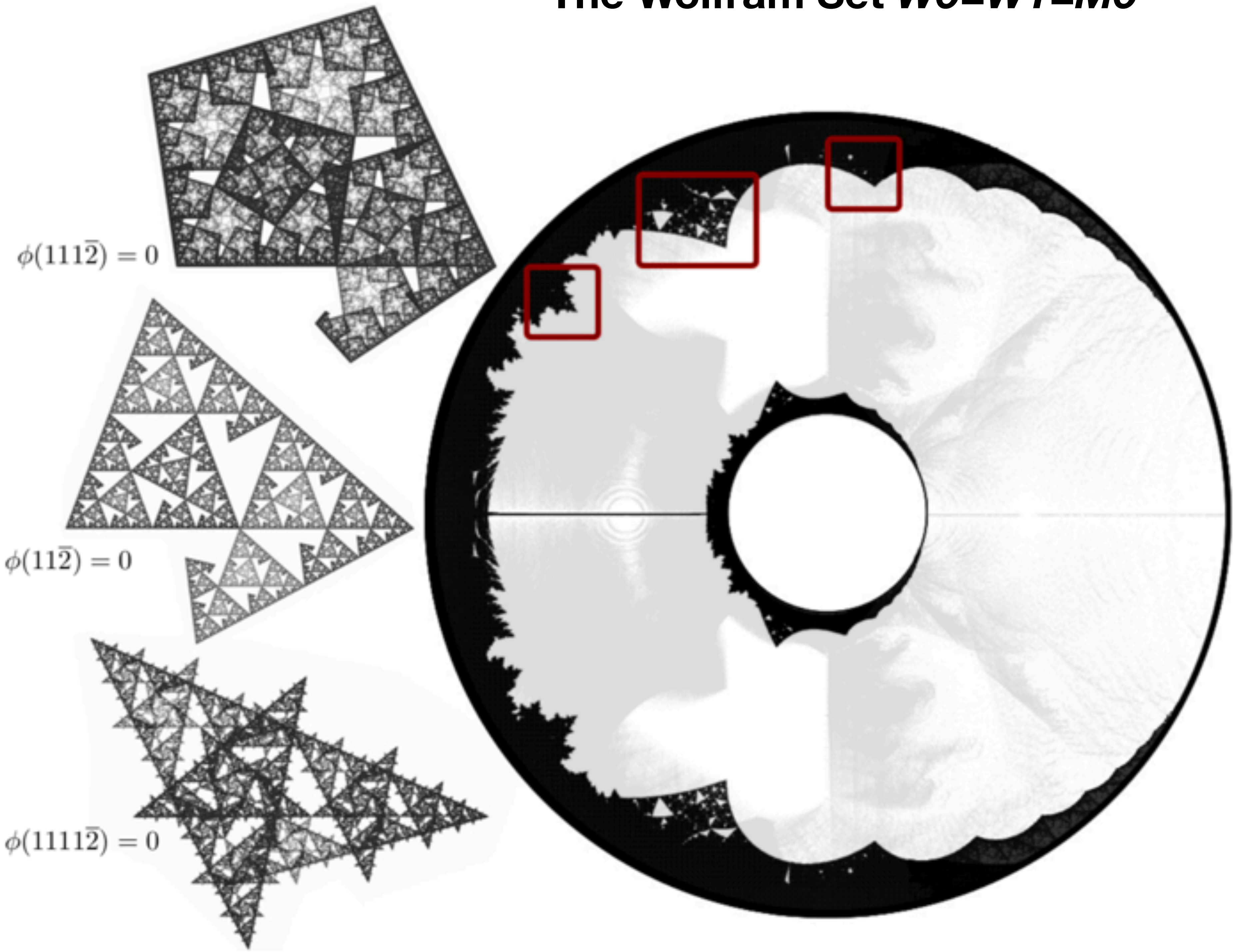


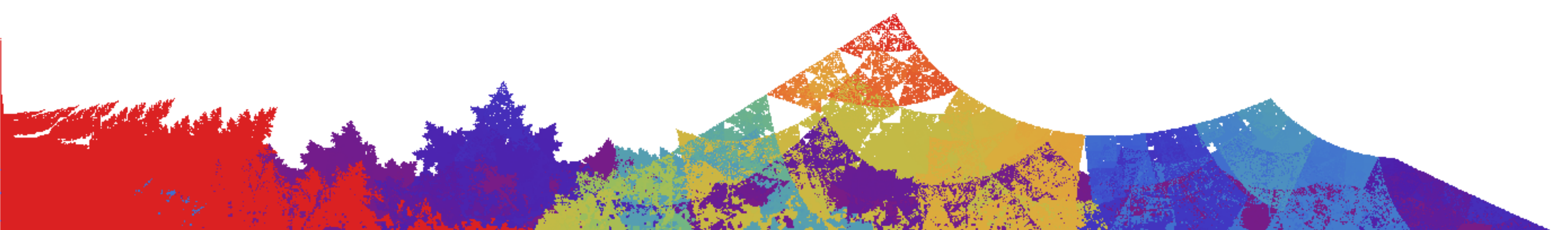
Figure 17: The unstable set $\mathcal{M} = \{c : Q_{\{c,c^*\}} \neq \emptyset\}$, and the root connectivity set $\mathcal{M}_0 = \{c : 0 \in F_{\{c,c^*\}}\}$.

The Wolfram Set $W_\lambda := \{c : \lambda \in F\{c, c^*\}\}$



The Wolfram Set *W0=W1=M0*





The Wolfram Set $W_\lambda := \{z : \lambda \in F_A(z)\}$ **where** $A(z) := \{z, 1/2, 1/4z\}$

$\lambda \in [0,2]$